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SOLUTION Exercise Sheet 1

Power Harvesting Factor

Consider a symmetrical, three-bladed ($B = 3$) wind turbine with rotor radius R. Assume a constant angular velocity Ω of the rotor and a uniform wind field with velocity $\boldsymbol{u}_\infty$ so that the dominant wind direction $\hat{\boldsymbol{i}}$ is along the turbine axis of rotation. We will also use a nondimensional spanwise position $\mu = r/R$ that is 0 at the blade root/rotor hub, and 1 at the blade tips.

1. What is the tip speed ratio λ of the turbine?

   The tip speed ratio $\lambda = \frac{\Omega R}{u_\infty}$ is the ratio between the blade speed u_b at the tip and the freestream wind speed u_∞.

   (We'll use the abreviation that $v = \|\boldsymbol{v}\|_2$ from here on.)

2. What is the local speed ratio λ_r at some spanwise location μ?

   The local speed ratio is the equivalent concept to the tip speed ratio, but considered at different spanwise positions μ. That is:

   $$\lambda_r = \mu\lambda = \mu\frac{\Omega R}{u_\infty}.$$

3. What is the effective wind (also called apparent velocity) $\boldsymbol{u}_\text{a}$ at the position μ?

   The apparent wind $\boldsymbol{u}_\text{a}$ is the difference between the freestream wind velocity $\boldsymbol{u}_\infty$ and the blade's motion $\boldsymbol{u}_\text{b}$ at the station. That is:

   $$\boldsymbol{u}_\text{a} = \boldsymbol{u}_\infty - \boldsymbol{u}_\text{b}.$$

   The velocity of the blade points in the tangential direction $\hat{\boldsymbol{j}}$, with magnitude λu_∞. That is:

   $$\boldsymbol{u}_\text{b} = \mu\Omega R\hat{\boldsymbol{j}} = \mu\lambda u_\infty\hat{\boldsymbol{j}}.$$

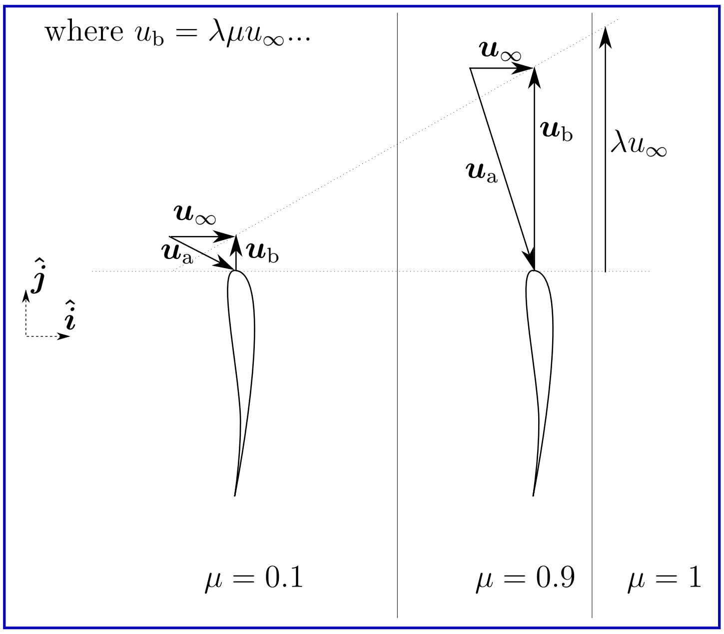
   So, the apparent velocity at position μ is:

   $$\boldsymbol{u}_\text{a} = u_\infty\hat{\boldsymbol{i}} - \mu\lambda u_\infty\hat{\boldsymbol{j}}$$

4. Sketch the velocity triangles for the following positions:

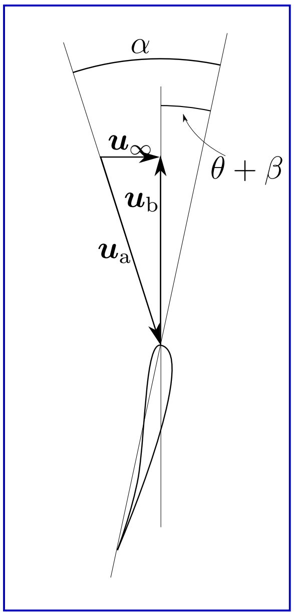
   (a) $\mu = 0.1$

   (b) $\mu = 0.9$



5. Assume that the blades are uniformly pitched with an angle β, but have a 'perfect' twist distribution $\theta(\mu)$ so that α always takes its design value of 6 degrees if $\beta = 0$. What is $\theta(\mu)$?

The angle of attack is the angle between the chord line and the apparent velocity.



This gives:

$$\alpha = \phi + \theta + \tan^{-1}\left(\frac{1}{\lambda\mu}\right) = \phi + \theta + \cot^{-1}(\lambda\mu).$$

If $\alpha = 6\pi/180$ rad, then:

$$\theta = \frac{1}{30}\left(-30\cot^{-1}(\lambda\mu) - 30\phi + \pi\right)$$

6. For arbitrary lift c_l and drag c_d coefficients, what is the aerodynamic force $\mathrm{d}\boldsymbol{F}_\mathrm{aero}$ for an infinitesimal segment of area $\mathrm{d}A$ around a position μ? Assume that the blades point straight, radially outwards.

   We know that the aerodynamic force is the sum of the lift and drag forces

$$\mathrm{d}\boldsymbol{F}_\mathrm{aero} = \mathrm{d}\boldsymbol{F}_\mathrm{L} + \mathrm{d}\boldsymbol{F}_\mathrm{D}.$$

   By using the definitions of the coefficients, we can see that:

$$\mathrm{d}\boldsymbol{F}_\mathrm{L} = c_\mathrm{l}\frac{1}{2}\rho \left\|\boldsymbol{u}_\mathrm{a}\right\|_2^2 \mathrm{d}A\hat{\boldsymbol{v}}l, \qquad \mathrm{d}\boldsymbol{F}_\mathrm{D} = c_\mathrm{d}\frac{1}{2}\rho \left\|\boldsymbol{u}_\mathrm{a}\right\|_2^2 \mathrm{d}A\hat{\boldsymbol{v}}d$$

   We know the orientations of these forces because the drag force must be along the apparent velocity, and the lift force must be perpedicular to the drag and the span.

$$\hat{\boldsymbol{v}}d = \frac{\boldsymbol{u}_\mathrm{a}}{\left\|\boldsymbol{u}_\mathrm{a}\right\|_2} = \frac{u_\infty\hat{\boldsymbol{i}} - \mu\lambda u_\infty\hat{\boldsymbol{j}}}{\left\|u_\infty\hat{\boldsymbol{i}} - \mu\lambda u_\infty\hat{\boldsymbol{j}}\right\|_2} = \frac{\hat{\boldsymbol{i}} - \mu\lambda\hat{\boldsymbol{j}}}{\sqrt{1+\mu^2\lambda^2}}$$

   To give a right-handed coordinate system $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{i}}$ in the sketch above: $\hat{\boldsymbol{k}}$ must point down into the page. Then:

$$\hat{\boldsymbol{v}}l = \frac{\boldsymbol{u}_\mathrm{a}\times\hat{\boldsymbol{k}}}{\left\|\boldsymbol{u}_\mathrm{a}\times\hat{\boldsymbol{k}}\right\|_2} = \frac{u_\infty\hat{\boldsymbol{j}} + \lambda\mu u_\infty\hat{\boldsymbol{i}}}{\left\|u_\infty\hat{\boldsymbol{j}} + \lambda\mu u_\infty\hat{\boldsymbol{i}}\right\|_2} = \frac{\hat{\boldsymbol{j}} + \mu\lambda\hat{\boldsymbol{i}}}{\sqrt{1+\mu^2\lambda^2}}$$

   Now we can put all of these expressions together:

$$\mathrm{d}\boldsymbol{F}_\mathrm{aero} = \frac{1}{2}\rho u_\infty^2\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\left(c_\mathrm{l}\left(\hat{\boldsymbol{j}} + \mu\lambda\hat{\boldsymbol{i}}\right) + c_\mathrm{d}\left(\hat{\boldsymbol{i}} - \mu\lambda\hat{\boldsymbol{j}}\right)\right)\mathrm{d}A$$

7. What is the mechanical power production $\mathrm{d}P(\mu)$ of that segment around position μ?

   The power is the force acting parallel to the blade's motion:

$$\mathrm{d}P = \mathrm{d}\boldsymbol{F}_\mathrm{aero}\cdot\boldsymbol{u}_\mathrm{b}.$$

   Since we know that the blade's motion is in the $\hat{\boldsymbol{j}}$ direction, we can use the above force expression:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^2\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\mathrm{d}A\left(c_\mathrm{l} - c_\mathrm{d}\mu\lambda\right)\left(\lambda\mu u_\infty\right) = \frac{1}{2}\rho u_\infty^3\lambda\mu\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\mathrm{d}A\left(c_\mathrm{l} - c_\mathrm{d}\mu\lambda\right).$$

   Just for abbreviation, let's define $\xi_n := c_\mathrm{l} - c_\mathrm{d}\mu\lambda$.

8. If the lift c_l and drag c_d coefficients can be found with the following relations, what is the power harvested by the blade segment around position μ?

$$c_\mathrm{l}(\mu) = 1.2\mu, \qquad \frac{c_\mathrm{l}}{c_\mathrm{d}}(\mu) = 100\mu$$

Let's start with the $\mathrm{d}P$ expression from above:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^3 \lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \mathrm{d}A \left(c_\mathrm{l} - c_\mathrm{d}\mu\lambda\right).$$

If we plug the above lift and drag ($c_\mathrm{d} = c_\mathrm{l}/(c_\mathrm{l}/c_\mathrm{d})$) expressions into this power statement then, it gives the following:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^3 \lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \mathrm{d}A \left(1.2\mu - \frac{1.2}{100}\mu\lambda\right).$$

9. What is the relationship between the power harvesting factor ζ and μ?

The power harvesting factor is the harvested power divided by the power density and the segment area. That is:

$$\zeta = \frac{\mathrm{d}P}{\frac{1}{2}\rho u_\infty^3 \mathrm{d}A} = \lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \xi_n$$

10. How would you go about finding the total power P harvested by the entire turbine? (*Hint: just give the procedure; don't follow it yet.*)

Everything in our problem so far has been symmetrical. That means that the total power must be the sum of all segment powers for all blades:

$$P = B\int_{\mu=0}^{\mu=1} \mathrm{d}P$$

Remember that $\mathrm{d}P = \mathrm{d}P(\mathrm{d}\mu)$. So, we'll have to integrate over μ.

11. How would you go about finding the power coefficient C_P of the entire turbine? Use the following definition: $\mathrm{d}A = c(\mu)\mathrm{d}\mu R$, where $c(\mu)$ is a chord length as a function of μ.

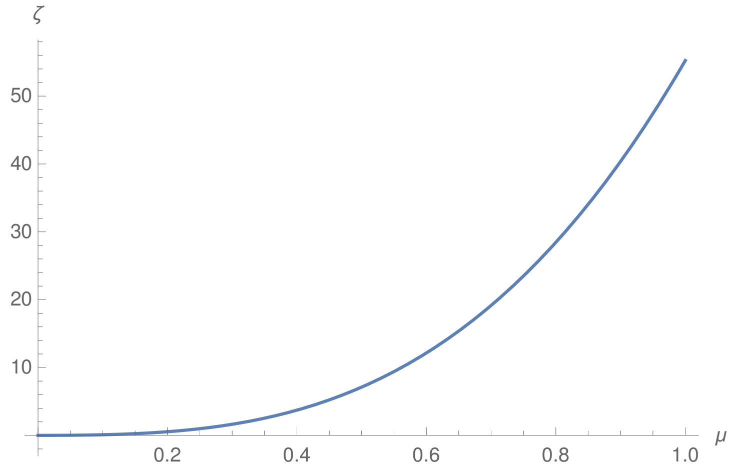
The power coefficient of the entire turbine is the total harvested power divided by the power density (at hub-height, though this is not relevant in a uniform wind field) and the total rotor area πR^2. That gives:

$$C_\mathrm{P} = \frac{P}{\frac{1}{2}\rho u_\infty^3\ \pi R^2} = \frac{B}{\pi R}\int_{\mu=0}^{\mu=1} \lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \xi_n c(\mu) d\mu$$

12. If we use the above model that we've described to this point, for some given parameter values ($\lambda = 7$, $c_0 = 0.15R$, $c_1 = 0.05R$, $u_\infty = 10$ m/s, $\rho = 1.225$ kg/m^3, $R = 50$ m and $B = 3$), can you find how much power the full turbine will extract?

*here assume that the chord is a linear interpolation between the chord c_1 at the tip and the chord c_0 at the root: $c(\mu) = c_0 + (c_1 - c_0)\mu$, what gives us $\mathrm{d}A = (c_0 + (c_1 - c_0)\mu)\,\mathrm{d}\mu R$

(a) plot the power harvesting factor ζ vs. μ



(b) find how much power the full turbine will extract

When we plug in our values into the dP expression, we get the following ugly numeric expression:

$$\mathrm{d}P \approx 1.2 \times 10^6 \mu^2 (1.5 - \mu)\sqrt{49\mu^2 + 1}$$

We can integrate this expression numerically between $\mu = 0$ and $\mu = 1$ to get:

$$P = B \int_0^1 \mathrm{d}P \approx 4.5 \cdot 10^6 \text{ W} = 4.5 \text{ MW}$$