Prof. Dr. Moritz Diehl, Nick Harder and Rachel Leuthold

SOLUTION Exercise Sheet 3: Momentum Theory

In this exercise sheet we'll look into Bernoulli's principle and explore the Momentum theory. Bernoulli's principle is a key component to the classic momentum theory. We'll see what sorts of assumptions go into each model and how to apply the models.

Bernoulli's principle

1. Let's define a thin tube-like control volume (CV) in which the fluid velocity is everywhere parallel to the tubular structure. Then, fluid only enters the CV at one end of the CV and leaves the CV at the other end. Let's call the entrance end 'face A', and the exit end 'face B'. The fluid passing through each face has a speed v, pressure p and height z, as shown in the sketch.

The first law of thermodynamics says that a change in the energy E inside the CV is due to: Q the heat added to the CV, W the work done by the fluid in the CV, and E_A and E_B the energy that, respectively, enters and leaves the CV through faces A and B.

The work W can be divided into some parts: W_{shaft} the shaft work removed from the CV by machinery such as turbines and pistons; W_{flow} the flow work expended by the fluid as it moves to fill the CV; W_{viscous} the viscous work lost to fiction.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{Q} - \dot{W} + \dot{E}_A - \dot{E}_B = \dot{Q} - \dot{W}_{\text{shaft}} - \dot{W}_{\text{flow}} - \dot{W}_{\text{viscous}} + \dot{E}_A - \dot{E}_B.$$

Let's assume, for the following question, that the mass flow rate $\dot{m}$ into the CV equals the mass flow rate out of the CV, so that the mass m contained within CV is always constant. Further, assume that the fluid within the CV is incompressible, so that the fluid density ρ is always constant. Then, we know that the flow work can be found to be: $\dot{W}_{\text{flow}} = \left(\frac{\dot{m}_B}{\rho_B} p_B - \frac{\dot{m}_A}{\rho_A} p_A\right) = \frac{\dot{m}}{\rho}(p_B - p_A)$.

(a) What single-word name is given to a CV described by the picture and first paragraph?

This is a 'streamline.' (If the CV is thicker, but made of a bundle of streamlines, then it is a 'streamtube.'

(b) Let's define the rate of energy entering the CV as $\dot{E}_A := \dot{U}_A + \dot{E}_{\text{kinetic},A} + \dot{E}_{\text{potential},A}$, where U is the internal energy, E_{kinetic} is the kinetic energy and $E_{\text{potential}}$ is the potential energy. Define the kinetic and potential energy entering the CV through face A to express $\dot{E}_A$

The rate of energy entering the CV is:

$$\dot{E}_A = \dot{U}_A + \dot{m}(\frac{1}{2}v_A^2 + g z_A)$$

(c) Repeat the above question for face B.

The rate of energy leaving the CV is:

$$\dot{E}_{\mathrm{B}} = \dot{U}_{\mathrm{B}} + \dot{m}(\frac{1}{2}v_{\mathrm{B}}^2 + gz_B)$$

(d) What assumptions have we made to this point?

We've assumed the following so far:

- the CV is a STREAMLINE where the fluid velocity is everywhere parallel to the streamline;
- there is NO RADIATIVE ENERGY TRANSFER (equivalently, NO ENERGY EXCHANGE ACROSS STREAMLINE WALLS);
- the fluid does ONLY SHAFT, FLOW, AND VISCOUS WORK;
- only POTENTIAL, KINETIC, AND INTERNAL ENERGY are transfered with the fluid;
- there are NO SOURCES/SINKS inside the CV (equivalently, CONSTANT MASS FLOW RATE); and
- the fluid is INCOMPRESSIBLE.

(e) What remaining assumptions do we need to make in order to arrive at Bernoulli's principle:

$$\frac{1}{2}\rho v^2 + \rho g z + p = \text{constant?}$$

We know so far:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{Q} - \dot{W}_{\text{shaft}} - \dot{W}_{\text{viscous}} + \dot{U}_{\mathrm{A}} - \dot{U}_{\mathrm{B}} + \dot{m}\left((\frac{1}{2}v_{\mathrm{A}}^2 + gz_{\mathrm{A}}) - \frac{1}{\rho}(p_{\mathrm{B}} - p_{\mathrm{A}}) - (\frac{1}{2}v_{\mathrm{B}}^2 + gz_{\mathrm{B}})\right)$$

If the following assumptions are true:

- flow is STEADY such that $\frac{\mathrm{d}(\cdot)}{\mathrm{d}t} = 0$;
- flow is ADIABATIC (no heat added) such that $\dot{Q} = 0$;
- CV contains NO MACHINERY such as turbines, pumps, etc, such that $\dot{W}_{\text{shaft}} = 0$;
- flow is INVSICID (viscosity is zero) such that $\dot{W}_{\text{viscous}} = 0$;
- there is NO INTERNAL ENERGY CHANGE (temperature) between the inflow and the exit faces such that $\dot{U}_{\mathrm{A}} - \dot{U}_{\mathrm{B}} = 0$;

then, we simplify the above expression to:

$$(\frac{1}{2}v_{\mathrm{A}}^2 + gz_{\mathrm{A}}) - \frac{1}{\rho}(p_{\mathrm{B}} - p_{\mathrm{A}}) - (\frac{1}{2}v_{\mathrm{B}}^2 + gz_{\mathrm{B}}) \approx 0$$

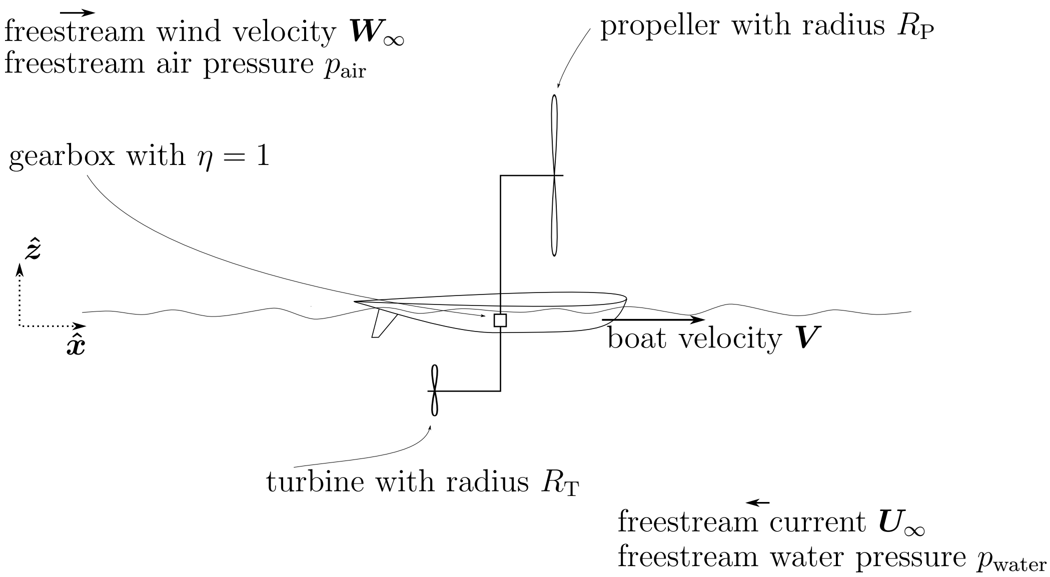
We can do some re-arranging to get:

$$\frac{1}{2}\rho v_{\mathrm{A}}^2 + \rho g z_{\mathrm{B}} + p_{\mathrm{A}} = \frac{1}{2}\rho v_{\mathrm{B}}^2 + \rho g z_{\mathrm{B}} + p_{\mathrm{B}}$$

Since we did not restrict where we cut the faces A and B, we see that $(\frac{1}{2}\rho v^2 + \rho g z + p)$ must be constant when travelling along the stream-line.

Classic Momentum Theory

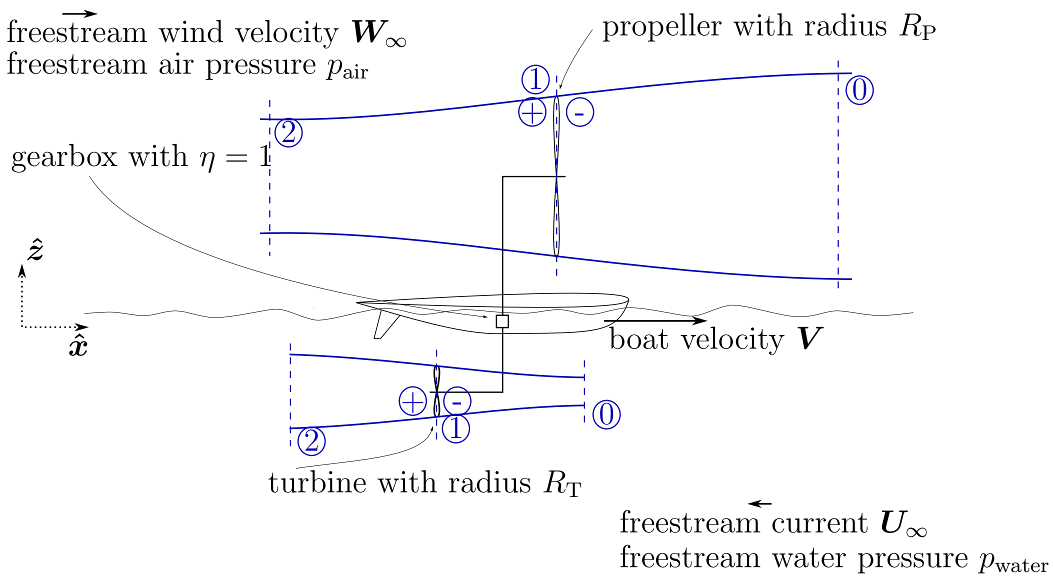
2. Let's tackle the momentum theory. To do this, let's consider a boat which extracts energy from the current to go faster than the wind. This boat is shown in the following sketch.



Assume a constant water density ρ_{water} and air density ρ_{air}, and that the boat is travelling parlllel to both the uniform freestream wind velocity $\boldsymbol{W}$, the uniform freestream current $\boldsymbol{U}$, and $\hat{\boldsymbol{x}}$ which is perpendicular to the acceleration of gravity.

(a) **Bernoulli and the streamtubes**

i. Make a sketch of the streamtubes around both the propeller and the turbine. Label the 'far-upstream' cross-section position as '0', the actuator-disk cross-section position as '1', and the 'far-downstream' cross-section position as '2'. Further, label the cross-section immediately upstream of '1' as '1^-', and the cross-section immediately downstream of '1' as '1^+'. (*Hint: remember which direction the fluid accelerates.*)
The propeller streamtube has areas $A_{\text{P},0}$, $A_{\text{P},1}$, and $A_{\text{P},2}$ at the specific cross-sections labelled. The turbine streamtube has areas $A_{\text{T},0}$, $A_{\text{T},1}$, and $A_{\text{T},2}$, at its cross-sections. The flow through the propeller streamtube has velocities $\boldsymbol{W}_0$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ at the cross-sections; the turbine streamtube has velocities $\boldsymbol{U}_0$, $\boldsymbol{U}_1$, $\boldsymbol{U}_2$.



ii. What is the mass flow rate through the turbine $\dot{m}_{\text{T}}$ and propller streamtubes $\dot{m}_{\text{P}}$?

The mass flow rate is the product of the fluid density and the speed of the fluid travelling through a given area.
Since we've said that all of the velocities in this picture are parallel, then the speed of the fluid through the streamtube cross-sections must be the magnitude of the vector.
That is:

$$\begin{aligned} \dot{m}_{\text{T}} &= \rho_{\text{water}} A_{\text{T},0} U_0 &= \rho_{\text{water}} A_{\text{T},1} U_1 &= \rho_{\text{water}} A_{\text{T},2} U_2, \\ \dot{m}_{\text{P}} &= \rho_{\text{air}} A_{\text{P},0} W_0 &= \rho_{\text{air}} A_{\text{P},1} W_1 &= \rho_{\text{air}} A_{\text{P},2} W_2. \end{aligned}$$

iii. Between which pairs of the 10 cross-sections (0, 1^-, 1, 1^+, 2 for turbine and propeller) can we plausibly argue that Bernoulli's principle holds?

We'll have most luck arguing for the assumptions behind Bernoulli's principle within individual streamtubes. But, we know from problem 1 that it will only work where we do not have turbomachinery extracting/adding work from/to the flow. So, there are only four pairs of cross-sections where there is any hope that a Bernoulli argument could be made:

$$0_{\text{T}} \text{ to } 1_{\text{T}}^-, \qquad 1_{\text{T}}^+ \text{ to } 2_{\text{T}}, \qquad 0_{\text{P}} \text{ to } 1_{\text{P}}^-, \qquad 1_{\text{P}}^+ \text{ to } 2_{\text{P}}$$

(Note that this is not saying that all of the assumptions made in problem 1 hold in these sections. Only that they are not obviously violated...)

iv. Use Bernoulli's expression to compare the dynamic pressure ($q = \frac{1}{2}\rho v^2 + p$, with a generic speed v and pressure p) between the pairs of cross-sections you selected above. (*Hint: what is the velocity immediately up- and down-stream of the actuator disk?*) (*Hint: what is the air pressure at the far-upstream and far-downstream cross-sections?*)

First, we should say that there is no average height difference along the streamtubes, because the boat velocity and the streamtubes are all perpendicular to the z axis. Then:

$$\begin{cases} \frac{1}{2}\rho_{\text{water}} U_0^2 + p_{\text{T},0} &= \frac{1}{2}\rho_{\text{water}} U_{1^-}^2 + p_{\text{T},1^-} \\ \frac{1}{2}\rho_{\text{water}} U_2^2 + p_{\text{T},2} &= \frac{1}{2}\rho_{\text{water}} U_{1^+}^2 + p_{\text{T},1^+} \\ \frac{1}{2}\rho_{\text{air}} W_0^2 + p_{\text{P},0} &= \frac{1}{2}\rho_{\text{air}} W_{1^-}^2 + p_{\text{P},1^-} \\ \frac{1}{2}\rho_{\text{air}} W_2^2 + p_{\text{P},2} &= \frac{1}{2}\rho_{\text{air}} W_{1^+}^2 + p_{\text{P},1^+} \end{cases}$$

$$\xrightarrow[p_{\mathrm{P},0} \approx p_{\mathrm{P},2} \approx p_{\mathrm{air}}]{\substack{W_{1^-} \approx W_{1^+} \approx W_1 \\ U_{1^-} \approx U_{1^+} \approx U_1 \\ p_{\mathrm{T},0} \approx p_{\mathrm{T},2} \approx p_{\mathrm{water}}}}$$

$$\begin{cases} \frac{1}{2}\rho_{\mathrm{water}} U_0^2 + p_{\mathrm{water}} & = & \frac{1}{2}\rho_{\mathrm{water}} U_1^2 + p_{\mathrm{T},1^-} \\ \frac{1}{2}\rho_{\mathrm{water}} U_2^2 + p_{\mathrm{water}} & = & \frac{1}{2}\rho_{\mathrm{water}} U_1^2 + p_{\mathrm{T},1^+} \\ \frac{1}{2}\rho_{\mathrm{air}} W_0^2 + p_{\mathrm{air}} & = & \frac{1}{2}\rho_{\mathrm{air}} W_1^2 + p_{\mathrm{P},1^-} \\ \frac{1}{2}\rho_{\mathrm{air}} W_2^2 + p_{\mathrm{air}} & = & \frac{1}{2}\rho_{\mathrm{air}} W_1^2 + p_{\mathrm{P},1^+} \end{cases}$$

We made the approximations that the flow velocity immediately up- and down-stream of the actuator disk are equivalent to the velocity at the disk itself, because our flow is incompressible, so there cannot be sudden shocks in the velocity.
We made the approximation that the far-upstream and far-downstream pressures have recovered to the free-stream pressure, because otherwise, the fluid in the streamtube would never reach equilibrium with the freestream. That would effectively mean that the streamtube would grow or shrink forever, and that (conceptually) sounds a lot like perpetual motion.

v. What is the relationship between U_0, U_2, $p_{\mathrm{T},1^-}$ and $p_{\mathrm{T},1^+}$?

Let's subtract the turbine Bernoulli relations from each other. This gives:

$$\frac{1}{2}\rho_{\mathrm{water}}(U_0^2 - U_2^2) = p_{\mathrm{T},1^-} - p_{\mathrm{T},1^+}.$$

vi. What is the relationship between W_0, W_2, $p_{\mathrm{P},1^-}$ and $p_{\mathrm{P},1^+}$?

Let's subtract the propeller Bernoulli relations from each other. This gives:

$$\frac{1}{2}\rho_{\mathrm{air}}(W_0^2 - W_2^2) = p_{\mathrm{P},1^-} - p_{\mathrm{P},1^+}.$$

(b) **the turbine actuator**

i. If the force across the actuator disk is a pressure difference over some area, what is the force $\boldsymbol{F}_{\mathrm{T}}$ exerted by the turbine on the flow? Please give a vector, not just a magnitude, within the coordinate system $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ shown in the sketch...

The force exerted by the turbine has a magnitude $(p_{\mathrm{T},1^-} - p_{\mathrm{T},1^+})\, A_{\mathrm{T}1}$ and must point along $\hat{\boldsymbol{x}}$ because the flow is being slowed down. Since we're pulling energy out of the flow at the actuator disk, there has to be a drop in pressure. Then $p_{\mathrm{T},1^-} \geq p_{\mathrm{T},1^+}$, so that the force is:

$$\boldsymbol{F}_{\mathrm{T}} = (p_{\mathrm{T},1^-} - p_{\mathrm{T},1^+})\, A_{\mathrm{T}1}\hat{\boldsymbol{x}}$$

And, we know from the previous questions that this pressure difference can be evaluated as:

$$\boldsymbol{F}_{\mathrm{T}} = \frac{1}{2}\rho_{\mathrm{water}}(U_0^2 - U_2^2) A_{\mathrm{T}1}\hat{\boldsymbol{x}}$$

ii. If the force across the actuator disk is equivalent to the rate of change in momentum of the flow within the streamtube, what is the force $\boldsymbol{F}_\text{T}$ exerted by the turbine on the flow?

The force on the flow is the mass-flow rate of the fluid within the streamtube mulitplied by the change in velocity from the small $\hat{\boldsymbol{x}}$ position to the large $\hat{\boldsymbol{x}}$ position. That is:

$$\boldsymbol{F}_\text{T} = \dot{m}_\text{T}(U_0 - U_2)\hat{\boldsymbol{x}} = \rho_\text{water} A_{\text{T},1} U_1 (U_0 - U_2)\hat{\boldsymbol{x}}.$$

iii. Let's define a turbine induction factor a_T such that $U_1 = U_0(1 - a_\text{T})$. Then, what is the relationship between U_2, a_T, and U_0?

We now have two relationships for $\boldsymbol{F}_\text{T}$. Let's set them equal, and see what we find...

$$\frac{1}{2}\rho_\text{water}(U_0^2 - U_2^2)A_{\text{T}1}\hat{\boldsymbol{x}} = \rho_\text{water} A_{\text{T},1} U_1 (U_0 - U_2)\hat{\boldsymbol{x}}$$

If we simplify, we get:

$$\frac{1}{2}(U_0 + U_2) = U_1$$

Now, plug in the induction factor definition, and re-arrange:

$$U_2 = 2U_1 - U_0 = 2(1 - a_\text{T})U_0 - U_0 = U_0(1 - 2a_\text{T}).$$

iv. How much power P_T does the turbine extract from the flow?

Power is force times velocity. So, the power exerted by the turbine on the flow must be:

$$P = \boldsymbol{F}_\text{T} \cdot \boldsymbol{U}_1 = -\frac{1}{2}\rho_\text{water}(U_0^2 - U_2^2)A_{\text{T}1}U_1$$

Remember that the flow through the turbine is moving in the $-\hat{\boldsymbol{x}}$ direction...

We can use the induction factor expressions from above to simplify:

$$P = -\frac{1}{2}\rho_\text{water} U_0^2 \left(1 - (1 - 2a_\text{T})^2\right) A_{\text{T}1} U_0 (1 - a_\text{T}) = -2\rho_\text{water} U_0^3 A_{\text{T},1} a_\text{T} (1 - a_\text{T})^2$$

Then the power P_T extracted from the flow is the negative of the amount exerted on the flow. So:

$$P_\text{T} = -P = 2\rho_\text{water} U_0^3 A_{\text{T},1} a_\text{T} (1 - a_\text{T})^2$$

(c) **the propeller actuator**

i. If the force across the actuator disk is a pressure difference over some area, what is the force $\boldsymbol{F}_\text{P}$ exerted by the propeller on the flow? Please give a vector, not just a magnitude, within the coordinate system $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ shown in the sketch...

The force exerted by the propeller has a magnitude $(p_{\text{P},1+} - p_{\text{P},1-})\, A_{\text{P}1}$ and must point along $-\hat{\boldsymbol{x}}$ because the flow is being accelerated. Since the propeller adds energy to the flow over the actuator disk, there must be a pressure jump. Then, $p_{\text{P},1+} \geq p_{\text{P},1-}$, this gives the force as:

$$\boldsymbol{F}_\text{P} = (p_{\text{P},1-} - p_{\text{P},1+})\, A_{\text{P},1}\hat{\boldsymbol{x}}$$

And, we know from the previous questions that this pressure difference can be evaluated as:

$$\boldsymbol{F}_{\mathrm{P}} = \frac{1}{2}\rho_{\mathrm{air}}(W_0^2 - W_2^2)A_{\mathrm{P},1}\hat{\boldsymbol{x}}$$

ii. If the force across the actuator disk is equivalent to the rate of change in momentum of the flow within the streamtube, what is the force $\boldsymbol{F}_{\mathrm{P}}$ exerted by the propeller on the flow?

The force on the flow is the mass-flow rate of the fluid within the streamtube multiplied by the change in velocity from the small $\hat{\boldsymbol{x}}$ position to the large $\hat{\boldsymbol{x}}$ position. That is:

$$\boldsymbol{F}_{\mathrm{P}} = \dot{m}_{\mathrm{P}}(W_0 - W_2)A_{\mathrm{P},1}\hat{\boldsymbol{x}} = \rho_{\mathrm{air}}A_{\mathrm{P},1}W_1(W_0 - W_2)\hat{\boldsymbol{x}}.$$

iii. Let's define a propeller induction factor a_{P} such that $W_1 = W_0(1 + a_{\mathrm{P}})$. Then, what is the relationship between W_2, a_{P}, and W_0?

We now have two relationships for $\boldsymbol{F}_{\mathrm{P}}$. Let's set them equal, and see what we find...

$$\boldsymbol{F}_{\mathrm{P}} = \frac{1}{2}\rho_{\mathrm{air}}A_{\mathrm{P},1}(W_0^2 - W_2^2)\hat{\boldsymbol{x}} = \rho_{\mathrm{air}}A_{\mathrm{P},1}W_1(W_0 - W_2)\hat{\boldsymbol{x}}.$$

If we simplify, we get:

$$\frac{1}{2}(W_0 + W_2) = W_1.$$

Now, plug in the induction factor definition, and re-arrange:

$$W_2 = 2W_1 - W_0 = 2(1 + a_{\mathrm{P}})W_0 - W_0 = W_0(1 + 2a_{\mathrm{P}}).$$

iv. How much power P_{P} does the propeller exert on the flow?

Power is force times velocity. So, the power exerted by the propeller on the flow must be:

$$P_{\mathrm{P}} = \boldsymbol{F}_{\mathrm{P}} \cdot \boldsymbol{W}_1$$

Remember that the flow through the propeller is along $-\hat{\boldsymbol{x}}$.

Then:

$$P_{\mathrm{P}} = -\frac{1}{2}\rho_{\mathrm{air}}(W_0^2 - W_2^2)A_{\mathrm{P},1}W_1$$

We can use the induction factor expressions from above to simplify:

$$P_{\mathrm{P}} = \frac{1}{2}\rho_{\mathrm{air}}W_0^2((1 + 2a_{\mathrm{P}})^2 - 1)A_{\mathrm{P},1}W_0(1 + a_{\mathrm{P}}) = 2\rho_{\mathrm{air}}W_0^3A_{\mathrm{P},1}a_{\mathrm{P}}(1 + a_{\mathrm{P}})^2$$