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Exercise Sheet 3: Momentum Theory

In this exercise sheet we'll look into Bernoulli's principle and explore the Momentum theory. Bernoulli's principle is a key component to the classic momentum theory. We'll see what sorts of assumptions go into each model and how to apply the models.

Bernoulli's principle

1. Let's define a thin tube-like control volume (CV) in which the fluid velocity is everywhere parallel to the tubular structure. Then, fluid only enters the CV at one end of the CV and leaves the CV at the other end. Let's call the entrance end 'face A', and the exit end 'face B'. The fluid passing through each face has a speed v, pressure p and height z, as shown in the sketch.

The first law of thermodynamics says that a change in the energy E inside the CV is due to: Q the heat added to the CV, W the work done by the fluid in the CV, and E_A and E_B the energy that, respectively, enters and leaves the CV through faces A and B.

The work W can be divided into some parts: W_shaft the shaft work removed from the CV by machinery such as turbines and pistons; W_flow the flow work expended by the fluid as it moves to fill the CV; W_viscous the viscous work lost to fiction.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{Q} - \dot{W} + \dot{E}_\mathrm{A} - \dot{E}_B = \dot{Q} - \dot{W}_\mathrm{shaft} - \dot{W}_\mathrm{flow} - \dot{W}_\mathrm{viscous} + \dot{E}_\mathrm{A} - \dot{E}_B.$$

Let's assume, for the following question, that the mass flow rate $\dot{m}$ into the CV equals the mass flow rate out of the CV, so that the mass m contained within CV is always constant. Further, assume that the fluid within the CV is incompressible, so that the fluid density ρ is always constant. Then, we know that the flow work can be found to be: $\dot{W}_\mathrm{flow} = \left(\frac{\dot{m}_B}{\rho_B} p_\mathrm{B} - \frac{\dot{m}_\mathrm{A}}{\rho_\mathrm{A}} p_\mathrm{A}\right) = \frac{\dot{m}}{\rho}(p_\mathrm{B} - p_\mathrm{A})$.

(a) What single-word name is given to a CV described by the picture and first paragraph?

(b) Let's define the rate of energy entering the CV as $\dot{E}_\mathrm{A} := \dot{U}_\mathrm{A} + \dot{E}_\mathrm{kinetic,A} + \dot{E}_\mathrm{potential,A}$, where U is the internal energy, E_kinetic is the kinetic energy and $E_\mathrm{potential}$ is the potential energy. Define the kinetic and potential energy entering the CV through face A to express $\dot{E}_A$

(c) Repeat the above question for face B.

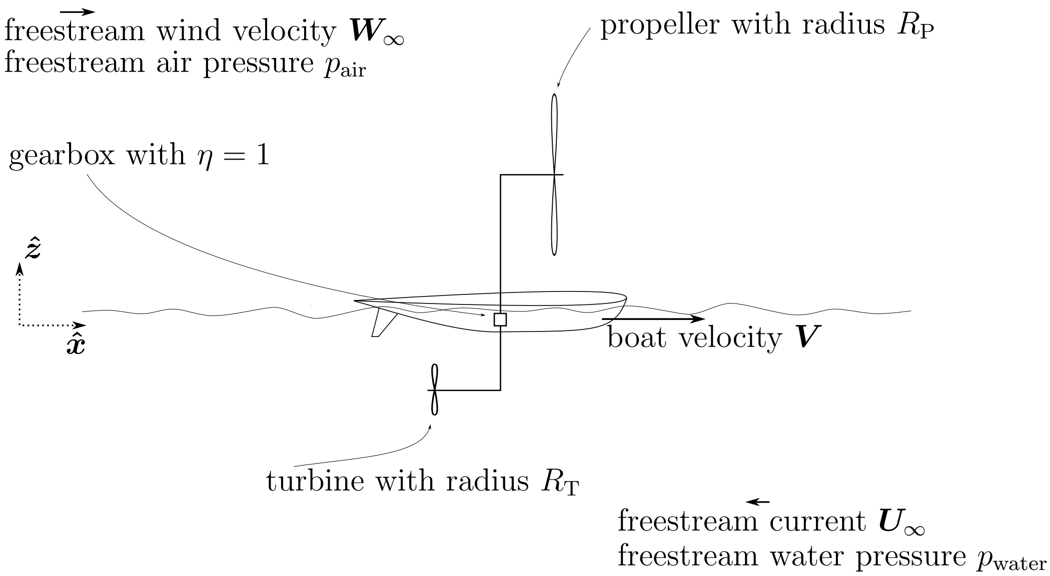
(d) What assumptions have we made to this point?

(e) What remaining assumptions do we need to make in order to arrive at Bernoulli's principle:

$$\frac{1}{2}\rho v^2 + \rho g z + p = \text{constant?}$$

Classic Momentum Theory

2. Let's tackle the momentum theory. To do this, let's consider a boat which extracts energy from the current to go faster than the wind. This boat is shown in the following sketch.



Assume a constant water density ρ_{water} and air density ρ_{air}, and that the boat is travelling parlllel to both the uniform freestream wind velocity $\boldsymbol{W}$, the uniform freestream current $\boldsymbol{U}$, and $\hat{\boldsymbol{x}}$ which is perpendicular to the acceleration of gravity.

(a) **Bernoulli and the streamtubes**

i. Make a sketch of the streamtubes around both the propeller and the turbine. Label the 'far-upstream' cross-section position as '0', the actuator-disk cross-section position as '1', and the 'far-downstream' cross-section position as '2'. Further, label the cross-section immediately upstream of '1' as '1^-', and the cross-section immediately downstream of '1' as '1^+'. (*Hint: remember which direction the fluid accelerates.*)
The propeller streamtube has areas $A_{\text{P},0}$, $A_{\text{P},1}$, and $A_{\text{P},2}$ at the specific cross-sections labelled. The turbine streamtube has areas $A_{\text{T},0}$, $A_{\text{T},1}$, and $A_{\text{T},2}$, at its cross-sections. The flow through the propeller streamtube has velocities $\boldsymbol{W}_0$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ at the cross-sections; the turbine streamtube has velocities $\boldsymbol{U}_0$, $\boldsymbol{U}_1$, $\boldsymbol{U}_2$.

ii. What is the mass flow rate through the turbine $\dot{m}_{\text{T}}$ and propller streamtubes $\dot{m}_{\text{P}}$?

iii. Between which pairs of the 10 cross-sections (0, 1^-, 1, 1^+, 2 for turbine and propeller) can we plausibly argue that Bernoulli's principle holds?

iv. Use Bernoulli's expression to compare the dynamic pressure ($q = \frac{1}{2}\rho v^2 + p$, with a generic speed v and pressure p) between the pairs of cross-sections you selected above. (*Hint: what is the velocity immediately up- and down-stream of the actuator disk?*) (*Hint: what is the air pressure at the far-upstream and far-downstream cross-sections?*)

v. What is the relationship between U_0, U_2, $p_{\text{T},1^-}$ and $p_{\text{T},1^+}$?

vi. What is the relationship between W_0, W_2, $p_{\text{P},1^-}$ and $p_{\text{P},1^+}$?

(b) **the turbine actuator**

i. If the force across the actuator disk is a pressure difference over some area, what is the force $\boldsymbol{F}_{\text{T}}$ exerted by the turbine on the flow? Please give a vector, not just a magnitude, within the coordinate system $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ shown in the sketch...

ii. If the force across the actuator disk is equivalent to the rate of change in momentum of the flow within the streamtube, what is the force $\boldsymbol{F}_\mathrm{T}$ exerted by the turbine on the flow?

iii. Let's define a turbine induction factor a_T such that $U_1 = U_0(1 - a_\mathrm{T})$. Then, what is the relationship between U_2, a_T, and U_0?

iv. How much power P_T does the turbine extract from the flow?

(c) **the propeller actuator**

i. If the force across the actuator disk is a pressure difference over some area, what is the force $\boldsymbol{F}_\mathrm{P}$ exerted by the propeller on the flow? Please give a vector, not just a magnitude, within the coordinate system $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}$ shown in the sketch...

ii. If the force across the actuator disk is equivalent to the rate of change in momentum of the flow within the streamtube, what is the force $\boldsymbol{F}_\mathrm{P}$ exerted by the propeller on the flow?

iii. Let's define a propeller induction factor a_P such that $W_1 = W_0(1 + a_\mathrm{P})$. Then, what is the relationship between W_2, a_P, and W_0?

iv. How much power P_P does the propeller exert on the flow?