Systems and Control II (SC2)
Albert-Ludwigs-Universität Freiburg – Wintersemester 2015/2016

Exercises 9: Z-transform, Discretization and Discrete-Continuous relation

(Thursday 07.01.2016 at 15:00 in Room SR 00 014)

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1. Consider the digital system shown in Fig. 1.

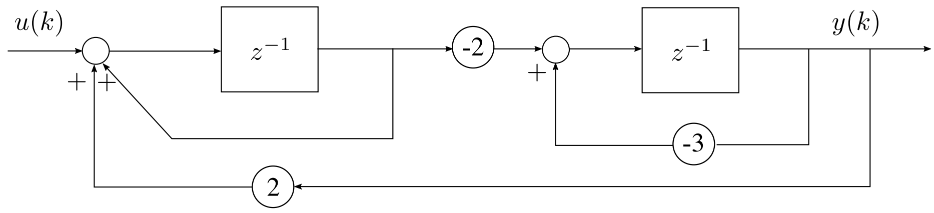

Figure 1: Diagram of a discrete time system

   (a) Determine the state space model for this system.
   (b) Derive the difference equation from the state space model.
   (c) Determine the transfer function $G_z(z)$ and calculate the system poles.
   (d) What is the steady state output $y_{ss}(k)$ of the system for an input signal $u(k) = 2\sigma_d(k) + 3\cos(\frac{\pi k}{2})$?

2. A process plant $G(s)$ is controlled by a digital PI-controller $K_z(z)$. The control signal goes through a zero-order-hold element before being fed to the plant. After the plant, a sampler digitizes the plant output $y(t)$ to a digital signal $y(k)$ that is then fed back to the controller, as shown in Fig. 2. The plant and the controller are described by

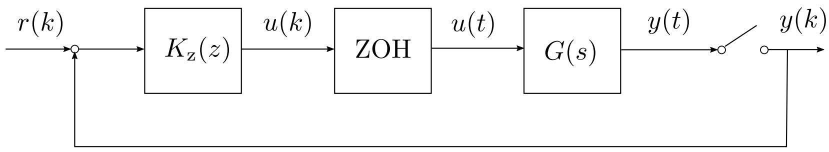

Figure 2: Digital control loop with plant $G(s)$

$$G(s) = \frac{A}{Ts+1} \quad \text{and} \quad K_z(z) = k_P(1 + \frac{T_p}{2T_I}\frac{z+1}{z-1}),$$

The plant gain $A = 10$ and the plant time constant $T = \frac{1}{12}$. The sampling time $T_P = 1\text{ms}$ and the controller parameters $k_P = 5$ and $T_I = 0.01\text{s}$.

   (a) Determine the transfer function $G_z(z) = \frac{Y_z(z)}{U_z(z)}$ of the discretized plant with ZOH. Use Table 1 with standard Laplace and z-transforms.
   (b) Compute the discrete closed-loop transfer function. Is the closed-loop system stable?

3. The discrete-time system $G_z(z)$ describes the transfer function $\frac{Y_z(z)}{U_z(z)}$ of the sampled-data system shown in Fig. 3 that consists of a zero order hold (ZOH), the continous-time system $G(s)$, and a sampler with sample time T_p that is synchronized with the ZOH.

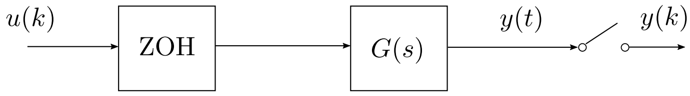

Figure 3: Digital control loop with plant $G(s)$

   (a) How do the poles and zeros of $G(s)$ relate to the poles and zeros of $G_z(z)$?
   (b) Fig. 4 shows different lines in the s-plane that represent possible pole locations of $G(s)$. Draw a diagram that shows the corresponding pole locations of $G_z(z)$ in the z-plane.

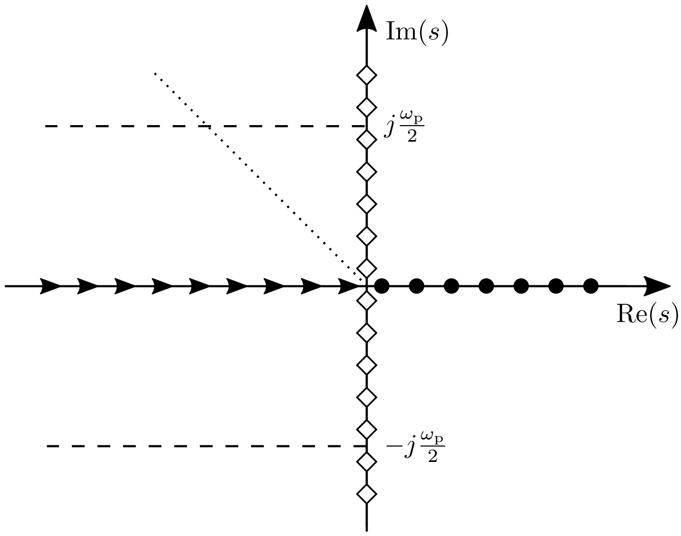

Figure 4: Lines in s-plane that symbolize different pole locations.

Nr.	function $f(t)$ with f(t) = for t¡0	$F(s) = \mathcal{L}\{f(t)\}$	$F(z) = \mathcal{Z}\{f(k)\}$ with $f(k) = f(kT_\mathrm{p})$	
1	$\sigma(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$	
2	t	$\frac{1}{s^2}$	$\frac{T_\mathrm{p}z}{(z-1)^2}$	
3	t^2	$\frac{2}{s^3}$	$\frac{T_\mathrm{p}^2 z(z+1)}{(z-1)^3}$	
4	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT_\mathrm{p}}}$	
5	te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{\alpha T_\mathrm{p} z}{(z-\alpha)^2}$	$\alpha = e^{-aT_\mathrm{p}}$
6	$1 - e^{at}$	$\frac{a}{s(s+a)}$	$\frac{(1-\alpha)z}{(z-1)(z-\alpha)}$	$\alpha = e^{-aT_\mathrm{p}}$
7	$\frac{1}{a}\left(at - 1 - e^{-at}\right)$	$\frac{a}{s^2(s+a)}$	$\frac{z[(aT_\mathrm{p}-1+\alpha)z+(1-\alpha-aT_\mathrm{p}\alpha)]}{a(z-1)^2(z-\alpha)}$	$\alpha = e^{-aT_\mathrm{p}}$
8	$(1-at)\,e^{-at}$	$\frac{s}{(s+a)^2}$	$\frac{z[z-\alpha(1+aT_\mathrm{p})]}{(z-\alpha)^2}$	$\alpha = e^{-aT_\mathrm{p}}$
9	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\frac{\beta z}{z^2-2\gamma z+1}$	$\beta = \sin(\omega T_\mathrm{p})$ $\gamma = \cos(\omega T_\mathrm{p})$
10	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\frac{z^2-\gamma z}{z^2-2\gamma z+1}$	$\gamma = \cos(\omega T_\mathrm{p})$
11	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{\alpha\beta z}{z^2-2\alpha\gamma z+\alpha^2}$	$\alpha = e^{-aT_\mathrm{p}}$ $\beta = \sin(\omega T_\mathrm{p})$ $\gamma = \cos(\omega T_\mathrm{p})$

Table 1: Table of Laplace- and z-Transforms