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SOLUTION Exercise Sheet 5: Mechanics of Wind Turbines

In this exercise sheet we'll explore the role of deflections and vibrations in wind turbine design, focusing on the blades and the tower. To accomplish this exploration, we will play briefly with simple Euler-Bernoulli beam theory, the Rayleigh energy method, and the Campbell diagram.

Blade deflection

1. In this problem, we would like to explore the blade deflection. Let's assume that the blade is approximately straight so that it lays more-or-less in the rotor tip plane, even when deflected.

   We will use the same three-bladed demonstration turbine. Turbine A is defined by the following parameters: the rotor radius $R = 50$m, and blades of constant chord $c = 5$m and constant profile shape. Turbine A is running in a free-stream wind of $u_\infty = 12$m/s with air density $\rho = 1.225$kg/m^3. For a solid symmetric airfoil, we can approximate the blade's second moment of area as $I_x \approx K_\mathrm{I} c^4 \tau^3$, where K_I approx 0.036 and $\tau = t_\mathrm{max}/c$ is the maximum airfoil thickness to chord length ratio. Assume that the airfoil non-dimensional thickness τ =24 percent.

   (a) We might make the assumption that the blade behaves as a slender beam. In that case, the downwind-direction blade deflection x could be found with Euler-Bernoulli beam theory from the distributed load q, the Young's modulus E and the cross-section second moment of area I_x:

   $$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(EI_x \frac{\mathrm{d}^2 x}{\mathrm{d}r^2}\right) = q$$

   There are some boundary conditions to this integral:

   $$x(0) = 0; \quad x'(0) = 0; \quad x''(R) = 0; \quad x'''(R) = 0$$

   Briefly, what do these boundary conditions mean?

   The B.C. $x(0) = 0$ means that there is no deflection at the blade root because the blade root is pinned to the nacelle;
   $x'(0) = 0$ means that the cantilevered beam lays perpendicular to the boundary;
   $x''(R) = 0$ says that there is no bending at the tip of the blade;
   and $x'''(R) = 0$ says that there is no shear force at the tip of the blade.

   (b) What is the relationship between the downwind-direction blade deflection at the tip and the rotor radius R?

   As we assume I_x to be constant along the blade, our differential equation is transformed to the following form:

   $$\frac{\mathrm{d}^4 x(r)}{\mathrm{d}r^4} = \frac{q}{EI_x}$$

   Using thrust equation:

   $$\mathrm{d}T(r) = C_\mathrm{T} q_\infty (2\pi r \mathrm{d}r)$$

   we derive:

   $$q(r) = \frac{C_\mathrm{T} q_\infty (2\pi r)}{B}$$

where $C_{\mathrm{T}} = 4a(1-a)$ with a at optimum equal to $\frac{1}{3}$
If we integrate the differential equation four times, including constants of integration, we get:

$$x(r) = r^2\left(\frac{C_{\mathrm{T}} q_\infty 2\pi r^3}{120 B E K_{\mathrm{I}} c t^3} + \frac{c_0 r}{6} + \frac{c_1}{2}\right) + c_2 r + c_3$$

Then, we can differentiate three times and plug in the boundary conditions above.

$$x'''(r) = \frac{C_{\mathrm{T}} q_\infty 2\pi r^2}{2 B E K_{\mathrm{I}} c t^3} + c_0 \Rightarrow x'''(R) = \frac{C_{\mathrm{T}} q_\infty 2\pi R^2}{2 B E K_{\mathrm{I}} c t^3} + c_0 = 0 \Rightarrow c_0 = -\frac{C_{\mathrm{T}} q_\infty 2\pi R^2}{2 B E K_{\mathrm{I}} c t^3}$$

Again,

$$x''(r) = \frac{C_{\mathrm{T}} q_\infty 2\pi r^3}{6 B E K_{\mathrm{I}} c t^3} + c_0 r + c_1 \Rightarrow x''(R) = \frac{C_{\mathrm{T}} q_\infty 2\pi R^3}{6 B E K_{\mathrm{I}} c t^3} - \frac{C_{\mathrm{T}} q_\infty 2\pi R^3}{2 B E K_{\mathrm{I}} c t^3} + c_1 = 0 \Rightarrow c_1 = \frac{C_{\mathrm{T}} q_\infty 2\pi R^3}{3 B E K_{\mathrm{I}} c t^3}$$

Again,

$$x'(r) = \frac{C_{\mathrm{T}} q_\infty 2\pi r^4}{24 B E K_{\mathrm{I}} c t^3} + \frac{c_0 r^2}{2} + c_1 r + c_2 \Rightarrow c_2 = 0$$

Again,

$$x(r) = \frac{C_{\mathrm{T}} q_\infty 2\pi r^5}{120 B E K_{\mathrm{I}} c t^3} + \frac{c_0 r^3}{6} + \frac{c_1 r^2}{2} + c_2 r + c_3 \Rightarrow c_3 = 0$$

So, we've found the deflection relationship as:

$$x(r) = \frac{C_{\mathrm{T}} q_\infty 2\pi r^2}{6 B E K_{\mathrm{I}} c t^3}\left(\frac{r^3}{20} - \frac{R^2 r}{2} + R^3\right)$$

(c) For "Turbine A", what is the ratio between the tip blade deflection and the rotor radius, if the blade is made of the following materials?

i. carbon-fiber composite ($E \approx 150$GPa), At the tip, $r = R$. We can also plug in our given Turbine A parameters, to get:

$$x_{\text{carbon}} = 0.1\text{m}$$

ii. fiberglass aka. glass-reinforced plastic ($E \approx 17$GPa), At the tip, $r = R$. We can also plug in our given Turbine A parameters, to get:

$$x_{\text{GRP}} = 0.9\text{m}$$

iii. polystyrene ($E \approx 3$GPa)? At the tip, $r = R$. We can also plug in our given Turbine A parameters, to get:

$$x_{\text{polystyrene}} = 5.8\text{m}$$

(d) What trade-offs might be relevant when selecting blade material?

There are certain features of blade materials that we are likely to care about:

- how much a material costs over the total amount of that material that is needed to make a 'safe' design.
- how much the material weighs over the total amount of the material that is needed to make a 'safe' design.
- how easily the material can be manufactured into the 'safe' design.
- and further lifetime concerns, such as succeptibility (of the material and design) to fatigure, etc.

(e) The fact that the blades are rotating will likley lead to a smaller deflection than predicted here. Briefly, why would that be? What is this phenomenon called?

This phenomenon is called 'centrifugal stiffening.' It occurs because the centrifugal force of the rotation will act to pull the blade flat into the plane of rotation.

(f) Qualitatively, what happens to the blade loading under the following conditions?

i. yawed flow

When the flow is symmetric, the apparent velocity is equal at all azimuthal angles. But, when the flow is asymmetric (as in the case of yaw), then as the blade travels along the azimuth, the flow will occasionally have a component that moves with the blade and occasionally a component that moves opposed to the blade.
When the flow moves opposed to the blade, the apparent velocity will be higher. This leads to higher aerodynamic forces and higher bending moments at the blade root.
When the flow moves with the blade, this happens in reverse, and the blade experiences a lower bending moment.
When the flow is yawed, then these parts of the azimuth with varying moments will be at the top and bottom pass of the blade (defined respectively as 0^o and 180^o azimuthal angle.)
Which side has the higher and which side the lower apparent velocity depends on the direction of yaw, but it is typical to define yaw angles as positive when they put the high force at 180^o (or bottom dead center.)
(For completeness: there is an induction effect that tends to shift this pattern to higher azimuthal angles, but it is mainly relevant at low wind speeds where the induction factors are high.)

ii. shaft tilt

When the rotor is tilted, we have the same behavior as with yaw, except shifted by 90^o. Again, because there is a component of the wind that lays in the blade-tip-plane. This component will cause an increase in apparent velocity when the blade moves downwards (azimuthal angle $\psi = 90^o$) and a decrease with the blade moves upwards ($\psi = 270^o$).

iii. wind shear

As you saw with the logarithmic wind profile, we expect that wind speeds will increase with height. This means that apparent velocities, forces and bending moments, will all vary sinusoidally with the blade's azimuthal angle, having a maximum when $\psi = 0\,\text{deg}$ and a minimum when $\psi = 180\,\text{deg}$.

iv. tower shadow

Further, the tower acts as an obstacle to the flow. This means that the flow immediately ahead of the tower will be slowed, and behind the tower will be even slower and turbulent. Then, a blade passing through $\psi = 180\,\text{deg}$ will see less wind velocity, consequently less force, and less bending moment.

Preliminary tower design

2. We would like to make a preliminary design of a wind turbine tower. This tower should support an un-yawed and un-tilted three-bladed wind turbine ('Turbine B'), with the following dimensions:

Table 1: wind turbine dimensions and properties for Turbine B

property	symbol	value
tower height	L	84 m
nacelle + hub mass	m_{nac}	143 tonnes
rotor radius	R	12 m
design tip speed ratio	λ_{rated}	5
cut-in wind speed	$u_{\text{cut-in}}$	3 m/s
rated wind speed	u_{rated}	12 m/s
cut-out wind speed	$u_{\text{cut-out}}$	25 m/s

Some other information that you might find useful is as follows:

Table 2: other potentially useful information

property	symbol	value
density of A36 structural steel	ρ_{steel}	$7.8{\cdot}10^3$ kg/m^3
Young's modulus of A36 structural steel	E_{steel}	200 GPa
yield stress of A36 structural steel	U_{steel}	250 MPa
air density	ρ_{air}	1.225 kg/m^3
typical wind turbine structural safety factor	f_{safety}	1.35

(a) **rotor thrust**

i. What is the design angular velocity Ω_{rated} of the wind turbine?

The design angular velocity Ω_{rated} can be found from the design tip speed ratio λ_{rated}, the rated wind speed u_{rated} and the rotor radius R. (All of these parameters are given in Table 1.)

$$\Omega_{\text{rated}} = \frac{\lambda_{\text{rated}} u_{\text{rated}}}{R}.$$

ii. What is the magnitude of the thrust force F on the rotor as a function of u_{rated}?

The thrust force F can be found based on the thrust coefficient:

$$F(u_{\text{rated}}) = C_{\text{T}} \left(\frac{1}{2} \rho (u_{\text{rated}})^3 \right) (\pi R^2)$$

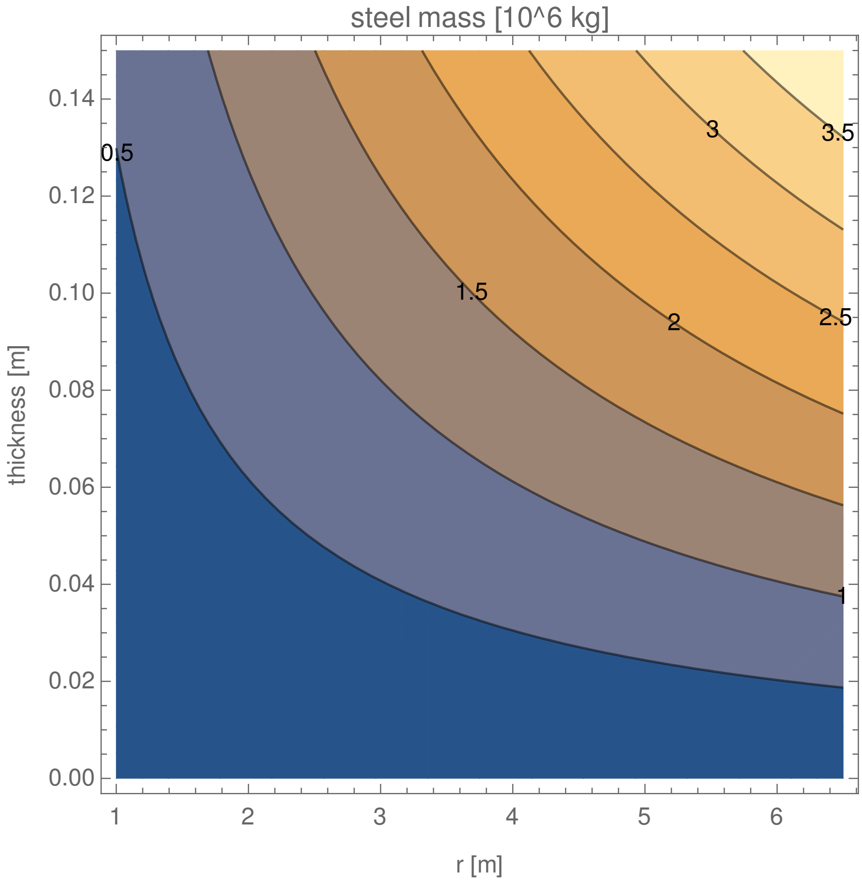
(b) **tower bending stress**

Let's consider the tower as a simple cantilevered beam, where the rotor thrust is acting at the top of the tower.

Let's assume that the tower is a thin walled tube with a constant cross-section along its length. This constant cross-section is an annulus, with an outer radius of r and a thickness τ.

i. Make a contour plot of the total mass of steel in the tower, based on $r \in [1\text{m}, 6\text{m}]$ and $\tau \in [0\text{m}, 0.15\text{m}]$.

The steel mass is $m_{\text{steel}} = \rho_{\text{steel}} L \pi (r^2 - (r - \tau)^2)$:



ii. You happen to know that the second moment of area of a filled circlular area with radius a is $\pi a^4/4$. What is the second moment of area I_x of the tower cross-section?

The tower will be bent along one of the symmetric axes of the annulus. Since the second moment of area is an integral over the area, we can construct I_{annulus} from I_{circle}. That is, the second moment of area:

$$I_x = I_{\text{outercircle}} - I_{\text{innercircle}} = \frac{\pi}{4} \left(r^4 - (r - \tau)^4 \right)$$

iii. What is the distance d between the beam's neutral axis and the outer radius?

The beam's neutral axis is the centroid of the beam's cross-section, which - since an annulus is symmetric - is the center of the annulus. Then, this distance is just r.

iv. What is the bending moment of the tower at the ground due only to the thrust on the rotor M_T?

We know the bending moment due to the rotor thrust is:

$$M_\text{T} = LF(u_\text{Ref}) = LC_\text{T}(u_\text{Ref}) \left(\frac{1}{2}\rho u_\infty(L, u_\text{Ref})^2\right)(\pi R^2).$$

v. What is the maximum stress σ_max due to bending on the tower?

The bending stress can be found from the bending moment, the distance to the neutral axis, and the moment of inertia:

$$\sigma_\text{max} = \frac{Mr}{I_x}$$

vi. Considering the safety factor f_safety, please devise a ratio ϕ which indicates whether the tower can safely support the maximum bending stress. Let's define $\phi < 1$ as safe, and $\phi > 1$ as unsafe.

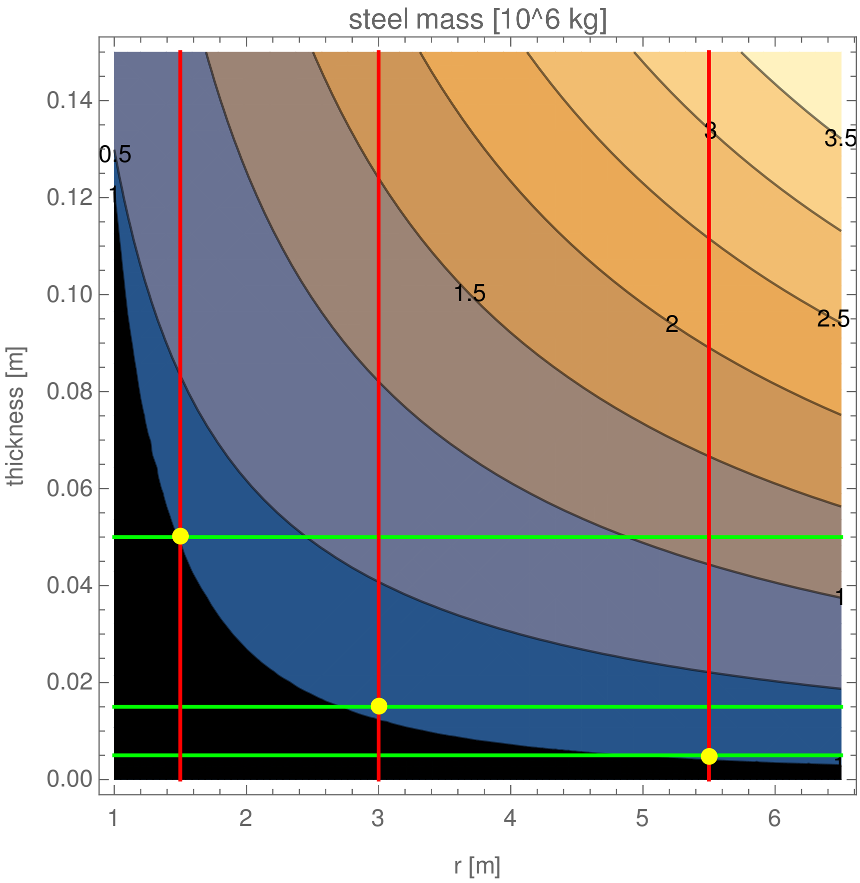
Let's define this 'safe' ratio ϕ as:

$$\phi = \frac{f_\text{safety}\sigma_\text{max}}{U_\text{steel}}$$

Notice that the safety factor has to 'inflate' the actually increased stress, because it is defined positive.

vii. For the following proposed tower outer diameters r, what thickness τ would you propose? Please motivate your choices. Also, please round thicknesses to the nearest 5mm.

What we can do, for the following diameters, is to plot the line of $\phi = 1$ on a plot of thickness vs. tower radius. We know that ϕ has to be smaller than one for the tower to be 'safe', so we can shade out the region of the plot where $\phi > 1$. (Here, I've done that over the same steel mass plot from before.
Then, to avoid over-designing the system (which would be expensive), we might try to chose the smallest allowed thickness for a 'safe' design. The intersections between thickness (green) and radius (red) give our design points, in the following plot...



Notice that we're not allowed to round our thicknesses 'down' because then ϕ becomes 'unsafe'. We can only round the thicknesses 'up' to the nearest 5mm. (I acknowledge that this is a fairly arbitrary number, but - in real life - sheet metal cannot be ordered in continuous thicknesses, but only in discrete units of thickness.)

A. $r = 5.5$ m

From the plot, we find a thickness $t = 0.005$m.

B. $r = 3.0$ m

From the plot, we find a thickness $t = 0.015$m.

C. $r = 1.5$ m

From the plot, we find a thickness $t = 0.05$m.

(c) **tower natural frequency**

Let's use Rayleigh's energy method to estimate the natural frequency of the tower. In this method, we assume that the strain energy from bending perfectly trades off with the kinetic energy of the tower's displacement x. We will again approximate the tower as a cantilevered beam.

Let's assume that the tower's displacement is sinusoidal in time:

$$x(t) = x_0 \sin(\omega t)$$

and that the tower remains approximately straight during its displacement.

Futher, we know that the strain energy from bending can be found as:

$$E_{\text{pot}} = \frac{1}{2} k x^2, \text{ where } k = 3 \frac{E_{\text{steel}} I_x}{L^3}.$$

i. What is $\dot{x}(t)$?

Let's differentiate:

$$\dot{x}(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = x_0 \omega \cos(\omega t)$$

ii. What is the kinetic energy due to the nacelle displacement T_{nac}?

We can find the kinetic energy with the standard expression:

$$T_{\mathrm{nac}} = \frac{1}{2} m_{\mathrm{nac}} \dot{x}(t)^2 = \frac{1}{2} m_{\mathrm{nac}} x_0^2 \omega^2 \cos^2(\omega t)$$

iii. What is the kinetic energy due to the displacement of the tower T_{t}? (*Hint: the tower is not massless...*) (*Hint: also, you might assume that the deflection of the tower is roughly proportional to the distance to the fixed point.*)

Here, we have to integrate:

$$T_{\mathrm{t}} = \int_0^L \frac{1}{2} m' \left(\dot{x}(t) \frac{y}{L} \right)^2 \mathrm{d}y$$

Here, m' is the mass per unit length of the tower: $m' = \rho_{\mathrm{steel}}(\pi)(r^2 - (r - \tau)^2)$. Also, y is the length along the tower.
The term $\left(\dot{x}(t) \frac{y}{L} \right)$ in the integrand is the deflection that is proportional to the distance to the fixed point.
Then, the kinetic energy gives:

$$T_{\mathrm{t}} = \left(\frac{m' \omega^2 y^3 x_0^2 \cos^2(\omega t)}{6L^2} \right) \Bigg]_0^L = \frac{m' L \omega^2 x_0^2 \cos^2(\omega t)}{6L^2}$$

iv. What is the total kinetic energy T of the swaying cantilevered beam?

Then, the total kinetic energy T is the sum of the tower and nacelle kinetic energies:

$$T = T_{\mathrm{nac}} + T_{\mathrm{t}} = \frac{1}{6} \left(3m_{\mathrm{nac}} + Lm' \right) \omega^2 x_0^2 \cos^2(\omega t)$$

v. What equation can you formulate, that would implicitly define the vibration frequency ω?

If all of the strain energy from bending gets transformed into kinetic energy, then the strain energy when the tower is at greatest deflection ($\sin(\omega t) = 1$) must equal the kinetic energy when the tower deflection is smallest ($\cos(\omega t) = 1$).
That is:

$$V(t = \pi/(2\omega)) = T(t = 0)$$

Alternatively:

$$\begin{aligned} \frac{3}{2} \frac{E_{\mathrm{steel}} I_x}{L^3} x_0^2 &= \frac{1}{6} \left(3m_{\mathrm{nac}} + Lm' \right) \omega^2 x_0^2 \\ \omega^2 &= 9 \frac{E_{\mathrm{steel}} I_x}{\left(3m_{\mathrm{nac}} + Lm' \right) L^3} \end{aligned}$$

vi. Please find ω. Starting from above, we get:

$$\omega^2 = \frac{3E_{\text{steel}}I_x}{(m_{\text{nac}} + m_{\text{t}}/3)\,L^3}$$

Where, $m_{\text{t}} = L\rho_{\text{steel}}\pi(r^2 - (r-\tau)^2)$, based on previous definitions...
Then:

$$\omega = \left(\frac{3E_{\text{steel}}I_x}{(m_{\text{nac}} + m_{\text{t}}/3)\,L^3}\right)^{\frac{1}{2}}$$

Notice that negative frequencies are not particularly meaningful, such that only the positive root of ω should be given.

vii. What is the natural frequency f_{nat} of the cantilevered tower?

We can convert from ω to f_{nat} as:

$$f_{\text{nat}} = \omega/(2\pi)$$

viii. What is the natural frequency of each of the three potential tower designs (defined by r and τ) that you determined in Exercise Sheet 3, Problem 2b)? (*Hint: If you do not have this solution, you can use the following combinations of* (r,τ)*:* $(1.5m, 0.05m)$, $(3.0m, 0.015m)$, $(5.5m, 0.005m)$.)

A. $r = 5.5$ m Plugging in the given values, gives: $f_{\text{nat}} \approx 0.6$ Hz.

B. $r = 3.0$ m Plugging in the given values, gives: $f_{\text{nat}} \approx 0.4$ Hz.

C. $r = 1.5$ m Plugging in the given values, gives: $f_{\text{nat}} \approx 0.2$ Hz.

(d) **Campbell diagram**

i. With what frequency (1P, 2P, 3P, ...) would you expect the tower to experience the following effects? What is this frequency (in Hertz), as a function of the wind turbine's rotor speed (in RPM)?

A. 'rotor-rotation' effects, such as having unequally dirty blades?

The load imbalance will occur at the same frequency as the entire rotor rotates. So, this occurs with a frequency of 1P (once periodic). The 1P frequency is equivalent to converting the RPM into Hertz:

$$f_{1\text{P}} = \text{RPM}\frac{1\,\text{min}}{60\text{s}}$$

B. 'blade-passing' effects, such as tower shadow?

Since $B = 3$ blades pass behind the shadow per rotation, the frequency of such effects is 3P.

$$f_{1\text{P}} = 3f_{1\text{P}} = 3\text{RPM}\frac{1\,\text{min}}{60\text{s}}$$

ii. Make a plot of frequency [Hz] vs rotor speed [RPM] that we will call the Campbell plot. Add the 'rotor-rotation' and 'blade-passing' frequencies into the Campbell plot. Include a 15 percent safety margin to either side of each curve.

See complete Campbell plot below...

iii. What is the design rotor speed (in RPM) of the wind turbine?

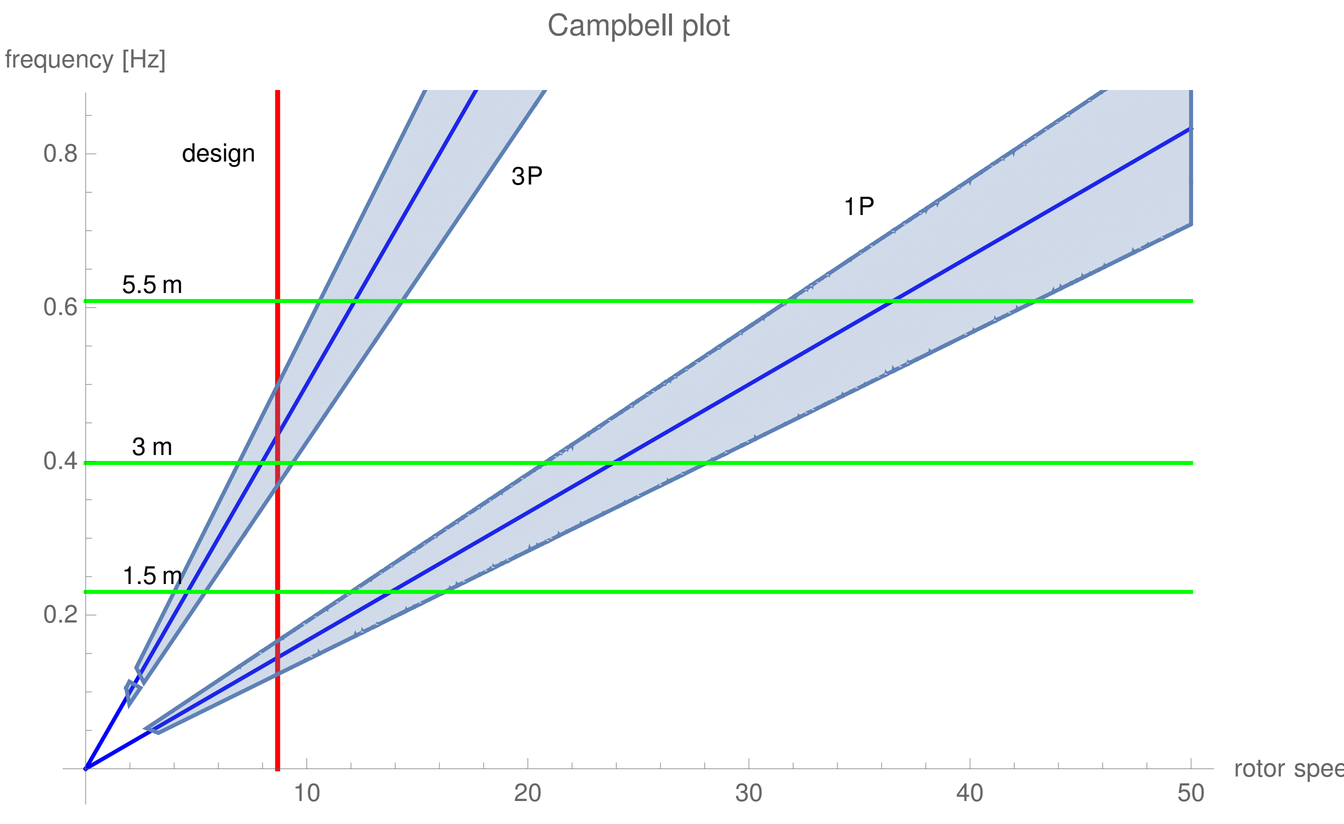
We can find the rotor speed from the design angular velocity Ω_{rated} of the rotor.

$$\text{RPM}_{\text{rated}} = \frac{60\text{s}}{1\,\text{min}} \frac{\Omega_{\text{rated}}}{2\pi}$$

iv. Please show the design rotor speed in the Campbell plot.

See complete Campbell plot below...

v. Please add the tower natural frequencies corresponding to your three possible tower designs (one for each of the outer diameters 5.5m, 3.0m and 1.5m) into the Campbell plot.



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vi. Which of the three investigated tower designs (outer diameters 5.5m, 3.0m, 1.5m) can be classified as the following? Please explain briefly.

A. soft-soft None of the investigated diameters gives a tower natural frequency less than the 1P frequency.

B. soft-stiff The 1.5m diameter tower has a natural frequency between the 1P and 3P frequency, and is consequently soft-stiff.

C. stiff-stiff The 5.5m diameter tower has a natural frequency above the 3P frequency, and is consequently stiff-stiff.

vii. Suggest some considerations you might have when chosing between your three proposed tower designs?

The 3.0m diameter tower is eliminated due to the risk of resonance. Then, the 5.5m diameter tower requires less steel overall (see the steel mass plot), but the 1.5m diameter tower would probably be easier to transport (think winding hairpin turns, in the mountains). The trade-off for selection would probably be determined by how easy it is to reach the site location, and how much steel (and machining) costs.