Finite Elements with Switch Detection (FESD) for numerical optimal control of Fillipov systems

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based on joint work with
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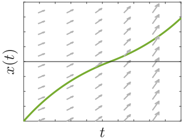
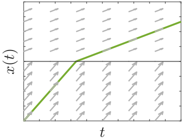
Nonsmooth Dynamics (NSD) - a classification

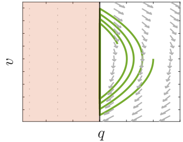
Regard ordinary differential equation (ODE) with a **nonsmooth** right-hand side (RHS).
Distinguish three cases:



NSD1: non-differentiable RHS, e.g., $\dot{x} = 1 + |x|$

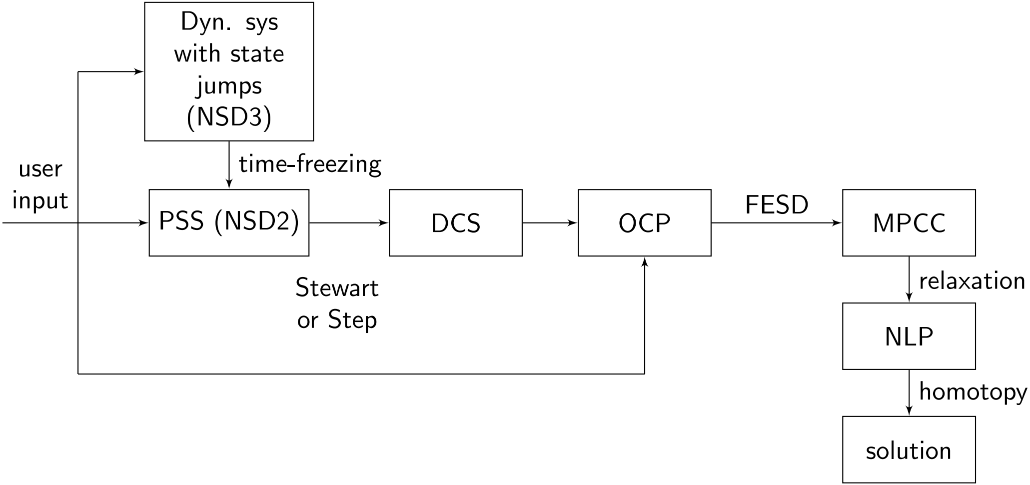


NSD2: state dependent switch of RHS, e.g., $\dot{x} = 2 - \mathrm{sign}(x)$



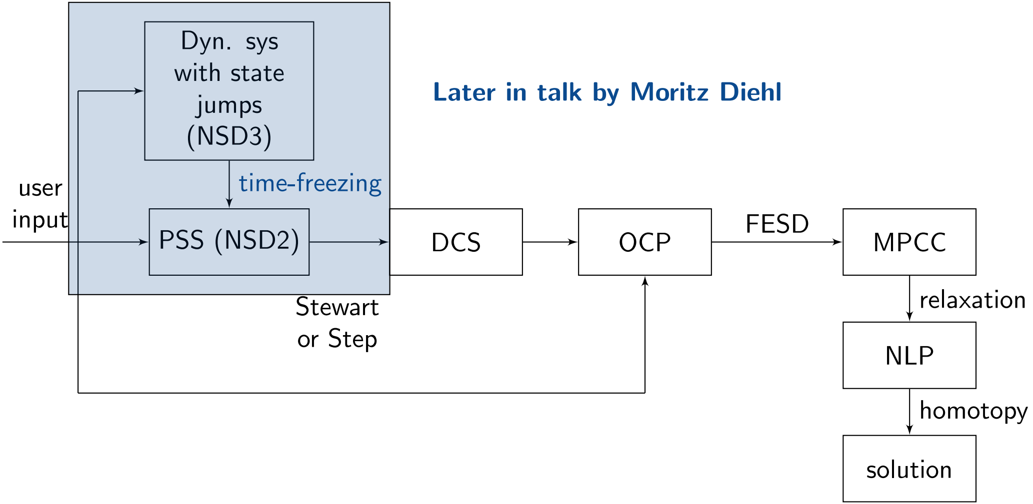
NSD3: state dependent jump, e.g., bouncing ball, $v(t_+) = -0.9\, v(t_-)$

Overview



PSS - piecewise smooth systems; DCS - dynamic complementarity system; OCP - optimal control problem; FESD - finite elements with switch detection; MPCC - mathematical program with complementarity constraints ; NLP - nonlinear program

Overview



Later in talk by Moritz Diehl

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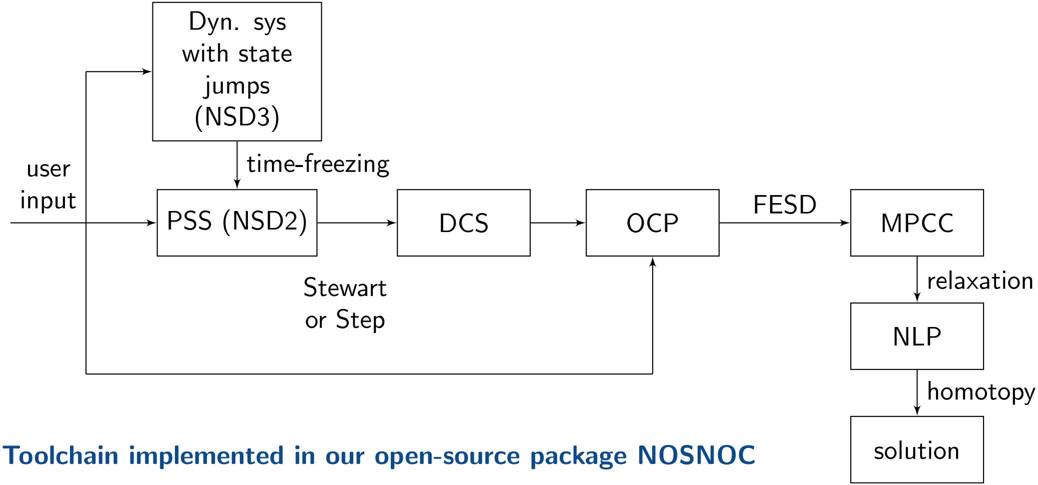
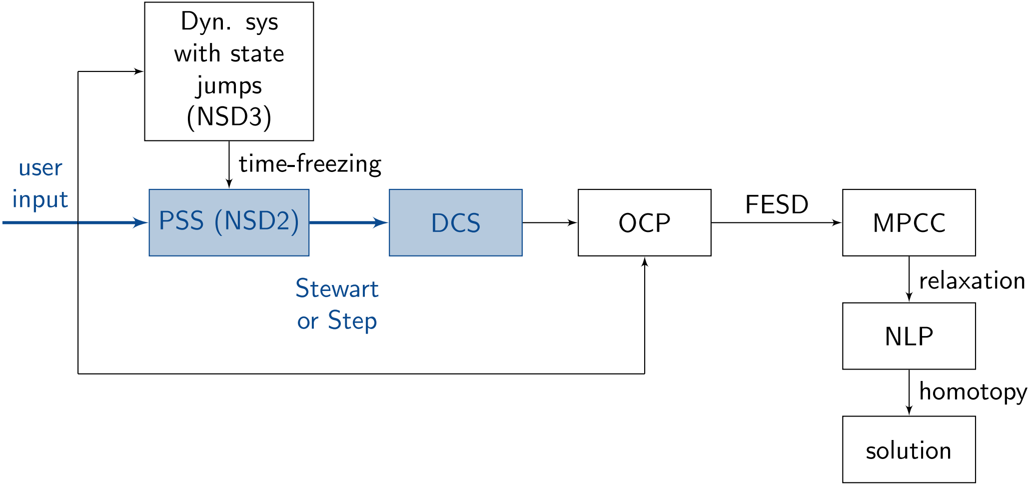


Toolchain implemented in our open-source package NOSNOC

PSS - piecewise smooth systems; DCS - dynamic complementarity system; OCP - optimal control problem; FESD - finite elements with switch detection; MPCC - mathematical program with complementarity constraints ; NLP - nonlinear program

Overview - Piecewise smooth and Filippov systems

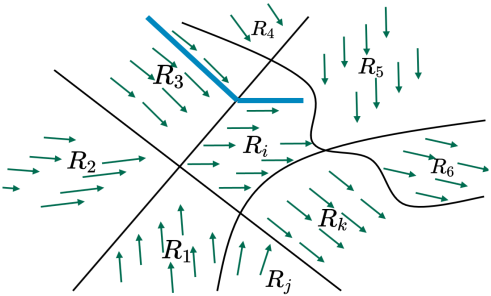


NSD2 Systems - state dependent switches

Regard **discontinuous** right-hand side, piecewise smooth on disjoint open regions $R_i \subset \mathbb{R}^{n_x}$

Discontinuous ODE (NSD2)

$$\dot{x} = f_i(x,u), \text{ if } x \in R_i, \quad i \in \{1,\dots,n_f\}$$



Numerical aims:

1. exactly detect switching times
2. obtain exact sensitivities across regions

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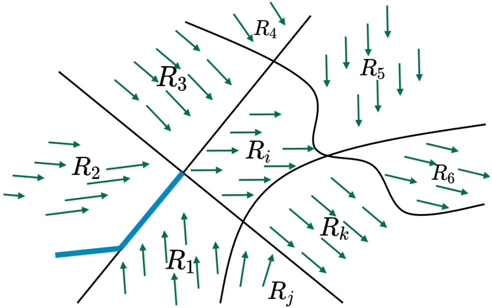


Numerical aims:

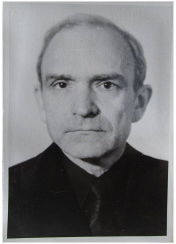
1. exactly detect switching times
2. obtain exact sensitivities across regions
3. appropriately treat evolution on boundaries (sliding mode → Filippov convexification)

Filippov Convexification

Dynamics not yet well-defined on region boundaries ∂R_i. Idea by A.F. Filippov (1923-2006): replace ODE by differential inclusion, using convex combination of neighboring vector fields.

Filippov Differential Inclusion

$$\dot{x} \in F_{\mathrm{F}}(x,u) := \Big\{ \sum_{i=1}^{n_f} f_i(x,u)\,\theta_i \;\Big|\; \sum_{i=1}^{n_f} \theta_i = 1, \quad \theta_i \geq 0, \quad i = 1, \dots n_f, \quad \theta_i = 0, \quad \text{if } x \notin \overline{R_i} \Big\}$$



Aleksei F. Filippov (1923-2006)
image source: wikipedia

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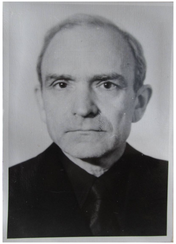

Aleksei F. Filippov
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- for interior points $x \in R_i$ nothing changes: $F_{\mathrm{F}}(x,u) = \{f_i(x,u)\}$
- Provides meaningful generalization on region boundaries.
  E.g. on $\overline{R_1} \cap \overline{R_2}$ both θ_1 and θ_2 can be nonzero

How to compute convex multipliers θ?

Answer in a remarkable paper by David E. Stewart from 1990

Numer. Math. 58, 299–328 (1990)

Numerische Mathematik

A high accuracy method for solving ODEs with discontinuous right-hand side

David Stewart
Department of Mathematics, University of Queensland, St. Lucia, Australia 4067

Received August 1, 1987/January 16, 1990

Summary. Ordinary Differential Equations with discontinuities in the state variables require a differential inclusion formulation to guarantee existence [8]. From this formulation a high accuracy method for solving such initial value problems is developed which can give any order of accuracy and "tracks" the discontinuities. The method uses an "active set" approach, and determines appropriate active sets from solutions to Linear Complementarity Problems. Convergence results are established under some non-degeneracy assumptions. The method has been implemented, and results compare favourably with previously published methods [7, 21].

How to compute convex multipliers θ?

Assume sets R_i given by [cf. Stewart, 1990]

$$R_i = \left\{ x \in \mathbb{R}^n \,\middle|\, g_i(x) < \min_{j \neq i} g_j(x) \right\}$$

How to compute convex multipliers θ?

Assume sets R_i given by [cf. Stewart, 1990]

$$R_i = \{x \in \mathbb{R}^n | g_i(x) < \min_{j \neq i} g_j(x)\ \}$$

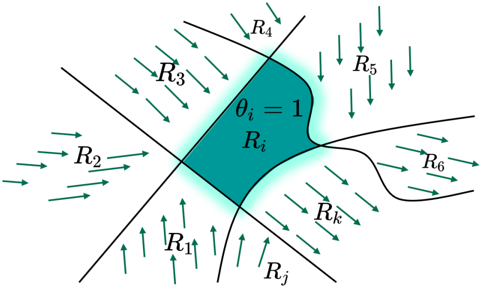
Linear program (LP) Representation

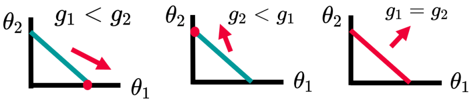
$$\dot{x} = \sum_{i=1}^{n_f} f_i(x,u)\,\theta_i \quad \text{with}$$

$$\theta \in \arg\min_{\tilde{\theta} \in \mathbb{R}^{n_f}} \quad \sum_{i=1}^{n_f} g_i(x)\,\tilde{\theta}_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n_f} \tilde{\theta}_i = 1$$

$$\tilde{\theta} \geq 0$$





Note that the boundary between R_i and R_j is defined by $\{x \in \mathbb{R}^n \mid 0 = g_i(x) - g_j(x)\}$.

LP representation

$$\dot{x} = F(x,u)\ \theta$$

$$\text{with} \quad \theta \in \underset{\tilde{\theta} \in \mathbb{R}^{n_f}}{\operatorname{argmin}} \quad g(x)^\top \tilde{\theta}$$
$$\text{s.t.} \quad 0 \leq \tilde{\theta}$$
$$1 = e^\top \tilde{\theta}$$

where

$$F(x,u) := [f_1(x,u), \dots, f_{n_f}(x,u)] \in \mathbb{R}^{n_x \times n_f}$$
$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$
$$e := [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}$$

From Filippov to dynamic complementarity systems

Using the KKT conditions of the parametric LP

LP representation

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Express equivalently by optimality conditions:

Dynamic Complementarity System (DCS)

$$\dot{x} = F(x,u)\,\theta \tag{1a}$$

$$0 = g(x) - \lambda - e\mu \tag{1b}$$

$$0 \leq \theta \perp \lambda \geq 0 \tag{1c}$$

$$1 = e^\top \theta \tag{1d}$$

Compact notation

$$\dot{x} = F(x,u)\,\theta$$

$$0 = G_{\mathrm{LP}}(x,\theta,\lambda,\mu),$$

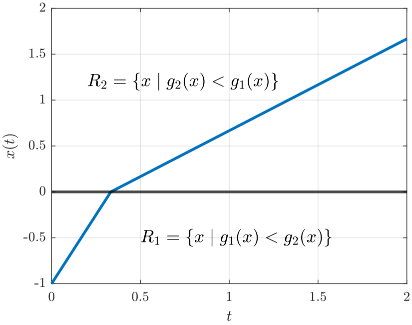
- $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^{n_f}$ are Lagrange multipliers
- (1c) $\Leftrightarrow \min\{\theta, \lambda\} = 0 \in \mathbb{R}^{n_f}$
- Together, (1b), (1c), (1d) determine the $(2n_f + 1)$ variables θ, λ, μ uniquely

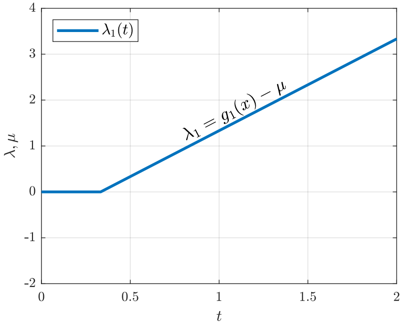
Interpretation of the DCS multipliers

DCS

$$\dot{x} = F(x,u)\,\theta$$
$$0 = g_i(x) - \lambda_i - \mu,\ i = 1,\dots,n_f$$
$$0 \leq \theta \perp \lambda \geq 0$$
$$1 = e^\top \theta$$

- If $x \in R_i$, then $\theta_i > 0$, $\lambda_i = 0$ (from complementarity)
- $\lambda_i = g_i(x) - \mu$ (from $\nabla_x \mathcal{L}(x, \lambda, \mu) = 0$)





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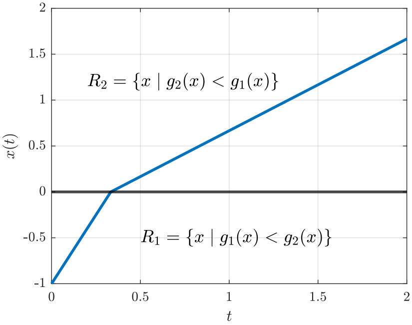

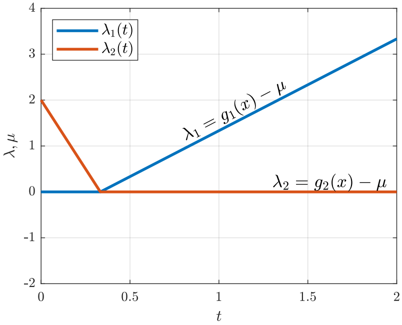

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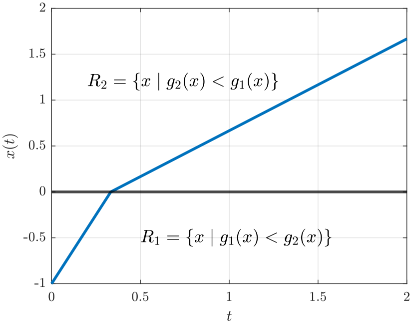
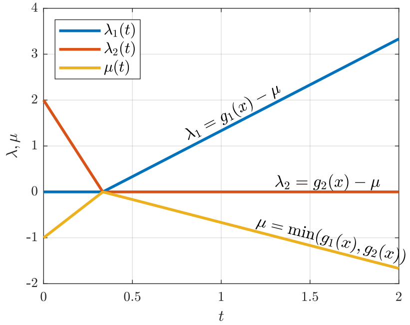
- $\lambda_i = g_i(x) - \min_j g_j(x)$ continuous functions!





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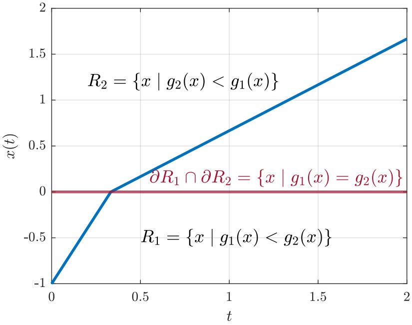
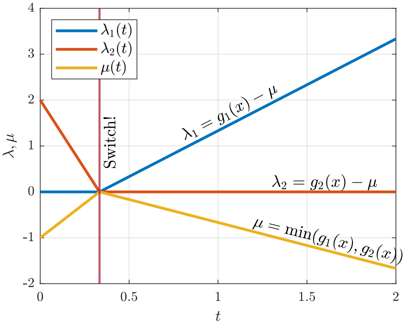
- At switch $\lambda_i = \lambda_j = 0 \implies g_i(x) - g_j(x) = 0$ (region boundary)





Step function representation

Step representation

$$\dot{x} = F(x,u)\,\theta$$
$$0 = G_{\mathrm{Step}}(x,\theta,\alpha,\lambda),$$

- similar properties as Stewart's representation
- with some modifications - FESD applicable
- more practical for some region shapes

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Vincent Acary *, Hidde de Jong, Bernard Brogliato

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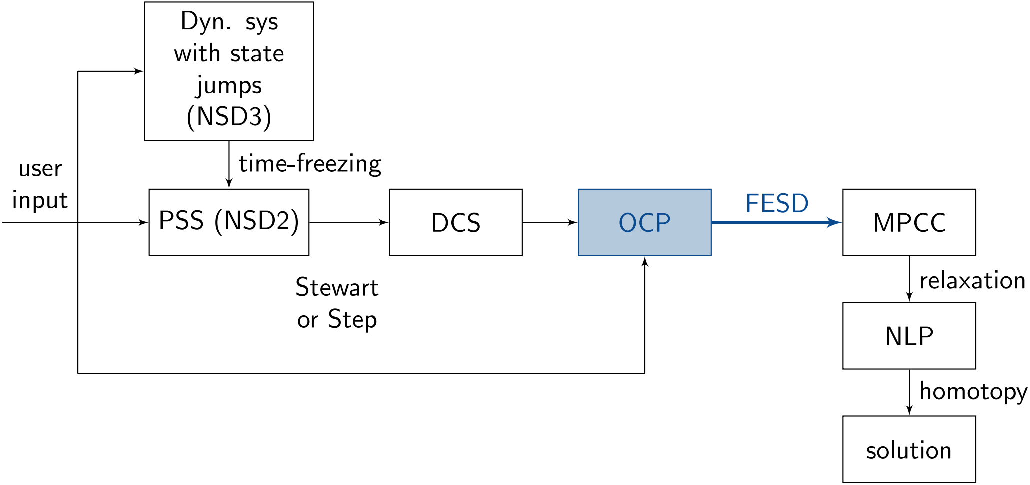
Numer. Math. (2011) 117:779–811
DOI 10.1007/s00211-011-0365-4

Numerische Mathematik

Sliding motion on discontinuity surfaces of high co-dimension. A construction for selecting a Filippov vector field

Luca Dieci · Luciano Lopez

Overview - Finite Elements with Switch Detection



Optimal control needs to solve Nonlinear Programs (NLPs)

Original optimal control problem in continuous time

$$
\begin{aligned}
\min_{\substack{x(\cdot),u(\cdot),\\ \theta(\cdot),\lambda(\cdot),\mu(\cdot)}} \quad & \int_0^T L(x,u)\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = F(x(t),u(t))\ \theta(t) \\
& 0 = G_{\mathrm{LP}}(x(t),\theta(t),\lambda(t),\mu(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

Assume smooth (convex) L, E, h, r

Nonsmooth dynamics make problem nonconvex

Direct methods discretize, then optimize

E.g., collocation or multiple shooting

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- Assume smooth (convex) L, E, h, r
- Nonsmooth dynamics make problem nonconvex
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- E.g., collocation or multiple shooting

Goal: discretized optimal control problem (an NLP)

$$\begin{aligned}\min_{x,z,u} \quad & \textstyle\sum_{k=0}^{N-1} \Phi_L(x_k,z_k,u_k) + E(x_N) \\ \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & x_{k+1} = \Phi_f^{\mathrm{dif}}(x_k,z_k,u_k) \\ & 0 = \Phi_f^{\mathrm{alg}}(x_k,z_k,u_k) \\ & 0 \geq \Phi_h(x_k,z_k,u_k),\ k = 0,\ldots,N-1 \\ & 0 \geq r(x_N)\end{aligned}$$

- Smooth convex Φ_L, E, Φ_h, r
- Variables $x = (x_0,\ldots)$, $z = (z_0,\ldots)$ and $u = (u_0,\ldots,u_{N-1})$ summarized in vector $w \in \mathbb{R}^{n_w}$
- Nonsmooth Φ_f^{alg}

Conventional discretization by Implicit Runge Kutta (IRK) method

Continuous time DCS

$$
\begin{aligned}
x(0) &= \bar{x}_0, \\
\dot{x}(t) &= v(t) \\
v(t) &= F(x(t), u(t))\,\theta(t) \\
0 &= g(x(t)) - \lambda(t) - e\mu(t) \\
0 &\leq \theta(t) \perp \lambda(t) \geq 0 \\
1 &= e^\top \theta(t), \quad t \in [0, T]
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Discrete time IRK-DCS equation

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \quad x_{k+1,0} = x_{k,0} + h \textstyle\sum_{n=1}^{s} b_n v_{k,n} \\
x_{k,j} &= x_{k,0} + h \textstyle\sum_{n=1}^{s} a_{jn} v_{k,n} \\
v_{k,j} &= F(x_{k,j}, u_{k,j})\,\theta_{k,j} \\
0 &= g(x_{k,j}) - \lambda_{k,j} - e\mu_{k,j} \\
0 &\leq \theta_{k,j} \perp \lambda_{k,j} \geq 0 \\
1 &= e^\top \theta_{k,j}, \quad j = 1, \dots, s, \quad k = 0, \dots, N-1
\end{aligned}
$$

Notation: $x_{k,r} \in \mathbb{R}^{n_x}, \theta_{k,r} \in \mathbb{R}^m$ etc. with:

- $k \in \{0, 1, \dots, N\}$ - index of integration step; step length $h := T/N$
- $j, n \in \{0, 1, \dots, s\}$ - index of intermediate IRK stage / collocation point
- a_{jn} and b_n - Butcher tableau entries of Implicit Runge Kutta method

Direct optimal control with a standard IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]

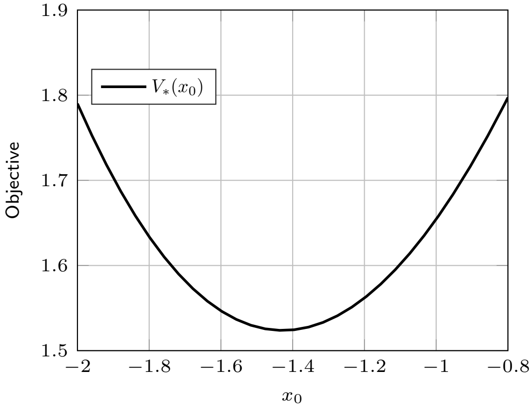
Continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^0([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2)-5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) = 2 - \text{sign}(x(t)), \quad t \in [0,2]$$

Free initial value $x(0)$ is the effective degree of freedom.
Denote by $V_*(x_0)$ the nonsmooth objective value for the unique feasible trajectory starting at $x(0) = x_0$.

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_*(x_0)$$



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Numer. Math. (2010) 114:653–695
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Numerische Mathematik

Optimal control of systems with discontinuous differential equations

David E. Stewart · Mihai Anitescu

(another remarkable paper by D. Stewart)

- discretize the OCP with standard IRK for DCS
- numerical sensitivities wrong independent of the step-size
- smoothing works only if step-size smaller than smoothing parameter

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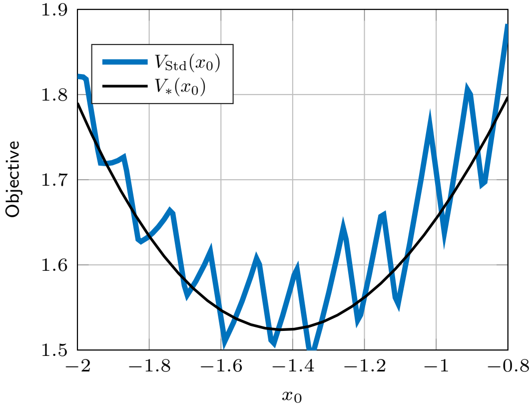
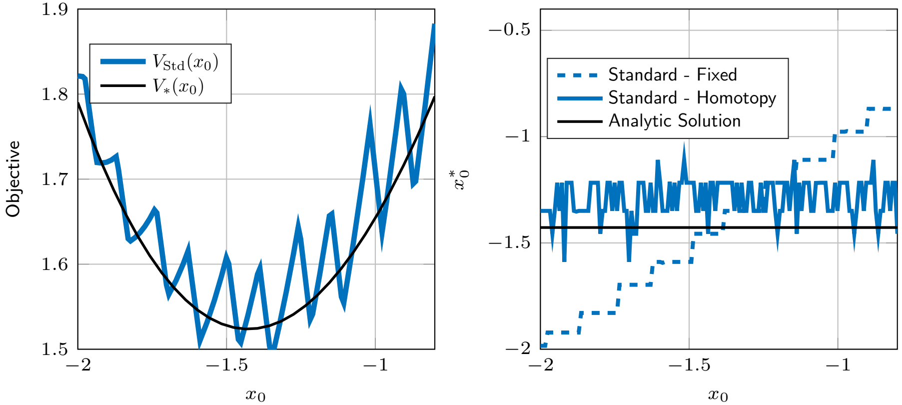
- numerical sensitivities wrong independent of the step-size
- smoothing works only if step-size smaller than smoothing parameter



Direct optimal control with a standard IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]



- Spurious local minima, optimizer gets trapped close to initialization
- Sensitivity correct if step-sizes smaller than smoothing parameter [Stewart & Anitescu, 2010] $\implies$ homotopy improves convergence
- Still, at best $O(h)$ accuracy can be expected

Conventional Collocation - illustrative example

Regard example with $x \in \mathbb{R}^2$ and constants $a, k, c > 0$:

$$\dot{x} = \begin{cases} f_1(x), & x_1 > 0, \\ f_2(x), & x_1 < 0. \end{cases}$$

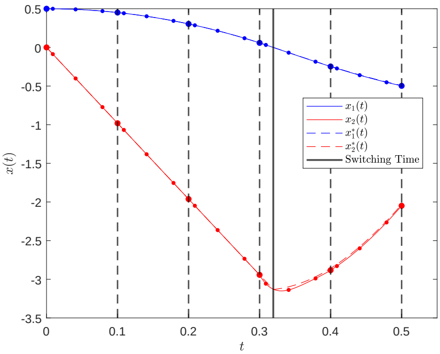
$$f_1(x) = \begin{pmatrix} x_2 \\ -a \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} x_2 \\ -kx_1 - cx_2 \end{pmatrix}$$

$$g_1(x) = -x_1,$$
$$g_2(x) = x_1,$$

$$\bar{x}_0 = [0.5\,,\, 0]^\top.$$

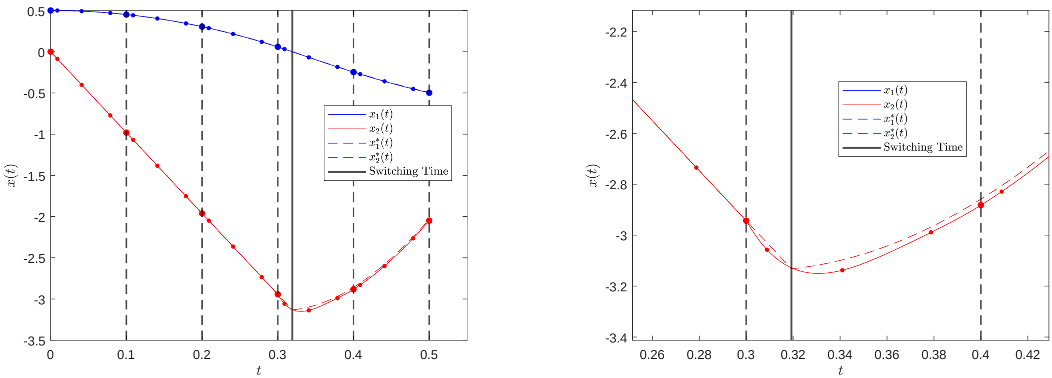
Solve with IRK Radau IIA method of **order 7**

$s = 4, \; N = 5, \; T = 0.5, \; h = 0.1$



Conventional Collocation - illustrative example

Zoom in



High integration accuracy of 7th order IRK method is lost in fourth time step.
Reason: we try to approximate a nonsmooth function by a (smooth) polynomial.

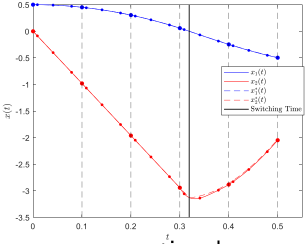
Question: could we ensure that switches happen only at element boundaries?
$\rightarrow$ Finite Elements with Switch Detection (FESD)

Finite Elements with Switch Detection (FESD)

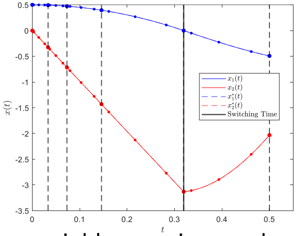
L-CSS 2022 & submitted to Num. Math.

FESD is a novel DCS discretization method based on three ideas:

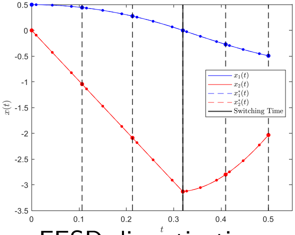
- make stepsizes h_k free, ensure $\sum_{k=0}^{N-1} h_k = T$ [cf. Baumrucker & Biegler, 2009]
- allow switches only at element boundaries, enforce via *cross-complementarities*
- remove spurious degrees of freedom via *step equilibration*



conventional discretization



variable stepsizes and cross-complementarities



FESD discretization with step equilibration

Conventional DCS and FESD discretization without step equilibration

Conventional discretization

$$x_{0,0} = \bar{x}_0, \quad h = T/N$$
$$x_{k+1,0} = x_{k,0} + h \textstyle\sum_{n=1}^{s} b_n v_{k,n}$$
$$x_{k,j} = x_{k,0} + h \textstyle\sum_{n=1}^{s} a_{jn} v_{k,n}$$
$$v_{k,j} = F(x_{k,j}, u_{k,j})\, \theta_{k,j}$$
$$0 = g(x_{k,j}) - \lambda_{k,j} - e\mu_{k,j}$$
$$0 \leq \theta_{k,j} \perp \lambda_{k,j} \geq 0$$
$$1 = e^\top \theta_{k,j}$$

for $j = 1, \dots, s$

and $k = 0, \dots, N-1$

FESD discretization without step equilibration

$$x_{0,0} = \bar{x}_0, \; \textstyle\sum_{k=0}^{N-1} h_k = T$$
$$x_{k+1,0} = x_{k,0} + h_k \textstyle\sum_{n=1}^{s} b_n v_{k,n}$$
$$x_{k,j} = x_{k,0} + h_k \textstyle\sum_{n=1}^{s} a_{jn} v_{k,n}$$
$$v_{k,j} = F(x_{k,j}, u_{k,j})\, \theta_{k,j}$$
$$0 = g(x_{k,j'}) - \lambda_{k,j'} - e\mu_{k,j'}$$
$$0 \leq \theta_{k,j} \perp \lambda_{k,j'} \geq 0 \quad \text{(cross-complementarities)}$$
$$1 = e^\top \theta_{k,j}$$

for $j = 1, \dots, s$ and $k = 0, \dots, N-1$

and $j' = 0, 1, \dots, s$

- N extra variables $(h_0, \dots, h_{N-1})$ restricted by one extra equality
- additional multipliers $\lambda_{k,0}, \mu_{k,0}$ are uniquely determined

Conventional DCS and FESD discretization with step equilibration

Conventional discretization

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \quad h = T/N \\
x_{k+1,0} &= x_{k,0} + h \textstyle\sum_{n=1}^{s} b_n v_{k,n} \\
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v_{k,j} &= F(x_{k,j}, u_{k,j})\, \theta_{k,j} \\
0 &= g(x_{k,j}) - \lambda_{k,j} - e\mu_{k,j} \\
0 &\leq \theta_{k,j} \perp \lambda_{k,j} \geq 0 \\
1 &= e^\top \theta_{k,j}
\end{aligned}
$$

for $j = 1, \dots, s$

and $k = 0, \dots, N-1$

FESD discretization with step equilibration

$$
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x_{0,0} &= \bar{x}_0, \; \textstyle\sum_{k=0}^{N-1} h_k = T \\
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v_{k,j} &= F(x_{k,j}, u_{k,j})\, \theta_{k,j} \\
0 &= g(x_{k,j'}) - \lambda_{k,j'} - e\mu_{k,j'} \\
0 &\leq \theta_{k,j} \perp \lambda_{k,j'} \geq 0 \quad \text{(cross-complementarities)} \\
1 &= e^\top \theta_{k,j} \\
0 &= \nu(\theta_{k'}, \theta_{k'+1}, \lambda_{k'}, \lambda_{k'+1}) \cdot (h_{k'} - h_{k'+1})
\end{aligned}
$$

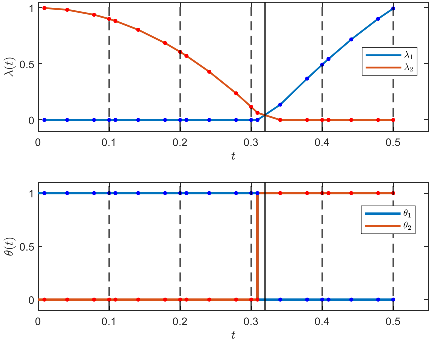
for $j = 1, \dots, s$ and $k = 0, \dots, N-1$

and $j' = 0, 1, \dots, s$ and $k' = 0, \dots, N-2$

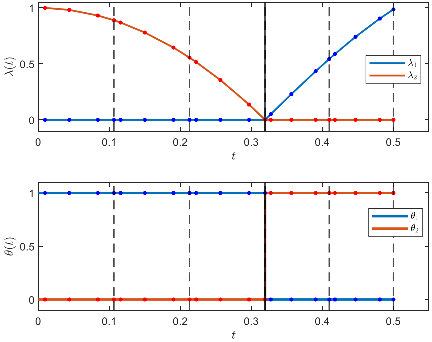
- N extra FESD variables $(h_0, \dots, h_{N-1})$ now locally uniquely determined by N constraints
- Indicator function $\nu(\theta_{k'}, \theta_{k'+1}, \lambda_{k'}, \lambda_{k'+1})$ only zero if a switch occurs

Multipliers in conventional and FESD discretization

Conventional Collocation:



FESD Discretization:



FESD's cross-complementarities exploit the fact that the multiplier $\lambda_i(t)$ is continuous in time

On boundary, $\lambda_i(t_k)$ must be zero if $\theta_i(t) > 0$ for any $t \in [t_{k-1}, t_{k+1}]$ on the adjacent intervals

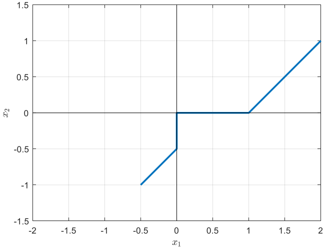
This implicitly imposes the constraint $g_i(x_k) - g_j(x_k) = 0$

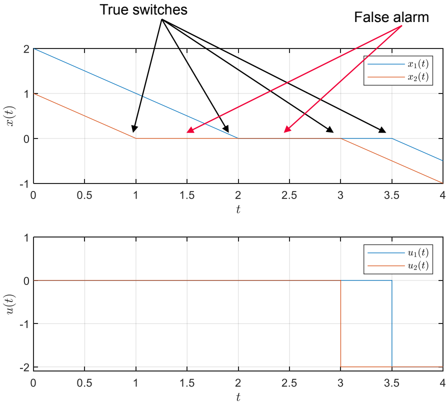
$\implies h_k$ adapts for exact switch detection

Optimal control example: solution trajectory with 3 sliding modes

Regard the following OCP

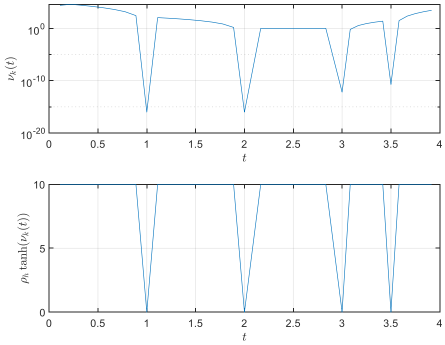
$$\begin{aligned}
\min_{x(.),u(.)} \quad & \int_0^4 u(t)^\top u(t)\mathrm{d}t \\
\text{s.t.} \quad & x(0) = (2,1), \\
& \dot{x}(t) \in -\mathrm{sign}(x(t)) + u(t),\ t \in [0,4], \\
& -2e_2 \leq u(t) \leq 2e_2,\ t \in [0,4], \\
& x(4) = (-1,-0.5).
\end{aligned}$$



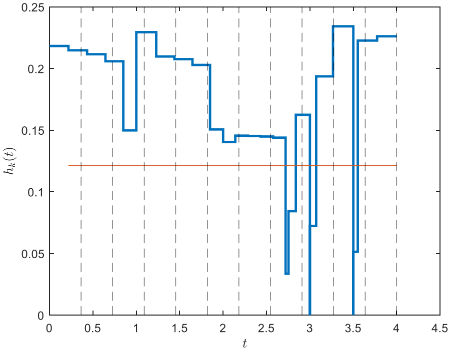


Numerical solution without equilibration

Indicator function over time:



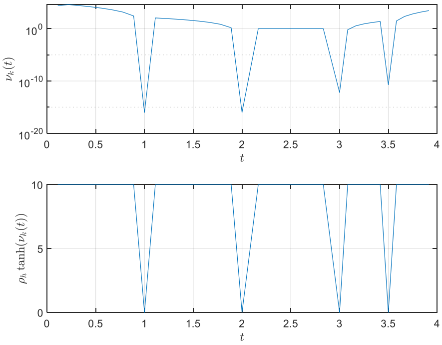
Step size over time:



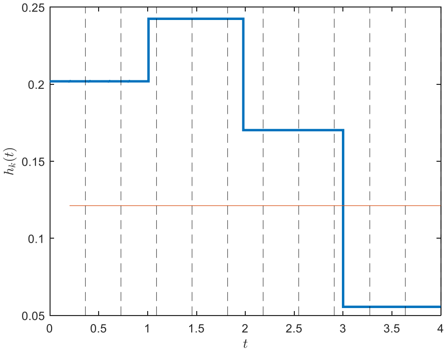
Optimizer varies step size randomly, potentially playing with integration errors.

Numerical solution with equilibration

Indicator function over time:



Step size over time:



Equidistant grid on each "switching stage". Jumps exactly at switching times.

Summary of theoretical results

Submitted to Num. Mat, arXiv:2205.05337

1. Convergence of the FESD method to a Filippov solution of the underlying system with accuracy $O(h^p)$ is proven. Here, p is the order of the underlying smooth IRK scheme.

Summary of theoretical results

Submitted to Num. Mat, arXiv:2205.05337

1. Convergence of the FESD method to a Filippov solution of the underlying system with accuracy $O(h^p)$ is proven. Here, p is the order of the underlying smooth IRK scheme.

2. Convergence of numerical sensitivities to the true value with $O(h^p)$ is given.
   The Stewart & Anitescu problem is resolved.

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1. Convergence of the FESD method to a Filippov solution of the underlying system with accuracy $O(h^p)$ is proven. Here, p is the order of the underlying smooth IRK scheme.
2. Convergence of numerical sensitivities to the true value with $O(h^p)$ is given. The Stewart & Anitescu problem is resolved.
3. An FESD problem needs to solve a nonlinear complementarity problem (NCP) to advance the integration. The solutions of these NCP are locally unique.

Numerical simulation example: unstable switched oscillator

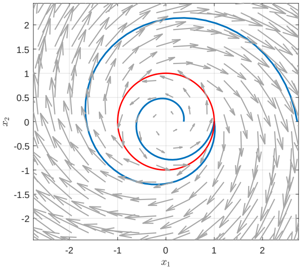
Regard an unstable nonsmooth oscillator

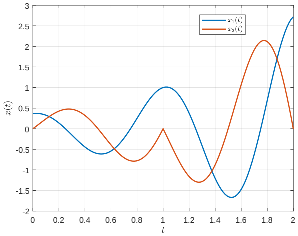
$$\dot{x}(t) = \begin{cases} A_1 x, & c(x) < 0, \\ A_2 x, & c(x) > 0, \end{cases}$$

with

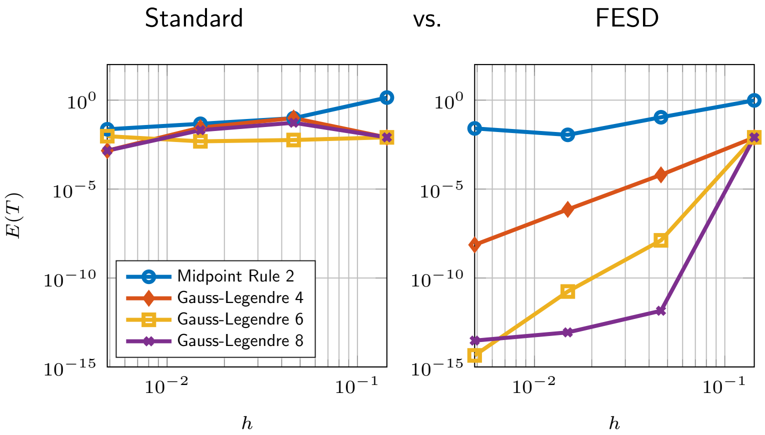
$$A_1 = \begin{bmatrix} 1 & \omega \\ -\omega & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & -\omega \\ \omega & 1 \end{bmatrix},$$

$$c(x) = x_1^2 + x_2^2 - 1, \; \omega = 2\pi, \; x(0) = [e^{-1} \; 0]^\top$$





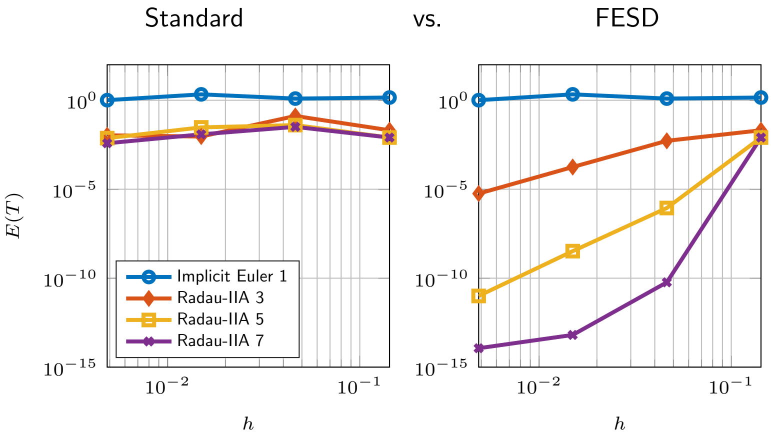
FESD recovers high integration order for switched systems



Integration error $E(T)$ at time $T = \pi/2$ vs. step-size h, for different IRK methods.

FESD discretization delivers versatile MPCC formulation with high integration order

FESD recovers high integration order for switched systems



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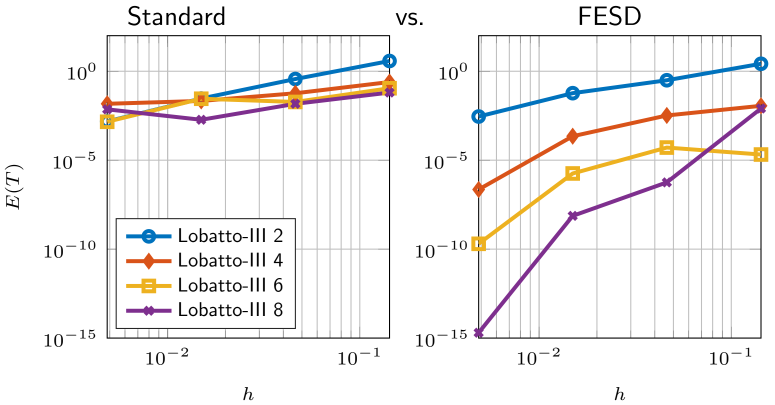


Integration error $E(T)$ at time $T = \pi/2$ vs. step-size h, for different IRK methods.

FESD discretization delivers versatile MPCC formulation with high integration order

Revisiting the OCP example - now with FESD

Tutorial example inspired by [Stewart & Anitescu, 2010]

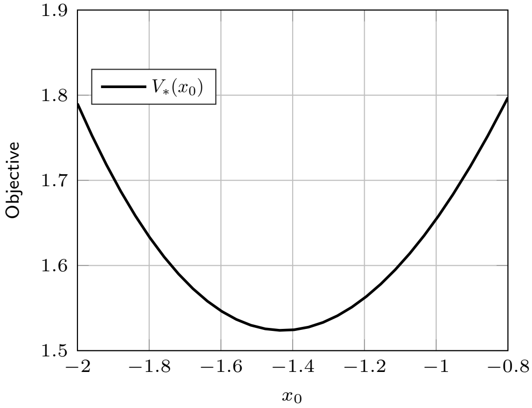
Continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^0([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) = 2 - \operatorname{sign}(x(t)), \quad t \in [0,2]$$

Free initial value $x(0)$ is the effective degree of freedom.
Denote by $V_*(x_0)$ the nonsmooth objective value for the unique feasible trajectory starting at $x(0) = x_0$.

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_*(x_0)$$



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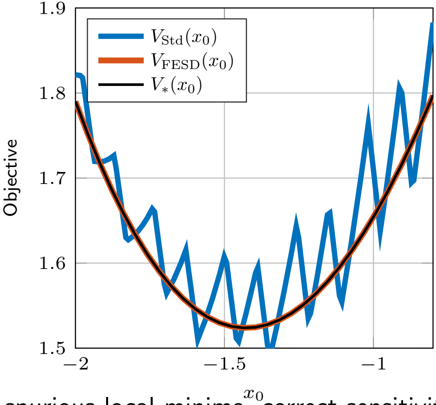
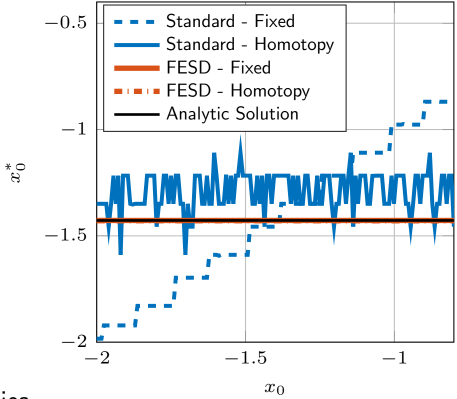
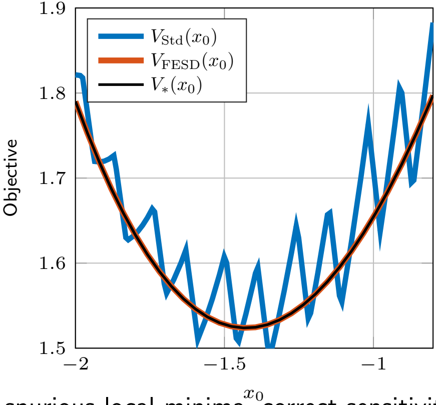


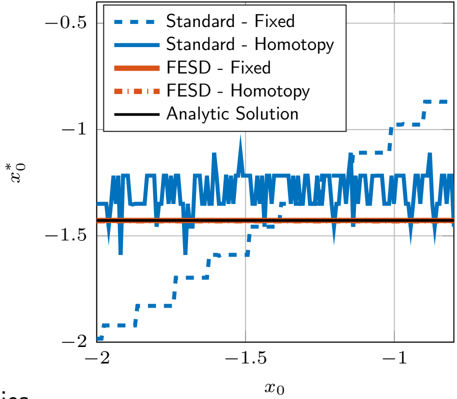


- No spurious local minima, correct sensitivities
- Convergence to the "true" local minima, both with homotopy and without it
- In contrast to the standard approach with accuracy $O(h)$, now we have $O(h^p)$

Revisiting the OCP example - now with FESD

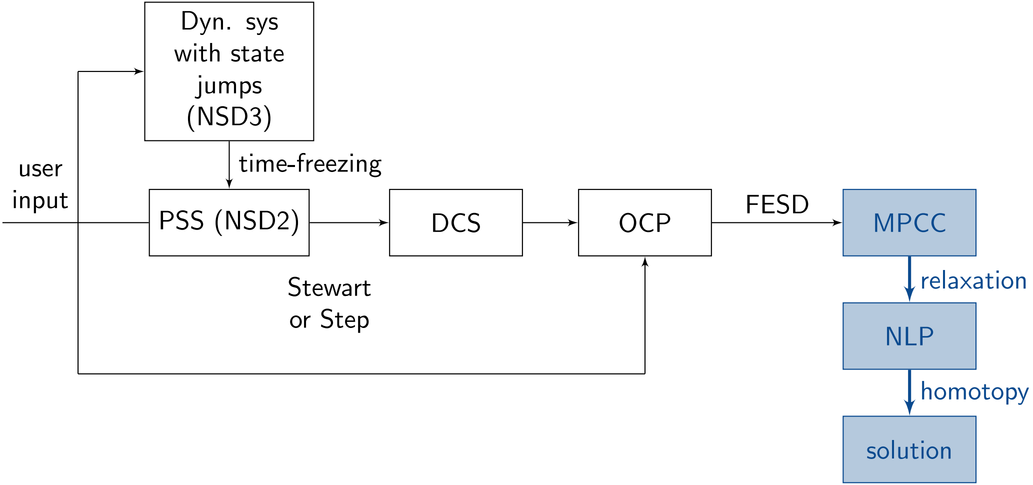
Tutorial example inspired by [Stewart & Anitescu, 2010]





- No spurious local minima, correct sensitivities
- Convergence to the "true" local minima, both with homotopy and without it
- In contrast to the standard approach with accuracy $O(h)$, now we have $O(h^p)$
- FESD resolves the accuracy and convergence issues

Overview - Solving discrete-time OCPs



Optimal control needs to solve Nonlinear Programs (NLPs)

Original optimal control problem in continuous time

$$
\begin{aligned}
\min_{\substack{x(\cdot),u(\cdot),\\ \theta(\cdot),\lambda(\cdot),\mu(\cdot)}} \quad & \int_0^T L(x,u)\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = F(x(t),u(t))\,\theta(t) \\
& 0 = G_{\mathrm{LP}}(x(t),\theta(t),\lambda(t),\mu(t)), \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

Assume smooth (convex) L, E, h, r
Nonsmooth dynamics make problem nonconvex
Direct methods discretize, then optimize
E.g., collocation or multiple shooting

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Assume smooth (convex) L, E, h, r
Nonsmooth dynamics make problem nonconvex
Direct methods discretize, then optimize
E.g., collocation or multiple shooting

Discretized optimal control problem (an MPCC)

$$\begin{aligned}\min_{x,z,u} \quad & \textstyle\sum_{k=0}^{N-1} \Phi_L(x_k,z_k,u_k) + E(x_N)\\ \text{s.t.} \quad & x_0 = \bar{x}_0\\ & x_{k+1} = \Phi_f^{\mathrm{dif}}(x_k,z_k,u_k)\\ & 0 = \Phi_f^{\mathrm{alg}}(x_k,z_k,u_k)\\ & 0 \geq \Phi_h(x_k,z_k,u_k),\ k=0,\ldots,N-1\\ & 0 \geq r(x_N)\end{aligned}$$

Smooth convex Φ_L, E, Φ_h, r
Variables $x=(x_0,\ldots)$, $z=(z_0,\ldots)$ and $u=(u_0,\ldots,u_{N-1})$ summarized in vector $w\in\mathbb{R}^{n_w}$
Nonsmooth Φ_f^{alg}, complementarity constraints

Nonlinear Programs (NLP)

Newton-type methods generate a sequence $w_0, w_1, w_2, \dots$ by linearizing and solving convex subproblems.

Summarized NLP

$$
\begin{aligned}
\min_{w \in \mathbb{R}^{n_w}} \quad & J(w) \\
\text{s.t.} \quad & 0 = F(w) \\
& 0 \geq H(w)
\end{aligned}
$$

Still assume smooth convex J, H.
Nonlinear F makes problem nonconvex.

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Still assume smooth convex J, H.
Nonlinear F makes problem nonconvex.

NLP with Complementarity Constraints

$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & J(w) \\ \text{s.t.} \quad & 0 = F(w) \\ & 0 \geq H(w) \\ & 0 \leq G_1(w) \perp G_2(w) \geq 0 \end{aligned}$$

There is significant nonconvex and nonsmooth structure in the NLP.

Mathematical Programs with Complementarity Constraints (MPCC)

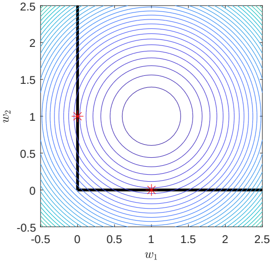
NLP with additional constraints of complementarity type: $x \perp y \Leftrightarrow x^\top y = 0$

MPCC as an NLP

$$\min_{w \in \mathbb{R}^{n_w}} \; J(w)$$

$$\begin{aligned} \text{s.t.} \quad & 0 = F(w) \\ & 0 \geq H(w) \\ & 0 \leq G_1(w) \\ & 0 \leq G_2(w) \\ & 0 \geq G_1(w)^\top G_2(w) \end{aligned}$$

Convex J, H and smooth F.
Smooth G_1, G_2.



Toy MPCC example:

$$\min_{w \in \mathbb{R}^2} \; (w_1 - 1)^2 + (w_2 - 1)^2$$

$$\text{s.t.} \;\; 0 \leq w_1 \perp w_2 \geq 0$$

Two local minimizers.
One local maximizer
(without constraint qualification)

Due to complementarity constraints, MPCC are nonsmooth and nonconvex.

MPCC solution by relaxation and homotopy

The homotopy MPCC approach [cf. Ferris 1999, Ralph&Wright 2004] generates sequence $w_0^*, w_1^*, w_2^*, \ldots$ by solving NLPs with decreasing $\sigma_0 > \sigma_1 > \sigma_2 > \ldots$, and NLP warm-starting.

Penalty subproblem for weight $1/\sigma_j$

$$\begin{aligned}
\min_{w \in \mathbb{R}^{n_w}} \quad & J(w) + \frac{1}{\sigma_j} G_1(w)^\top G_2(w) \\
\text{s.t.} \quad & 0 = F(w) \\
& 0 \geq H(w) \\
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& 0 \leq G_2(w)
\end{aligned}$$

Need good NLP solver (SCP, SQP, Interior Point, ...)

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$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & J(w) + \frac{1}{\sigma_j} G_1(w)^\top G_2(w) \\ \text{s.t.} \quad & 0 = F(w) \\ & 0 \geq H(w) \\ & 0 \leq G_1(w) \\ & 0 \leq G_2(w) \end{aligned}$$

Need good NLP solver (SCP, SQP, Interior Point, ...)

Relxed subproblem for parameter σ_j

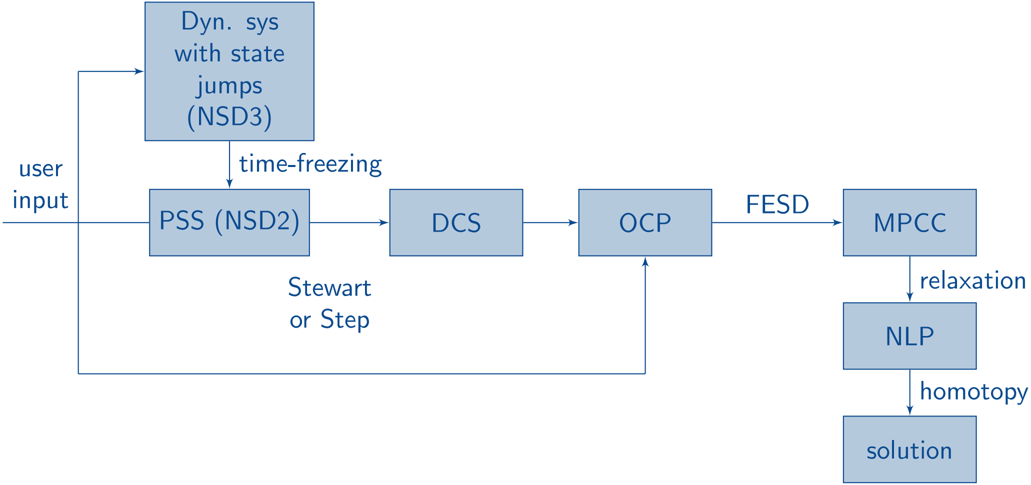
$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & J(w) \\ \text{s.t.} \quad & 0 = F(w) \\ & 0 \geq H(w) \\ & 0 \leq G_1(w) \\ & 0 \leq G_2(w) \\ & \sigma_j \geq G_1(w)^\top G_2(w) \end{aligned}$$

Crucial: start NLP solver at previous solution w_{j-1}^*.

One can often find "good" local minima with the homotopy method.

NOSNOC: NOnSmooth Numerical Optimal Control

The whole tool chain is available in our open-source package NOSNOC



NOSNOC: `https://github.com/nurkanovic/nosnoc`

NOSNOC: NOnSmooth Numerical Optimal Control

Open-source package based on MATLAB, CasADi and IPOPT

Key features

1. automatic reformulation of systems with state jumps into switched systems via the time-freezing reformulation
2. automatic discretization of the OCP via FESD (high accuracy)
3. solution methods for the resulting discrete-time OCP via continuous optimization in a homotopy (no integers)

NOSNOC: `https://github.com/nurkanovic/nosnoc`

OCP example

Benchmark example with entering/leaving sliding mode

OCP with sliding modes

$$\min_{x(\cdot),u(\cdot)} \quad \int_0^4 u(t)^\top u(t) + v(t)^\top v(t)\,\mathrm{d}t$$

$$\text{s.t.} \quad x(0) = (\frac{2\pi}{3}, \frac{\pi}{3}, 0, 0)$$

$$\dot{x}(t) = \begin{bmatrix} -\mathrm{sign}(c(x(t))) + v(t) \\ u(t) \end{bmatrix}$$

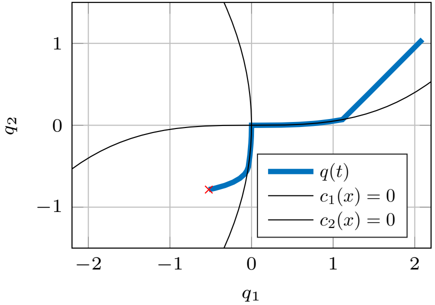
$$-2e \leq v(t) \leq 2e$$

$$-10e \leq u(t) \leq 10e \quad t \in [0,4],$$

$$q(T) = (-\frac{\pi}{6}, -\frac{\pi}{4})$$

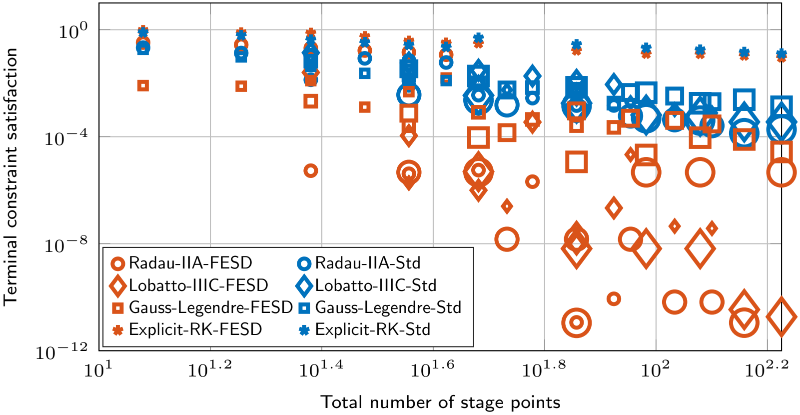
States $q, v \in \mathbb{R}^2$ and control $u \in \mathbb{R}^2$, $x = (q, v)$

Switching functions $c(x) = \begin{bmatrix} q_1 + 0.15q_2^2 \\ 0.05q_1^3 + q_2 \end{bmatrix}$



FESD vs standard IRK - number of function evaluations

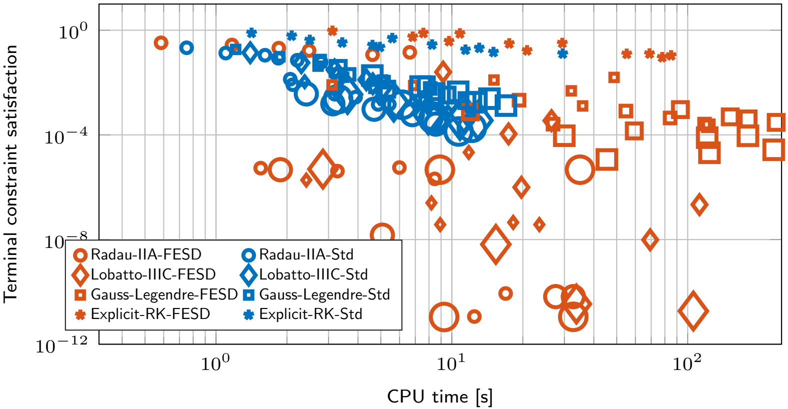
Benchmark on an optimal control problem with nonlinear sliding modes



Terminal constraint satisfaction vs. number of stage points

FESD vs standard IRK - CPU Time

Benchmark on an optimal control problem with nonlinear sliding modes



Terminal constraint satisfaction vs. CPU time

FESD one million times more accurate than Std. for CPU time of ≈ 2 s

Conclusions and outlook

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- Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for nonsmooth systems of level NSD2
- FESD resolves many of the issues that standard methods have: integration accuracy, convergence of sensitivities
- Main difficulty: solving the Mathematical Programs with Complementarity Constraints (MPCC)

Conclusions and outlook

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Outlook

- Improve on MPCC methods, test other existing relaxation methods (work in progress, soon available in NOSNOC)
- Properties of FESD-MPCC solutions. Are all stationary points strongly stationary points?
- Combinatorial methods for MPCC arising in nonsmooth optimal control
- Efficient NCP solvers for FESD subproblems

Related References

- A time-freezing approach for numerical optimal control of nonsmooth differential equations with state jumps.
  A. Nurkanović, T. Sartor, S. Albrecht, and M. Diehl, IEEE Cont. Sys. Lett., 2021.
- Continuous optimization for control of hybrid systems with hysteresis via time-freezing
  A. Nurkanović and M. Diehl, IEEE Cont. Sys. Lett., 2022.
- The Time-Freezing Reformulation for Numerical Optimal Control of Complementarity Lagrangian Systems with State Jumps.
  A. Nurkanović, S. Albrecht, B. Brogliato, and M. Diehl, arXiv preprint 2022
- Set-Valued Rigid Body Dynamics for Simultaneous Frictional Impact.
  Mathew Halm and Michael Posa, arXiv Preprint, 2021.

Thank you very much for your attention!