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SOLUTION Exercise Sheet *4*: Blade Element Momentum Method

In this problem, we want to see how the Blade Element Momentum (BEM) method gives the total thrust on a wind turbine.

To do this, consider an infinitesimally thin annulus (with radius r) sliced from a three-bladed ($B = 3$) wind turbine rotor of radius R. Aassume for the following problem that tip losses can be neglected. We will also again use $\mu = r/R$ the normalized radial position of the annulus.

The effective velocity at the rotor annulus is called $\boldsymbol{W}(r) = W(\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{t}})$, where $\hat{\boldsymbol{x}}$ points along the axis of rotation in the downwind direction, and $\hat{\boldsymbol{t}}$ points tangentially in the direction of rotation. Assume that the problem is axially symmetric so that all the blades behave identically.

We will use a demonstration turbine called 'Turbine A.' Turbine A is defined by the following parameters: tip speed ratio $\lambda = 7$, the local chord solidity $\sigma(r) = 8/(441\mu)$, the rotor radius $R = 50$m, the effective velocity angle $\phi = 5$deg, and the 2D lift and drag coefficients $c_\text{l} = 1$ and $c_\text{d} = 0.01$. Turbine A is running in a freestream wind of $u_\infty = 12$m/s with air density $\rho = 1.225$kg/m^3.

1. **Geometry**

(a) What is the area $\mathrm{d}A$ of the annulus, if the annulus has a thickness of $\mathrm{d}r$?

The area is the area of an annulus, with a thickness $\mathrm{d}r$:

$$\mathrm{d}A = 2\pi r \mathrm{d}r$$

(b) Assume that that the rotor is in a uniform flow field with a freestream wind $\boldsymbol{u}_\infty$ that is aligned with the rotor axis. What is the freestream dynamic pressure q_∞?

The freestream dynamic pressure is:

$$q_\infty = \frac{1}{2}\rho_\text{air} \|\boldsymbol{u}_\infty\|_2^2$$

(c) Find the magnitude of the effective velocity W in terms of some parameters of the wind turbine system: the freestream velocity $u_\infty = \|\boldsymbol{u}_\infty\|_2$, the tip speed ratio λ, the annulus radius r, rotor radius R, and the induction factors.

We know the components of $\boldsymbol{W}$:

$$\boldsymbol{W} = u_\infty(1-a)\hat{\boldsymbol{x}} + r\Omega(1+a')\hat{\boldsymbol{t}}.$$

Since $\Omega = u_\infty\lambda/R$, the magnitude $W = \|\boldsymbol{W}\|_2$ can be found to be:

$$W = u_\infty\left((1-a)^2 + (1+a')^2\lambda^2(r/R)^2\right)^{\frac{1}{2}}$$

(d) What is the effective dynamic pressure $q_\text{e}(r)$ based on the magnitude of the effective wind velocity?

The effective dynamic pressure is found the same way the freestream dynamic pressure was:

$$q_\text{e}(r) = \frac{1}{2}\rho W^2 = \frac{1}{2}\rho u_\infty^2\left((1-a)^2 + (1+a')^2\lambda^2(r/R)^2\right)$$

(e) Let's define the chord solidity $\sigma(r)$ as:

$$\sigma(r) = \frac{B}{2\pi\mu}\frac{c}{R}.$$

If $c(r)$ is the chord length of the blade at the annulus, what is the area $\mathrm{d}S$ of the blade section at the annulus?

We know that the area is the product of the chord and the annulus thickness: $\mathrm{d}S = c(r)\mathrm{d}r$. Using the definition of chord solidity, this simplifies to:

$$\mathrm{d}S = \frac{2\pi r}{B}\sigma \mathrm{d}r$$

2. **Momentum expressions**

(a) What is $\mathrm{d}T(r)$ ($\mathrm{d}F_\mathrm{A}$ in the lecture), the change in axial momentum in the flow due to that annulus, in terms of axial a and tangential a' induction factors?

We know the change in axial momentum based on the thrust coefficient:

$$\mathrm{d}T(r) = C_\mathrm{T} q_\infty \mathrm{d}A(r)$$

where the thrust coefficient reads as:

$$C_\mathrm{T} = 4a(1-a)$$

If we wanted to expand this, we would get:

$$\mathrm{d}T(r) = (\frac{1}{2}\rho u_\infty^2)(2\pi r \mathrm{d}r)4a(1-a)$$

(b) What is $\mathrm{d}Q(r)$, the change in angular momentum in the flow due to that annulus, in terms of axial a and tangential a' induction factors?

The change in angular momentum can be found with:

$$\mathrm{d}Q = 4a'(1-a)(\lambda\frac{r}{R})q_\infty r \mathrm{d}A(r) = 4a'(1-a)\lambda\rho u_\infty^2 \pi r^2 \frac{r}{R}\mathrm{d}r$$

3. **Blade element expressions**

(a) If you know that the blade section experiences lift ($\mathrm{d}\boldsymbol{L}$) and drag ($\mathrm{d}\boldsymbol{D}$) forces, what is the thrust $\mathrm{d}T(r)$ on the blade section (for one blade)?

Given the angle ϕ, we know that:

$$\mathrm{d}T = \|\mathrm{d}\boldsymbol{L}\|_2 \cos\phi + \|\mathrm{d}\boldsymbol{D}\|_2 \sin\phi$$

(b) Under the same conditions, what is the the torque $\mathrm{d}Q(r)$ on the blade element?

The torque will be the cross product of the moment arm and the force, so:

$$\mathrm{d}Q = (\|\mathrm{d}\boldsymbol{L}\|_2 \sin\phi - \|\mathrm{d}\boldsymbol{D}\|_2 \cos\phi)\, r$$

(c) Use the 2D lift and drag coefficients c_l and c_d to write your blade element thrust and torque expressions in terms of the defining parameters: B, u_∞, λ, r, R, a, a', ϕ, and σ.

As $\|\mathrm{d}\boldsymbol{L}\|_2 = c_\mathrm{l} q_\mathrm{e} \mathrm{d}S$ and $\|\mathrm{d}\boldsymbol{D}\|_2 = c_\mathrm{d} q_\mathrm{e} \mathrm{d}S$, we can rewrite $\mathrm{d}T$ and $\mathrm{d}Q$:

$$\begin{aligned}
\mathrm{d}T &= \frac{\sigma}{BR^2}\pi r \left((1+a')^2\lambda^2 r^2 + (1-a)^2 R^2\right) \rho u_\infty^2 (c_\mathrm{l}\cos\phi + c_\mathrm{d}\sin\phi)\mathrm{d}r \\
\mathrm{d}Q &= \frac{\sigma}{BR^2}\pi r^2 \left((1+a')^2\lambda^2 r^2 + (1-a)^2 R^2\right) \rho u_\infty^2 (c_\mathrm{l}\sin\phi - c_\mathrm{d}\cos\phi)\mathrm{d}r
\end{aligned}$$

4. **The full rotor thrust**

(a) You happen to learn that the induction factors can be approximated as:

$$a(\mu) \approx (0.8 + 28\mu)\cdot 10^{-2}, \qquad a'(\mu) \approx (0.3 + 0.6/\mu + 2.9\mu)\cdot 10^{-3}$$

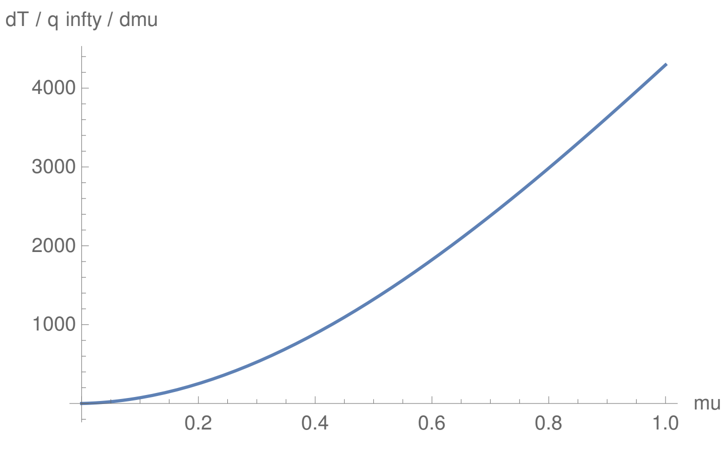
What is the thrust distribution over μ on one blade?

Let's take the momentum expression for rotor thrust and divide it by B to get the thrust distribution for one blade. (Again, the momentum and blade element expressions should be equal except for the factor of B, so let's take the one that's 'simpler'.)
Then,

$$\mathrm{d}T(\mu) = \frac{4}{B}(1-a)a(\rho u_\infty^2)(\pi\mu R^2 \mathrm{d}\mu)$$

* Appologies for the mix-up in notation, here...
Replacing a and a', and using our Turbine A parameters, we get:

$$\mathrm{d}T(\mu) = q_\infty(170\mu + 5800\mu^2 - 1600\mu^3)\mathrm{d}\mu$$



(b) For Turbine A, what is the thrust on the whole rotor?

Then, we can find T by integrating $\mathrm{d}T$ for all B blades:

$$T = B \int_0^1 \mathrm{d}T(\mu) \approx 4.2 \cdot 10^5 N$$

(c) For Turbine A, What is the thrust coefficient C_T for the full rotor?

Using the definition of the thrust coefficient:

$$C_\mathrm{T} = \frac{T}{q_\infty \pi R^2} \approx 0.6$$