Some useful facts about ellipsoids and their application to robust optimal control

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syscop talk

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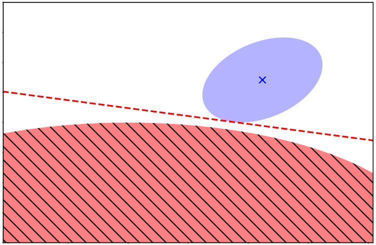
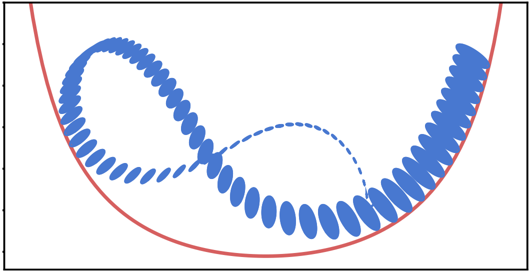
Introduction

1. Some basic knowledge about ellipsoids
2. Some stochastics
3. A little bit of robust optimization
4. Uncertainty description of dynamical systems





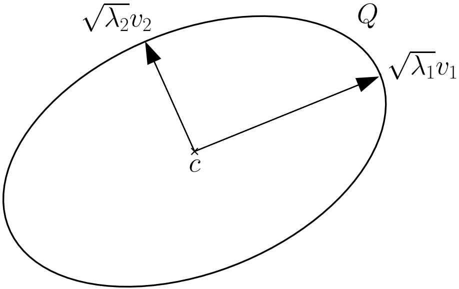
Definition and construction of ellipsoids

- Define ellipsoid by center $c \in \mathbb{R}^n$ and shape matrix $Q \in \mathbb{S}^n_{++}$ $(Q \succ 0)$

$$\begin{aligned}\mathcal{E}(Q,c) &:= \{x \in \mathbb{R}^n \mid \|x-c\|_{Q^{-1}} \leq 1\} \\ &= \{x \in \mathbb{R}^n \mid (x-c)^\top Q^{-1}(x-c) \leq 1\}\end{aligned}$$

- Denote by λ_i, v_i, $i = 1, \dots, n$, the eigenvalues / -vectors (normalized) of Q.
- Eigendecomposition $Q = V \Lambda V^\top$ with $\Lambda = \mathrm{diag}(\lambda)$, $VV^\top = I$

$$\mathcal{E}(Q,c) = \{V\Lambda^{\frac{1}{2}}w + c \mid w \in \mathbb{R}^n,\ w^\top w \leq 1\}$$



Definition and construction of ellipsoids (cont.)

- Relax to $Q \in \mathbb{S}^n_+$ $(Q \succeq 0)$
- $\lambda_i = 0$ for some $i \to$ degenerate ellipsoid
- For non-invertible Q, we can still use

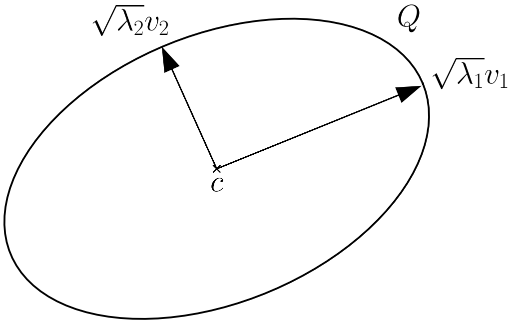
$$\mathcal{E}(Q, c) := \{Q^{\frac{1}{2}} w + c \mid w \in \mathbb{R}^n,\ w^\top w \leq 1\}$$

  with $Q = Q^{\frac{1}{2}} Q^{\frac{1}{2}}$ (unique, $Q^{\frac{1}{2}} = V \Lambda^{\frac{1}{2}} V^\top$)
- or just any $W \in \mathbb{R}^{n \times n}$ with $Q = W W^\top$

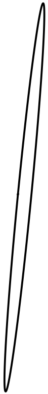
$$\mathcal{E}(Q, c) = \{W w + c \mid w \in \mathbb{R}^n,\ w^\top w \leq 1\}$$

- or non-square $W \in \mathbb{R}^{n \times m}$ with $Q = W W^\top$

$$\mathcal{E}(W W^\top, c) = \{W w + c \mid w \in \mathbb{R}^m,\ w^\top w \leq 1\}$$



Size of ellipsoids



- Most obvious measure of size is volume (with $\bar{V}_n$ volume of unit ball $\mathcal{E}(I_n)$)

$$\mathrm{Vol}(\mathcal{E}(Q)) = \bar{V}_n \prod_{i=1}^{n} \sqrt{\lambda_i} = \bar{V}_n \sqrt{\det Q}$$

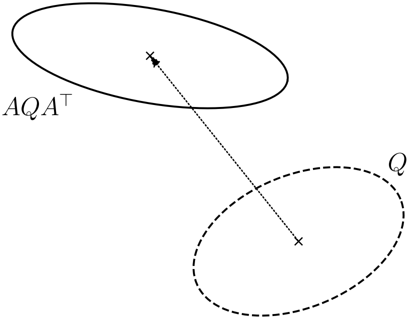
$\det Q = \det V \Lambda V^\top = \det V V^\top \Lambda = \det \Lambda = \prod_{i=1}^{n} \lambda_i$

- Often more intuitive measure

$$\sum_{i=1}^{n} \sqrt{\lambda_i}^2 = \sum_{i=1}^{n} \lambda_i = \operatorname{Tr} Q$$

$\operatorname{Tr} Q = \operatorname{Tr} V \Lambda V^\top = \operatorname{Tr} V^\top V \Lambda = \operatorname{Tr} \Lambda = \sum_{i=1}^{n} \lambda_i$

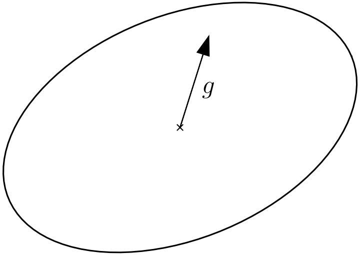
Affine transformation



$$A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,$$

$$\begin{aligned}
& A\mathcal{E}(Q,c) + b, \\
:= & \{Ax + b \mid x \in \mathcal{E}(Q,c)\} \\
= & \{A(Q^{\frac{1}{2}}w + c) + b \mid w \in \mathcal{E}(I_n)\} \\
= & \{AQ^{\frac{1}{2}}w + Ac + b \mid w \in \mathcal{E}(I_n)\} \\
= & \mathcal{E}(AQ^{\frac{1}{2}}{Q^{\frac{1}{2}}}^\top A^\top, Ac + b) \\
= & \mathcal{E}(AQA^\top, Ac + b)
\end{aligned}$$

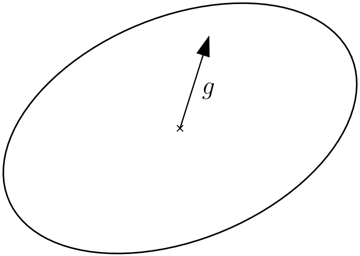
Maximum over linear function



$$\max_{x \in \mathbb{R}^n} \; g^\top x \quad \text{s.t.} \quad x \in \mathcal{E}(Q)$$
$$= \max_{w \in \mathbb{R}^n} \; g^\top Q^{\frac{1}{2}} w \quad \text{s.t.} \quad w \in \mathcal{E}(I)$$
$$= \max_{w \in \mathbb{R}^n} \; \bar{g}^\top w \quad \text{s.t.} \quad w \in \mathcal{E}(I)$$

$$w^* = \frac{\bar{g}}{\|\bar{g}\|} = \frac{Q^{\frac{1}{2}} g}{\sqrt{g^\top Q g}}, \quad x^* = Q^{\frac{1}{2}} w^* = \frac{Qg}{\sqrt{g^\top Q g}}$$
$$g^\top x^* = \frac{g^\top Q g}{\sqrt{g^\top Q g}} = \sqrt{g^\top Q g}$$

Support function



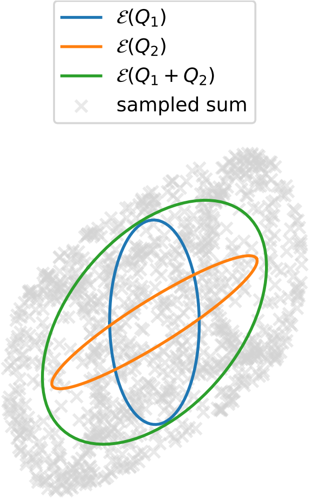
- Any non-empty compact convex set $\mathcal{S} \subset \mathbb{R}^n$ can be defined via its support function:

$$V(g) = \max_{x \in \mathbb{R}^n} \; g^\top x \quad \text{s.t.} \quad x \in \mathcal{S}$$

- Important tool for analysis of convex sets
- For ellipsoid:

$$\begin{aligned} V(g) &= \max_{x \in \mathbb{R}^n} \; g^\top x \quad \text{s.t.} \quad x \in \mathcal{E}(Q) \\ &= \sqrt{g^\top Q g} \end{aligned}$$

Sum of two ellipsoids



- Minkowski sum

$$\begin{aligned} &\mathcal{E}(Q_1, c_1) + \mathcal{E}(Q_2, c_2) \\ := &\{x_1 + x_2 \mid x_1 \in \mathcal{E}(Q_1, c_1),\ x_2 \in \mathcal{E}(Q_2, c_2)\} \end{aligned}$$

- not an ellipsoid (in general)

Overapproximating sum of ellipsoids by ellipsoid

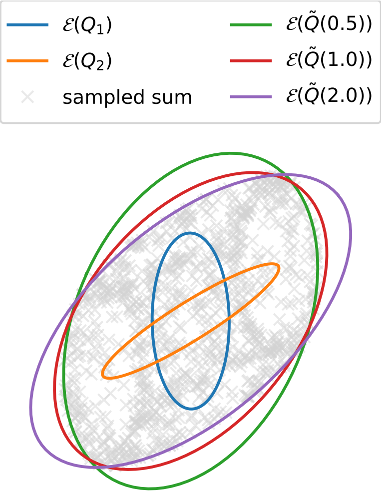
- Aim: find Q such that $\mathcal{E}(Q) \supseteq \mathcal{E}(Q_1) + \mathcal{E}(Q_2)$
- More general: Find Q such that $\mathcal{E}(Q) \supseteq \sum_{k=1}^{K} \mathcal{E}(Q_k)$
- Construct family of outer approximations parametrized by $\alpha \in \mathbb{R}_{++}^K$

$$Q(\alpha) = \left(\sum_{k=1}^{K} \alpha_k \right) \sum_{k=1}^{K} \frac{1}{\alpha_k} Q_k \qquad \Rightarrow \qquad \mathcal{E}(Q(\alpha)) \supseteq \sum_{k=1}^{K} \mathcal{E}(Q_k) \quad \forall \alpha \in \mathbb{R}_{++}^K$$

- Possible to assume / require $\sum_{k=1}^{K} \alpha_k = 1$ w.l.o.g.
- Parametrized outer approximation is tight (but not complete)

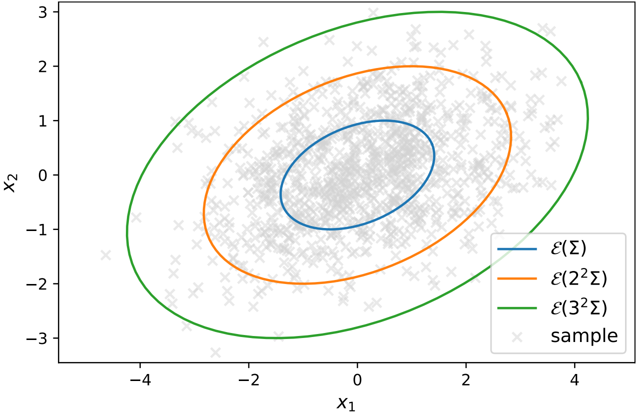
$$\bigcap_{\alpha \in \mathbb{R}_{++}^K} \mathcal{E}(Q(\alpha)) = \sum_{k=1}^{K} \mathcal{E}(Q_k)$$

Overapproximating sum of ellipsoids by ellipsoid (cont.)



- Consider $K = 2$
  - $Q(\alpha) = \frac{1}{\alpha_1} Q_1 + \frac{1}{\alpha_2} Q_2$ with $\alpha_1 + \alpha_2 = 1$
  - Reparametrize: $\alpha_2 = 1 - \alpha_1$, $\beta = \frac{1}{1-\alpha_1} > 0$
  - $\tilde{Q}(\beta) = (1 + \frac{1}{\beta}) Q_1 + (1 + \beta) Q_2$
- In general: Choose α according to some criterion
  - e.g., such that $\mathcal{E}(Q(\alpha))$ has minimal size
  - or such that approximation is tight in a given direction $p \in \mathbb{R}^n$ (approximation touches true sum)
  - $\rightarrow$ convex optimization problem

Ellipsoids in multivariate normal distribution

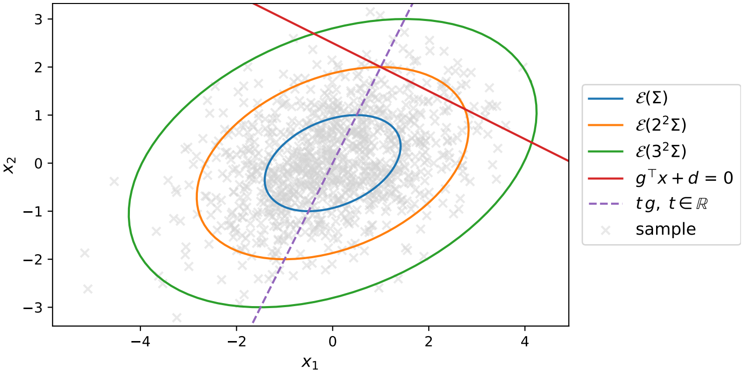


- Consider n-dimensional normal distribution $\mathcal{N}(0, \Sigma)$ with variance $\Sigma \in \mathbb{S}^n_+$
- Probability density function (pdf) has ellipsoidal level lines
- Ellipsoidal $s\sigma$ confidence regions

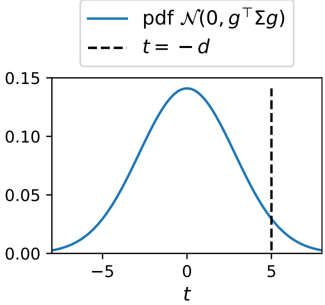
$$P_{x\sim\mathcal{N}(0,\Sigma)}(x \in \mathcal{E}(s^2\Sigma)) = f(s^2, n)$$

($f(s^2, n)$ is cumulative density of χ^2_n dist.)

$f(s^2, n)$	$n = 1$	$n = 2$	$n = 3$	$n = 5$
$s = 1$	0.683	0.393	0.199	0.037
$s = 2$	0.954	0.865	0.739	0.451
$s = 3$	0.997	0.989	0.971	0.891



$$
\begin{aligned}
&P_{x\sim\mathcal{N}(0,\Sigma)}(g^\top x + d \leq 0) = ?\\
&= P_{t\sim\mathcal{N}(0,g^\top\Sigma g)}(t + d \leq 0)\\
&= P_{t\sim\mathcal{N}(0,g^\top\Sigma g)}(t \leq -d)
\end{aligned}
$$



Approximate robust optimization

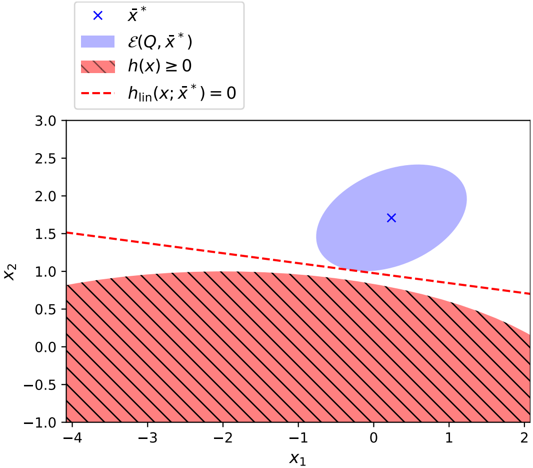
$$
\begin{aligned}
& \min_{\bar{x} \in \mathbb{R}^n} f(\bar{x}) \quad \text{s.t.} \quad 0 \geq h(x) \quad \forall x \in \mathcal{E}(Q, \bar{x}) \qquad (h : \mathbb{R}^n \to \mathbb{R}) \\
= & \min_{\bar{x} \in \mathbb{R}^n} f(\bar{x}) \quad \text{s.t.} \quad 0 \geq \max_{x \in \mathcal{E}(Q, \bar{x})} h(x) \\
\approx & \min_{\bar{x} \in \mathbb{R}^n} f(\bar{x}) \quad \text{s.t.} \quad 0 \geq \max_{x \in \mathcal{E}(Q, \bar{x})} h(\bar{x}) + \nabla h(\bar{x})^\top (x - \bar{x}) \\
= & \min_{\bar{x} \in \mathbb{R}^n} f(\bar{x}) \quad \text{s.t.} \quad 0 \geq h(\bar{x}) + \max_{\Delta x \in \mathcal{E}(Q)} \nabla h(\bar{x})^\top \Delta x \\
= & \min_{\bar{x} \in \mathbb{R}^n} f(\bar{x}) \quad \text{s.t.} \quad 0 \geq h(\bar{x}) + \sqrt{\nabla h(\bar{x})^\top Q \nabla h(\bar{x})}
\end{aligned}
$$

Approximate robust optimization – example

$$\min_{\bar{x} \in \mathbb{R}^2} \quad \bar{x}^\top \bar{x}$$
$$\text{s.t.} \quad x \notin \mathcal{E}(V, c) \qquad \forall x \in \mathcal{E}(Q, \bar{x})$$

$$\min_{\bar{x} \in \mathbb{R}^2} \quad \bar{x}^\top \bar{x}$$
$$\text{s.t.} \quad (x - c)^\top V^{-1}(x - c) \geq 1 \quad \forall x \in \mathcal{E}(Q, \bar{x})$$

Obtain approximate solution $\bar{x}^*$ by robustifying against linearized constraint



Approximate robust optimization – chance constraint variation

$$
\begin{aligned}
& \min_{\bar{x} \in \mathbb{R}^n} \; f(\bar{x}) \quad \text{s.t.} \quad P_{x \sim \mathcal{N}(\bar{x},Q)}\{h(x) \leq 0\} \geq \bar{p} \\
\approx\; & \min_{\bar{x} \in \mathbb{R}^n} \; f(\bar{x}) \quad \text{s.t.} \quad P_{x \sim \mathcal{N}(\bar{x},Q)}\{h(\bar{x}) + \nabla h(\bar{x})^\top (x - \bar{x}) \leq 0\} \geq \bar{p} \\
=\; & \min_{\bar{x} \in \mathbb{R}^n} \; f(\bar{x}) \quad \text{s.t.} \quad P_{\Delta x \sim \mathcal{N}(0,Q)}\{h(\bar{x}) + \nabla h(\bar{x})^\top \Delta x \leq 0\} \geq \bar{p} \\
=\; & \min_{\bar{x} \in \mathbb{R}^n} \; f(\bar{x}) \quad \text{s.t.} \quad h(\bar{x}) + \gamma(\bar{p}) \sqrt{\nabla h(\bar{x})^\top Q \nabla h(\bar{x})} \leq 0
\end{aligned}
$$

$$
\begin{gathered}
\gamma(\bar{p}) = 1 \Leftrightarrow \bar{p} = 0.841 \\
\gamma(\bar{p}) = 2 \Leftrightarrow \bar{p} = 0.977 \\
\gamma(\bar{p}) = 3 \Leftrightarrow \bar{p} = 0.999
\end{gathered}
$$

(cumulative density of normal dist. up to the $\gamma(\bar{p})\sigma$ level)

Uncertain linear dynamical systems

$$x_{k+1} = Ax_k + Bu_k + \Gamma w_k$$

- Stochastic setting

$$x_k \sim \mathcal{N}(\bar{x}_k, P_k),\ w_k \sim \mathcal{N}(0, \Sigma)$$
$$\Rightarrow x_{k+1} \sim \mathcal{N}(\underbrace{A\bar{x}_k + Bu_k}_{\bar{x}_{k+1}}, \underbrace{AP_kA^\top + \Gamma\Sigma\Gamma^\top}_{P_{k+1}})$$

  - Uncertainty set still ellipsoidal :)

- Robust setting

$$x_k \in \mathcal{E}(P_k, \bar{x}_k),\ w_k \in \mathcal{E}(\Sigma)$$
$$\Rightarrow x_{k+1} \in \mathcal{E}(AP_kA^\top, \underbrace{A\bar{x}_k + Bu_k}_{\bar{x}_{k+1}}) + \mathcal{E}(\Gamma\Sigma\Gamma^\top)$$

  - Uncertainty set not ellipsoidal :(

Uncertain linear dynamical systems – robust case

$$x_{k+1} = Ax_k + Bu_k + \Gamma w_k$$

$$x_k \in \mathcal{E}(P_k, \bar{x}_k),\ w_k \in \mathcal{E}(\Sigma) \Rightarrow x_{k+1} \in \mathcal{X}_{k+1}, \quad \text{not ellipsoidal}$$

- Option 1: Ellipsoidal overapproximation $P_{k+1}(\alpha) \supseteq \mathcal{X}_{k+1}$
  - choose arbitrary $\alpha \Rightarrow$ probably bad quality approximation
  - choose best α according to some relevant criterion $\Rightarrow$ optimization problem
- Option 2: Modify assumption on initial uncertainty

$$\begin{bmatrix} x_k \\ w_k \end{bmatrix} \in \mathcal{E}\left(\begin{bmatrix} P_k & 0 \\ 0 & \Sigma \end{bmatrix}, \begin{bmatrix} \bar{x}_k \\ 0 \end{bmatrix}\right) \quad \Rightarrow \quad x_{k+1} \in \mathcal{E}(\underbrace{AP_kA_k^\top + \Gamma\Sigma\Gamma^\top}_{P_{k+1}}, \underbrace{A\bar{x}_k + Bu_k}_{\bar{x}_{k+1}})$$

  - For N-step prediction assume $(x_0, w_0, \ldots, w_{N-1}) \in \mathcal{E}(\operatorname{diag}(P_0, \Sigma, \ldots, \Sigma), (\bar{x}_0, 0, \ldots, 0))$
  - Justification of assumption?

Nonlinear dynamical systems

- Consider a stochastic nonlinear dynamical system

$$x_0 = \bar{\bar{x}}_0, \qquad x_{k+1} = f_k(x_k, u_k, w_k), \qquad k = 0, \dots, N-1. \tag{1}$$

- Noise $w = (w_0, \dots, w_{N-1})$ drawn from ellipsoid $w \in \mathcal{E}(\sigma^2 I)$.
  - uncertainty scaling parameter $\sigma \geq 0$
  - $w \in \mathcal{E}(\sigma^2 I)$ instead of $w \in \mathcal{E}(\sigma^2 W)$ w.l.o.g.
- We are interested in robust constraint satisfaction for all possible trajectories

$$h(x_k, u_k) \leq 0, \quad \forall x_k \in \mathcal{X}_k(u), \quad k = 0, \dots, N, \tag{2}$$

$$h(x_N) \leq 0, \quad \forall x_N \in \mathcal{X}_N(u), \tag{3}$$

where $\mathcal{X}_k(u)$, $k = 0, \dots, N$, is the set of all reachable states at k given controls u.

Approximations

- Model uncertainty tube by ellipsoids around nominal trajectory $\bar{x}$, $\bar{u}$

$$\bar{x}_0 = \bar{\bar{x}}_0, \qquad \bar{x}_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \dots, N-1, \tag{4}$$

$$x_k \in \mathcal{E}(P_k, \bar{x}_k), \qquad k = 0, \dots, N. \tag{5}$$

- Propagate ellipsoids according to dynamics linearized at $\bar{x}$, $\bar{u}$

$$P_0 = 0, \qquad P_{k+1} = A_k P_k A_k^\top + \sigma^2 \Gamma_k \Gamma_k^\top, \qquad k = 0, \dots, N-1. \tag{6}$$

where $A_k = \frac{\partial f_k}{\partial x_k}(\bar{x}_k, \bar{u}_k, 0), \qquad \Gamma_k = \frac{\partial f_k}{\partial w_k}(\bar{x}_k, \bar{u}_k, 0)$.

Simplifications (cont.)

- Approximate robust constraint satisfaction through linearization (componentwise)

$$
\begin{aligned}
0 \geq\ & h_k^i(x_k, \bar{u}_k) && \forall x_k \in \mathcal{E}(P_k, \bar{x}_k) \\
\geq\ & \max_{x_k \in \mathcal{E}(P_k, \bar{x}_k)} h_k^i(x_k, \bar{u}_k) \\
\approx\ & h_k^i(\bar{x}_k, \bar{u}_k) + \sqrt{\nabla_x h_k^i(\bar{x}_k, \bar{u}_k)^\top P_k \nabla_x h_k^i(\bar{x}_k, \bar{u}_k)}, && i = 1, \ldots, n_{h_k}, \\
& && k = 0, \ldots, N-1, \\
0 \geq\ & h_N^i(\bar{x}_N) + \sqrt{\nabla_x h_N^i(\bar{x}_N)^\top P_N \nabla_x h_N^i(\bar{x}_N)}, && i = 1, \ldots, n_{h_N}.
\end{aligned}
$$

- Variation: Single-chance constraints
  - Interpret P_k as variance of normal distribution
  - Multiply back-off by $\gamma(\bar{p})$ to ensure satisfaction of **this specific** constraint with at least probability $\bar{p}$ (approximately)
  - Probability that **no** constraint is violated is lower

Open-loop Robustified NMPC problem

$$\min_{\bar{x}, \bar{u}, P} \quad \sum_{k=0}^{N-1} l(\bar{x}_k, \bar{u}_k) + E(\bar{x}_N) \tag{8a}$$

$$\text{s.t.} \quad \bar{x}_0 = \bar{\bar{x}}_0, \tag{8b}$$

$$\bar{x}_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0), \quad k = 0, \dots, N-1, \tag{8c}$$

$$P_0 = 0, \tag{8d}$$

$$P_{k+1} = A_k P_k A_k^\top + \sigma^2 \Gamma_k \Gamma_k^\top, \quad k = 0, \dots, N-1, \tag{8e}$$

$$0 \geq h_k^i(\bar{x}_k, \bar{u}_k) + \sqrt{\nabla_x h_k^i(\bar{x}_k, \bar{u}_k)^\top P_k \nabla_x h_k^i(\bar{x}_k, \bar{u}_k)} \quad \begin{array}{l} i = 1, \dots, n_h, \\ k = 0, \dots, N-1, \end{array}, \tag{8f}$$

$$0 \geq h_N^i(\bar{x}_N) + \sqrt{\nabla_x h_N^i(\bar{x}_N)^\top P_N \nabla_x h_N^i(\bar{x}_N)} \quad i = 1, \dots, n_{h_N}, \tag{8g}$$

where $P = (P_0, \dots, P_N)$

Some references

- Material for this talk
  - B. Houska. *Robust Optimization of Dynamic Systems* , PhD thesis, KU Leuven, 2011
  - J. Gillis. *Practical methods for approximate robust periodic optimal control of nonlinear mechanical systems*, PhD thesis, KU Leuven, 2015
  - ... and a lot of wikipedia / stackoverflow / random slides
- To at least mention our most recent results and closely related ones
  - L. Hewing, J. Kabzan, and M. N. Zeilinger. *Cautious model predicitive control using gaussian process regression*. IEEE Transaction on Control Systems Technology, 28(6):2736–2743, 2020
  - X. Feng, S. Di Cairano, R. Quirynen. *Inexact Adjoint-based SQP Algorithm for Real-Time Stochastic Nonlinear MPC*, Proceedings of the IFAC World Congress, 2020
  - A. Zanelli, J. Frey, **F. Messerer**, M. Diehl. *Zero-Order Robust Nonlinear Model Predictive Control with Ellipsoidal Uncertainty Sets*, IFAC-NMPC 2021
  - **F. Messerer**, M. Diehl. *An Efficient Algorithm for Tube-based Robust Nonlinear Optimal Control with Optimal Linear Feedback*, 2021