

The Lifted Newton Method and Its Use for Nonlinear Optimization

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joint work with Jan Albersmeyer

(picture from 2010 at KU Leuven)



Overview

- **Idea of Lifted Newton Method**
- An algorithmic trick to exploit structure "for free"
- Application to optimization
- Convergence analysis
- Numerical Tests

Lifted Newton Method described in:

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THE LIFTED NEWTON METHOD AND ITS APPLICATION IN OPTIMIZATION*

JAN ALBERSMEYER[†] AND MORITZ DIEHL[‡]

Simplified Setting: Root Finding Problem

Aim: solve root finding problem

$$F(u) = 0$$

where $F \in \mathcal{C}^1(\mathbb{R}^{n_u}, \mathbb{R}^{n_u})$ is composed of several nonlinear subfunctions:

Simplified Setting: Root Finding Problem

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$$F(u) = 0$$

where $F \in \mathcal{C}^1(\mathbb{R}^{n_u}, \mathbb{R}^{n_u})$ is composed of several nonlinear subfunctions:

Algorithm 1: Function with output of intermediate variables

Input : $u \in \mathbb{R}^{n_u}$

Output: $x_1 \in \mathbb{R}^{n_1}, \dots, x_m \in \mathbb{R}^{n_m}, F \in \mathbb{R}^{n_u}$

begin

 for $i = 1, 2, \dots, m$ do

$x_i = f_i(u, x_1, x_2, \dots, x_{i-1});$

 end for

$F := f_F(u, x_1, x_2, \dots, x_m);$

end

Idea: "Lift" root finding problem into a higher dimensional space...

Lifted Problem

The equivalent, "lifted" problem is:

$$G(u, x) = 0$$

with

$$G(u, x) = \begin{pmatrix} f_1(u) & - & x_1 \\ f_2(u, x_1) & - & x_2 \\ \vdots & & \\ f_m(u, x_1, \dots, x_{m-1}) & - & x_m \\ f_F(u, x_1, \dots, x_m) & & \end{pmatrix}$$

Why to lift and increase the system size?

- Lifting is a generalization of the well-known "multiple shooting" technique for solution of
 - boundary value problems [Osborne 1969]
 - parameter estimation problems in ODE [Bock 1987, Schloeder 1988]
 - optimal control problems [Bock and Plitt 1984, Gill et al. 2000, Schaefer 2005]
- Lifting often offers advantages in terms of
 - sparsity exploitation (not today's focus)
 - larger region of convergence
 - **more freedom for initialization (e.g. measurements)**
 - **faster local contraction rate**

Motivating Toy Example

- Original scalar root finding problem:

$$F(u) := u^{16} - 2 = 0$$

- Lifted formulation yields a nonlinear equation system with 5 equations in 5 variables:

$$x_1 := u^2 \quad x_2 := x_1^2$$

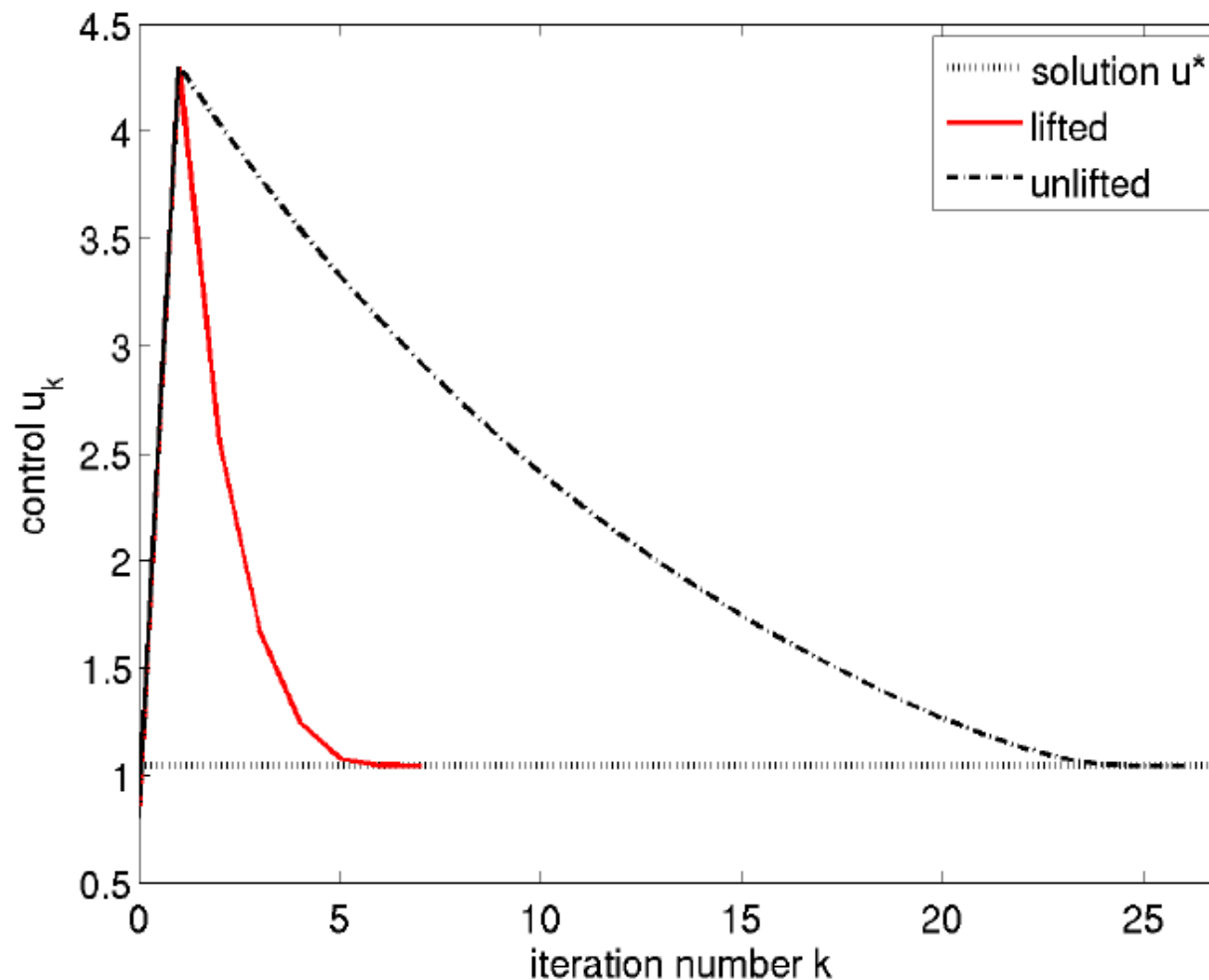
$$x_3 := x_2^2 \quad x_4 := x_3^2$$

$$F := x_4 - 2$$

- Compare lifted and unlifted Newton iterates with same initial guess. Obtain intermediate lifted variables by function evaluation.

Motivating Toy Example

- First iteration is identical, as we initialized identically
- Lifted Newton method converges in 7 instead of 26 iterations!



Lifted Problem is much larger ... is it more expensive?

In each lifted Newton iteration

$$\begin{pmatrix} x^{k+1} \\ u^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ u^k \end{pmatrix} + \begin{pmatrix} \Delta x^k \\ \Delta u^k \end{pmatrix}$$

we have to solve a large linear system:

$$\begin{pmatrix} \Delta x^k \\ \Delta u^k \end{pmatrix} = - \left[\frac{\partial G}{\partial (u, x)}(x^k, u^k) \right]^{-1} G(x^k, u^k)$$

It is large and structured,...

... but exploitation algorithms are difficult to develop and implement,
cf. Dissertations [Schloeder1988, Schaefer2005] in context of
multiple shooting

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Algorithmic Trick: Preliminaries 1

Original "user function":

Algorithm 1: Function with output of intermediate variables

Input : $u \in \mathbb{R}^{n_u}$
Output: $x_1 \in \mathbb{R}^{n_1}, \dots, x_m \in \mathbb{R}^{n_m}, F \in \mathbb{R}^{n_u}$
begin
 for $i = 1, 2, \dots, m$ **do**
 $x_i = f_i(u, x_1, x_2, \dots, x_{i-1});$
 end for
 $F := f_F(u, x_1, x_2, \dots, x_m);$
end

"Lifted residual" (after minor code additions):

Algorithm 2: Residual function $G(u, y)$

Input : $u, y_1, \dots, y_m$
Output: $G_1, \dots, G_m, F$
begin
 for $i = 1, 2, \dots, m$ **do**
 $x_i = f_i(u, x_1, x_2, \dots, x_{i-1});$
 $G_i = x_i - y_i;$
 $x_i = y_i;$
 end for
 $F = f_F(u, x_1, x_2, \dots, x_m);$
end

Algorithmic Trick: Preliminaries 2

Write

$$G(u, x) = \begin{pmatrix} H(u, x) - x \\ f_F(u, x) \end{pmatrix}$$

with

$$H(u, x) := \begin{pmatrix} f_1(u) \\ f_2(u, x_1) \\ \vdots \\ f_m(u, x_1, \dots, x_{m-1}) \end{pmatrix}$$

in each lifted Newton iteration, we have to solve:

$$\begin{aligned} H(u, x) - x + \left(\frac{\partial H}{\partial x}(u, x) - \mathbb{I}_{n_x} \right) \Delta x + \frac{\partial H}{\partial u}(u, x) \Delta u &= 0 \\ f_F(u, x) + \frac{\partial f_F}{\partial x}(u, x) \Delta x + \frac{\partial f_F}{\partial u}(u, x) \Delta u &= 0 \end{aligned}$$

How to solve this linear system efficiently ?

Well-Known Technique: "Condensing"

To solve $H(u, x) - x + \left(\frac{\partial H}{\partial x}(u, x) - \mathbb{I}_{n_x} \right) \Delta x + \frac{\partial H}{\partial u}(u, x) \Delta u = 0$

$$f_F(u, x) + \frac{\partial f_F}{\partial x}(u, x) \Delta x + \frac{\partial f_F}{\partial u}(u, x) \Delta u = 0$$

can eliminate $\Delta x = \underbrace{- \left(\frac{\partial H}{\partial x}(u, x) - \mathbb{I}_{n_x} \right)^{-1} (H(u, x) - x)}_{=:a} + \underbrace{- \left(\frac{\partial H}{\partial x}(u, x) - \mathbb{I}_{n_x} \right)^{-1} \frac{\partial H}{\partial u}(u, x)}_{=:A} \Delta u$

and solve "condensed" system in original

degrees of freedom only: $\Delta u = -B^{-1} b$

and expand solution again: $\Delta x = a + A \Delta u$

But how to compute a, b, A, B efficiently ?

Basis of new trick: an auxiliary function

Define $Z(u, d)$

as implicit function satisfying a perturbed fixed point equation (by vector d)

$$H(u, z) - z - d = 0$$

PROPOSITION: if $d = H(u, x) - x$ then

$$\frac{\partial Z}{\partial u}(u, d) = - \left(\frac{\partial H}{\partial x}(u, x) - \mathbb{I}_{n_x} \right)^{-1} \frac{\partial H}{\partial u}(u, x)$$

and

$$\frac{\partial Z}{\partial d}(u, d) = \left(\frac{\partial H}{\partial x}(u, x) - \mathbb{I}_{n_x} \right)^{-1}$$

Partial derivatives of Z deliver part of solution of Newton system ! But how to obtain Z ?

Obtain Z by another minor code modification

- Can show that Z is obtained by one evaluation of the following function:

Algorithm 3: Modified function $Z(u, d)$

Input : $u, d_1, \dots, d_m$

Output: $z_1, \dots, z_m, F$

begin

for $i = 1, 2, \dots, m$ **do**

$x_i = f_i(u, x_1, x_2, \dots, x_{i-1});$

$z_i = x_i - d_i;$

$x_i = z_i;$

end for

$F = f_F(u, x_1, x_2, \dots, x_m);$

end

Costs of generating and solving linear system

Using Z , can easily compute via directional derivatives a, A, b, B as

$$a = -\frac{\partial Z}{\partial d}(u, d) d$$

$$A = \frac{\partial Z}{\partial u}(u, d)$$

$$b = f_F(u, x) + \frac{\partial f_F}{\partial x}(u, x)a$$

$$B = \frac{\partial f_F}{\partial u}(u, x) + \frac{\partial f_F}{\partial x}(u, x)A$$

and then compute the Newton step

$$\Delta u = -B^{-1} b$$

$$\Delta x = a + A \Delta u$$

Computational effort per iteration:

- computing A & B (in one combined forward sweep)
- factoring B

Extra efforts for lifting:

- computing vector a (one extra directional derivative)
- matrix vector product $A \Delta u$ (could also be done w/o evaluating A)

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Two lifted algorithms

- Lifted Gauss-Newton Method
- Lifted SQP Method

Lifted Gauss-Newton

- Original Problem:

$$\begin{array}{ll}\min_u & \frac{1}{2} \|F_1(u)\|_2^2 \\ \text{s.t.} & F_2(u) \begin{Bmatrix} = \\ \geq \end{Bmatrix} 0.\end{array}$$

- Lifted Problem:

$$\begin{array}{ll}\min_{u,x} & \frac{1}{2} \|f_{F_1}(u, x)\|_2^2 \\ \text{s.t.} & f_{F_2}(u, x) \begin{Bmatrix} = \\ \geq \end{Bmatrix} 0 \\ & H(u, x) - x = 0.\end{array}$$

(obtained by lifting $F(u) := (F_1(u)^T, F_2(u)^T)^T$)

Lifted Gauss-Newton: Quadratic Subproblems

- Linearized lifted problem = structured QP subproblem:

$$\begin{aligned} \min_{\Delta u, \Delta x} \quad & \frac{1}{2} \left\| f_{F_1}(u_k, x_k) + \frac{\partial f_{F_1}}{\partial(u, x)}(u_k, x_k) \begin{pmatrix} \Delta u \\ \Delta x \end{pmatrix} \right\|_2^2 \\ \text{s.t.} \quad & f_{F_2}(u_k) + \frac{\partial f_{F_2}}{\partial(u, x)}(u_k, x_k) \begin{pmatrix} \Delta u \\ \Delta x \end{pmatrix} \begin{cases} = \\ \geq \end{cases} 0 \\ & H(u_k, x_k) - x_k + \frac{\partial H}{\partial(u, x)}(u_k, x_k) \begin{pmatrix} \Delta u \\ \Delta x \end{pmatrix} - \Delta x = 0 \end{aligned}$$

Lifted Gauss-Newton: Quadratic Subproblems

- Linearized lifted problem = structured QP subproblem:

$$\begin{aligned}
 \min_{\Delta u, \Delta x} \quad & \frac{1}{2} \left\| f_{F_1}(u_k, x_k) + \frac{\partial f_{F_1}}{\partial(u, x)}(u_k, x_k) \begin{pmatrix} \Delta u \\ \Delta x \end{pmatrix} \right\|_2^2 \\
 \text{s.t.} \quad & \\
 & f_{F_2}(u_k) + \frac{\partial f_{F_2}}{\partial(u, x)}(u_k, x_k) \begin{pmatrix} \Delta u \\ \Delta x \end{pmatrix} \begin{cases} = \\ \geq \end{cases} 0 \\
 & H(u_k, x_k) - x_k + \frac{\partial H}{\partial(u, x)}(u_k, x_k) \begin{pmatrix} \Delta u \\ \Delta x \end{pmatrix} - \Delta x = 0
 \end{aligned}$$

- Condensed QP (solved by dense QP solver):

$$\begin{aligned}
 \min_{\Delta u} \quad & \frac{1}{2} \|b_1 + B_1 \Delta u\|_2^2 \\
 \text{s.t.} \quad & \\
 & b_2 + B_2 \Delta u \begin{cases} = \\ \geq \end{cases} 0
 \end{aligned}$$

easily get $\dot{b} = (b_1^T, b_2^T)^T$ and $B = (B_1^T, B_2^T)^T$ as lifted derivatives of
 $F(u) := (F_1(u)^T, F_2(u)^T)^T$

Lifted SQP Method: Problem Statement

- For notational simplicity, regard first only unconstrained optimization problem:

$$\min_u \varphi(u)$$

- Aim: find root of first order necessary conditions for optimality:

$$F(u) := \nabla_u \varphi(u) = 0$$

Lifted Problem

- Lifted problem formulation:

$$\begin{array}{ll} \min_{u,w} & f_\varphi(u, w_1, w_2, \dots, w_m) \\ \text{s.t.} & \\ & g(u, w) = \begin{pmatrix} f_1(u) & - & w_1 \\ f_2(u, w_1) & - & w_2 \\ \vdots & & \\ f_m(u, w_1, \dots, w_{m-1}) & - & w_m \end{pmatrix} = 0 \end{array}$$

- First order necessary conditions for optimality of lifted problem

$$\nabla_u \mathcal{L}(u, w, \lambda) = \nabla_u f_\varphi(u, w) + \nabla_u g(u, w) \lambda = 0$$

$$\nabla_w \mathcal{L}(u, w, \lambda) = \nabla_w f_\varphi(u, w) + \nabla_w g(u, w) \lambda = 0$$

$$\nabla_\lambda \mathcal{L}(u, w, \lambda) = g(u, w) = 0.$$

- Lifted Newton = full space SQP method (with exact Hessian) ?

Gradient evaluation via adjoint differentiation

To compute $F(u) := \nabla_u \varphi(u)$ perform the following code:

$$w_1 = f_1(u) \tag{3.8a}$$

$$w_2 = f_2(u, w_1) \tag{3.8b}$$

$\vdots$

function value

$$w_m = f_m(u, w_1, \dots, w_{m-1}) \tag{3.8c}$$

$$y \equiv f_\varphi(u, w_1, \dots, w_m) \tag{3.8d}$$

$$\bar{w}_m = \nabla_{w_m} f_\varphi \tag{3.8e}$$

$$\bar{w}_{m-1} = \nabla_{w_{m-1}} f_\varphi + \nabla_{w_{m-1}} f_m \bar{w}_m \tag{3.8f}$$

$\vdots$

$$\bar{w}_1 = \nabla_{w_1} f_\varphi + \sum_{i=2}^m \nabla_{w_1} f_i \bar{w}_i \tag{3.8g}$$

gradient

$$\bar{u} = \nabla_u f_\varphi + \sum_{i=1}^m \nabla_u f_i \bar{w}_i. \tag{3.8h}$$

Gradient evaluation via adjoint differentiation

To compute $F(u) := \nabla_u \varphi(u)$ perform the following code:

$$w_1 = f_1(u) \quad (3.8a)$$

$$w_2 = f_2(u, w_1) \quad (3.8b)$$

$\vdots$

$$w_m = f_m(u, w_1, \dots, w_{m-1}) \quad (3.8c)$$

function value

$$y \equiv f_\varphi(u, w_1, \dots, w_m) \quad (3.8d)$$

$$\bar{w}_m = \nabla_{w_m} f_\varphi \quad (3.8e)$$

$$\bar{w}_{m-1} = \nabla_{w_{m-1}} f_\varphi + \nabla_{w_{m-1}} f_m \bar{w}_m \quad (3.8f)$$

$\vdots$

$$\bar{w}_1 = \nabla_{w_1} f_\varphi + \sum_{i=2}^m \nabla_{w_1} f_i \bar{w}_i \quad (3.8g)$$

gradient

$$\bar{u} = \nabla_u f_\varphi + \sum_{i=1}^m \nabla_u f_i \bar{w}_i. \quad (3.8h)$$

Lifted Newton = Full Space SQP

THEOREM: If we lift $F(u) := \nabla_u \varphi(u)$ with respect to all intermediate variables: $x = (w_1, \dots, w_m, \bar{w}_m, \dots, \bar{w}_1)$

then full space SQP and lifted Newton iterations are identical, with $\lambda \equiv \bar{w}$

COROLLARY: Same equivalence holds for lifting of constrained problems:

$$\boxed{\begin{array}{l} \min_u \varphi(u) \\ \text{subject to} \\ h(u) = 0, \end{array}} \longrightarrow \boxed{\begin{array}{l} \min_{u,w} f_\varphi(u, w) \\ \text{subject to} \\ g(u, w) = 0, \\ f_h(u, w) = 0, \end{array}}$$

if we lift Lagrange gradient and constraint:

$$\begin{pmatrix} \nabla_u \mathcal{L}^{\text{orig}} \\ h \end{pmatrix}$$

(need full derivatives only w.r.t. u , not μ , due to symmetry of KKT systems)

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Why does lifting often improve convergence speed ?

Simple model problem: chain of scalar functions

- Regard a sequence of scalar functions:

$$x_1 = f_1(u), x_2 = f_2(x_1), \dots, x_m = f_m(x_{m-1}), \text{ and } f_F(x_m) \equiv f_{m+1}(x_m)$$

- after affine transformations, can assume that solution is zero and

$$f_i(x) = x + b_i(x)^2 + O(|x|^3)$$

- under these circumstances, the non-lifted function is:

$$F(u) = f_{m+1}(f_m(\dots f_1(u) \dots)) = u + \left(\sum_{i=1}^{m+1} b_i \right) u^2 + O(|u|^3)$$

Convergence speed of Non-Lifted Newton

- Derivative is given by

$$F'(u) = 1 + 2\left(\sum_{i=1}^{m+1} b_i\right)u + O(|u|^2)$$

- It is easy to show that non-lifted Newton iterations contract like:

$$u^{[k+1]} = \left(\sum_{i=1}^{m+1} b_i\right)(u^{[k]})^2 + O(|u^{[k]}|^3)$$

i.e. quadratic convergence with contraction constant $\left(\sum_{i=1}^{m+1} b_i\right)$

Lifted Newton Convergence

- Lifted residual is

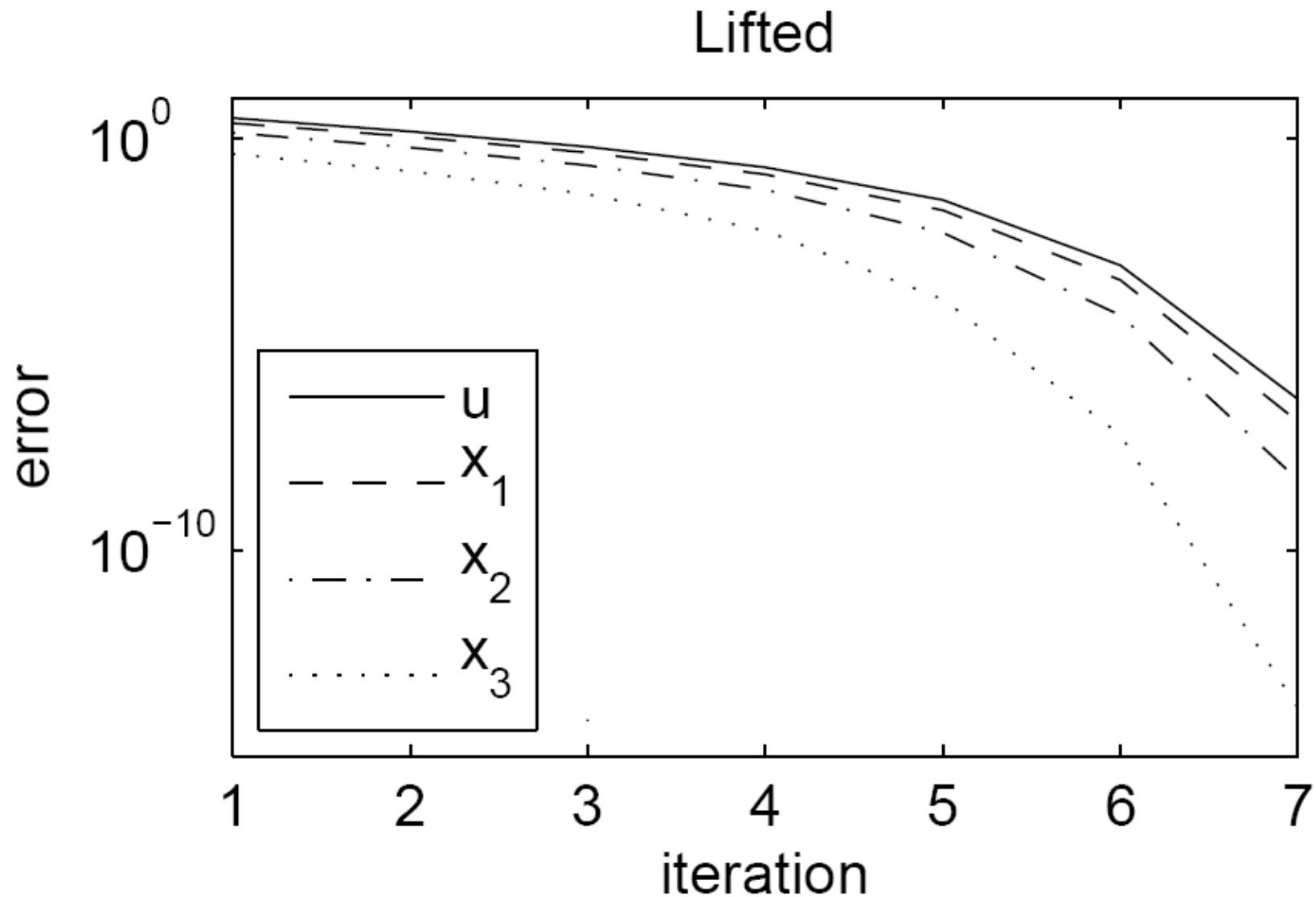
$$G(u, x) = \begin{pmatrix} u + b_1 u^2 & - & x_1 \\ x_1 + b_2 x_1^2 & - & x_2 \\ \vdots & & \\ x_{m-1} + b_m x_{m-1}^2 & - & x_m \\ x_m + b_{m+1} x_m^2 & & \end{pmatrix} + O\left(\left\|\begin{pmatrix} u \\ x \end{pmatrix}\right\|^3\right)$$

THEOREM: Lifted Newton iterations contract in a "staggered" way:

$$\begin{pmatrix} u \\ x_1 \\ \vdots \\ x_{m-1} \\ x_m \end{pmatrix}^{[k+1]} = \begin{pmatrix} b_1 (u^{[k]})^2 + \sum_{i=2}^{m+1} b_i (x_{i-1}^{[k]})^2 \\ \sum_{i=2}^{m+1} b_i (x_{i-1}^{[k]})^2 \\ \vdots \\ b_m (x_{m-1}^{[k]})^2 + b_{m+1} (x_m^{[k]})^2 \\ b_{m+1} (x_m^{[k]})^2 \end{pmatrix} + O\left(\left\|\begin{pmatrix} u \\ x \end{pmatrix}^{[k]}\right\|^3\right)$$

(if all b_i have the same sign, last variable is "leader" and contracts fastest)

Convergence for motivating toy example



Practical Conclusions from Theorem

Two cases:

- Same curvature $\rightarrow$ lifted Newton better.

E.g. in toy example, or dynamic simulation codes where several times

- same time stepping function is called
- at similar values

- Opposite curvature $\rightarrow$ unlifted Newton better.

Example: $b_1 = 1$ and $b_2 = -1$, e.g. $F(u) = u$ decomposed as

$$f_1(u) = \frac{1}{2}(1 + u)^2 - \frac{1}{2} \text{ and } f_2(x) = \sqrt{1 + 2x} - 1$$

After first iteration, unlifted Newton is already converged, lifted not!

Toy Optimal Control Problem for Illustration

$$\begin{aligned} \min_{x(\cdot), u(\cdot)} \int_0^3 |x(t)|^2 + |u(t)|^2 dt \\ \text{s.t.} \end{aligned}$$

$$\dot{x}(t) = x(t) (x(t) + 1) + u(t)$$

$$x(0) = x_0$$

$$x(3) = 0$$

$$|x(t)| \leq 1$$

$$|u(t)| \leq 1. \quad t \in [0, 3]$$

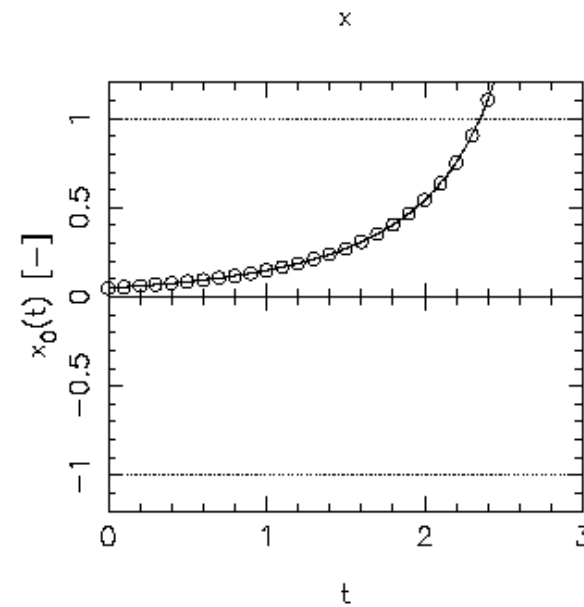
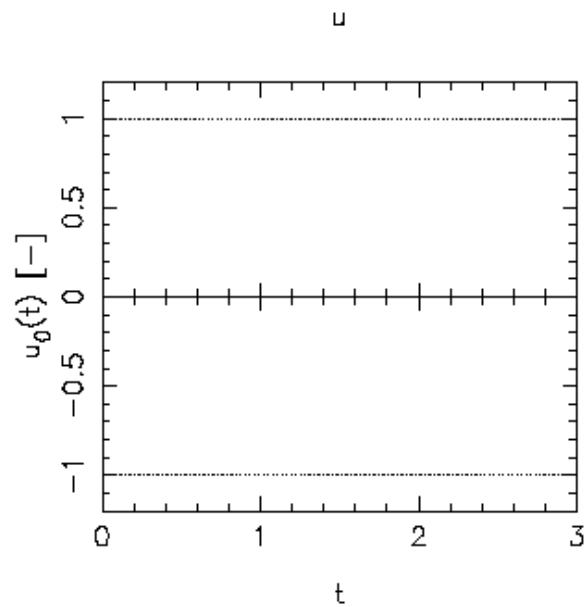
Mildly nonlinear and unstable system.

Use ODE solver with sensitivities (e.g. in acados, ACADO or MUSCOD-II).

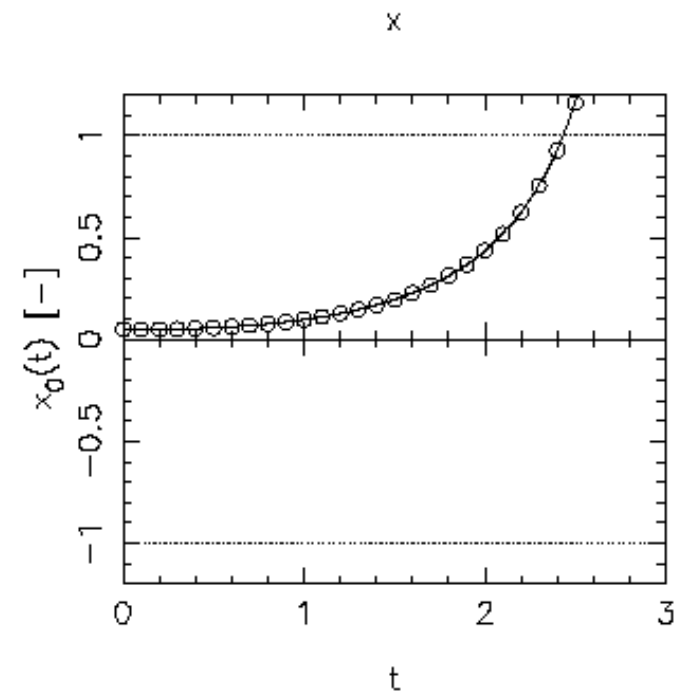
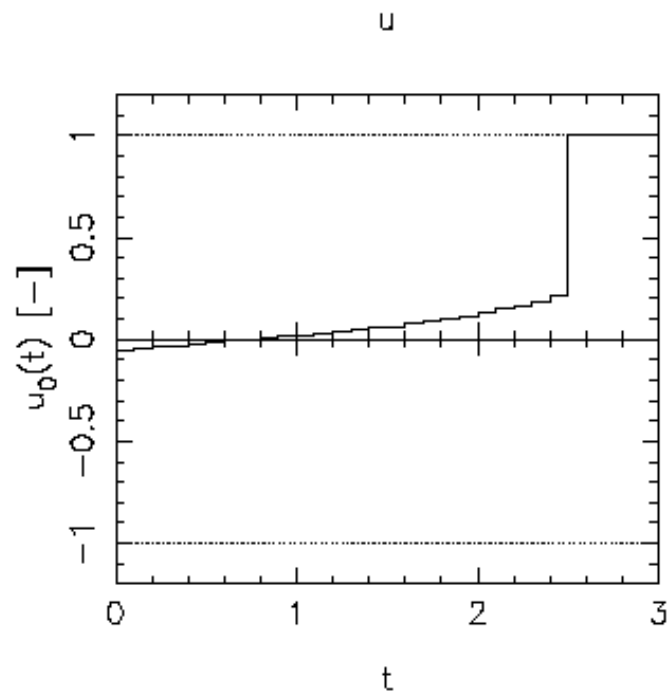
Compare unlifted and lifted Gauss-Newton method.

Unlifted Gauss-Newton

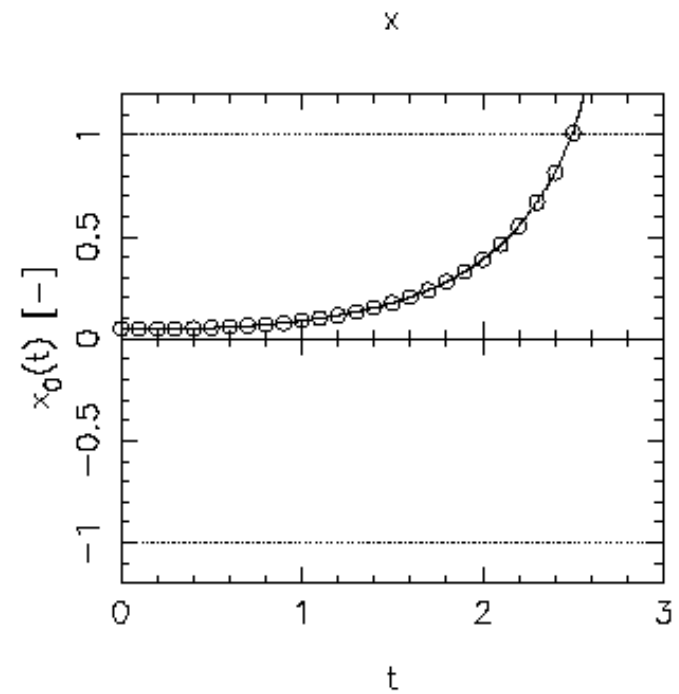
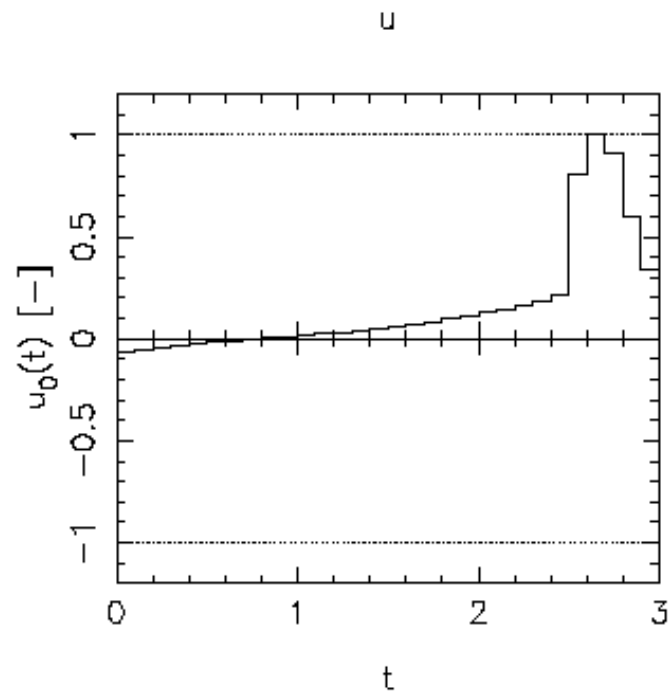
- Choose $N = 30$ equal control intervals.
- Initialize with steady state controls $u(t) \equiv 0$.



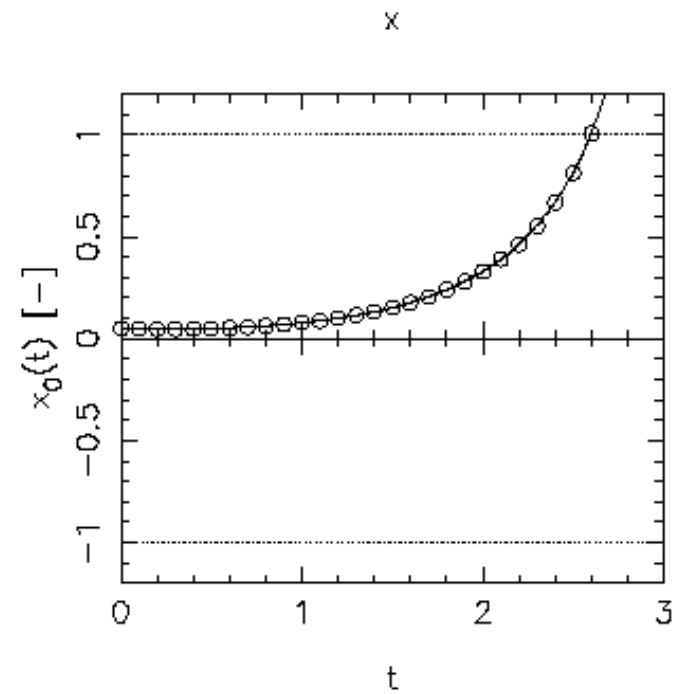
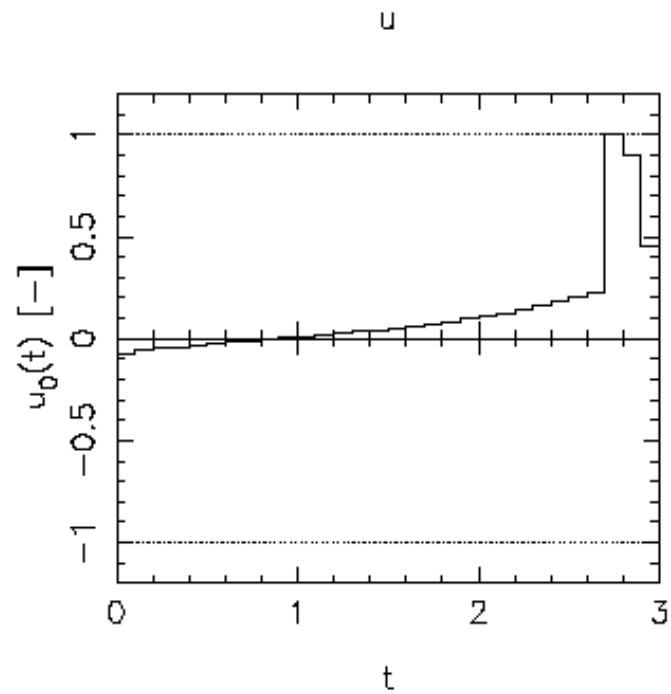
Unlifted: First Iteration



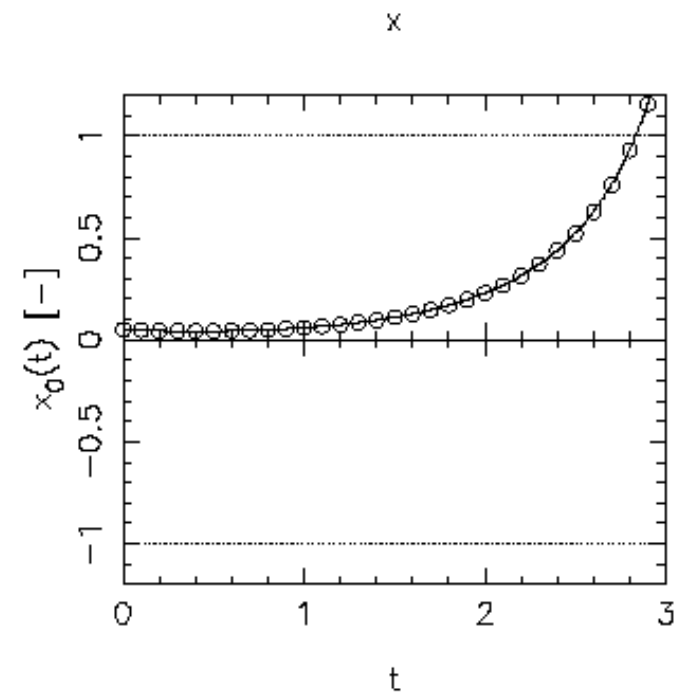
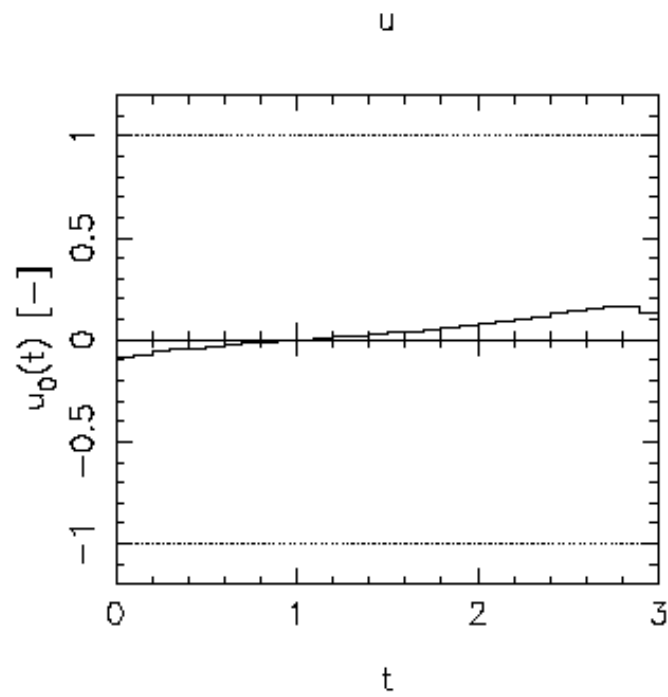
Unlifted: Second Iteration



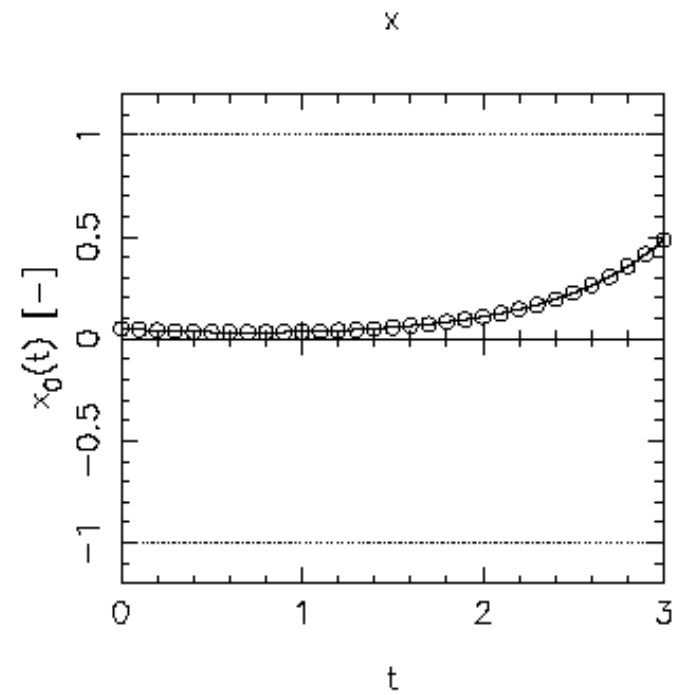
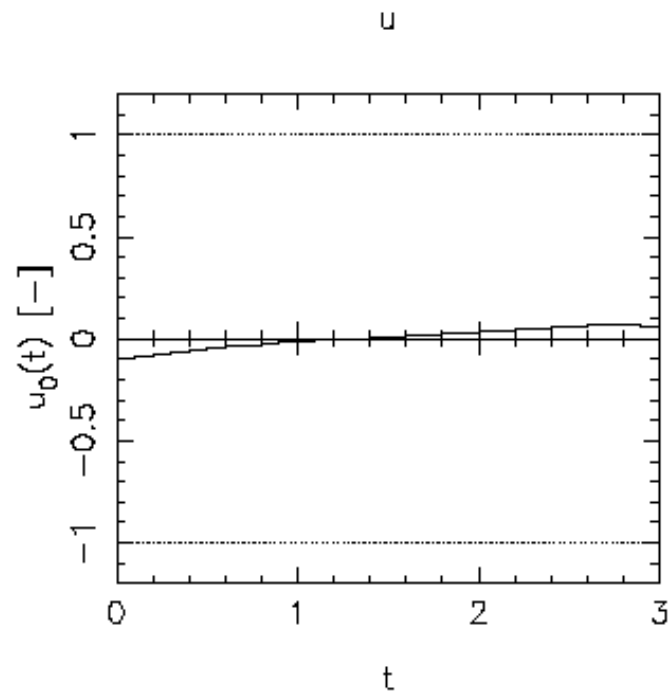
Unlifted: Third Iteration



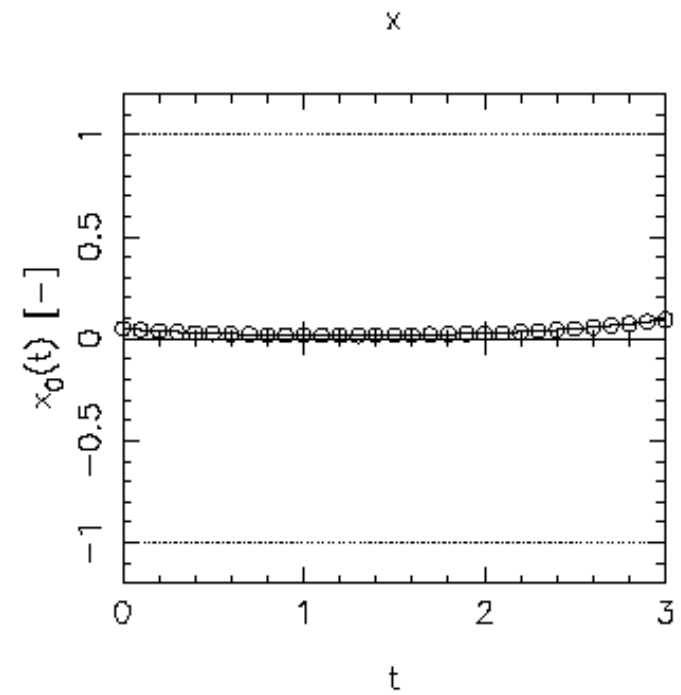
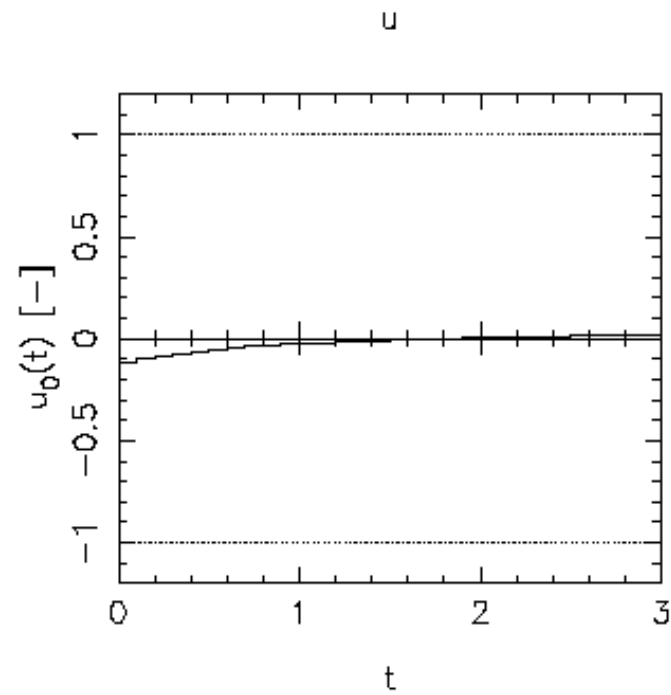
Unlifted: 4th Iteration



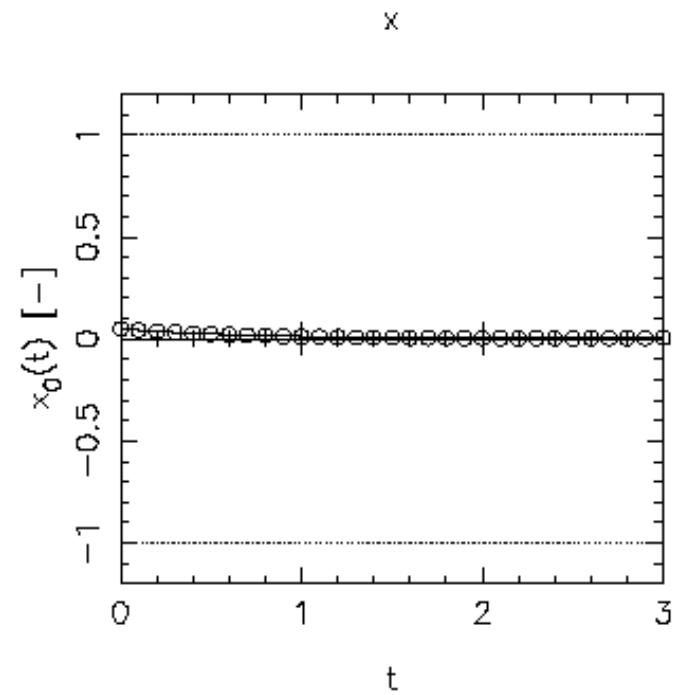
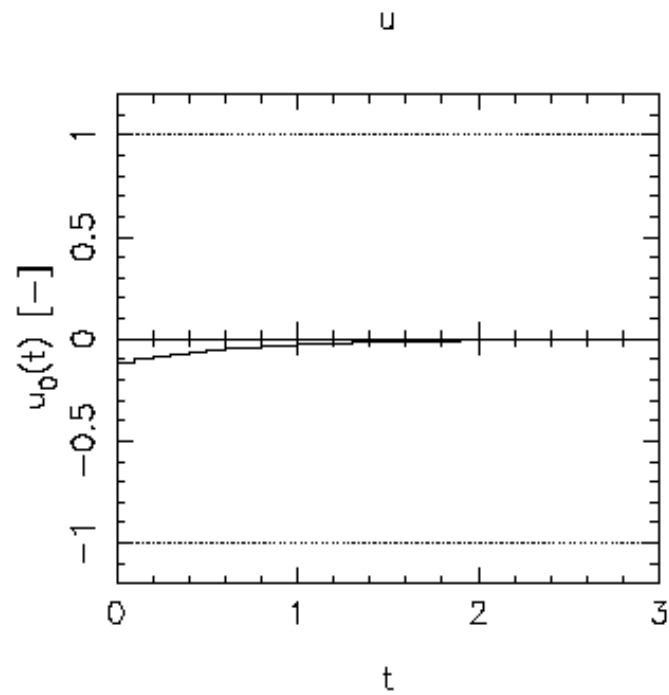
Unlifted: 5th Iteration



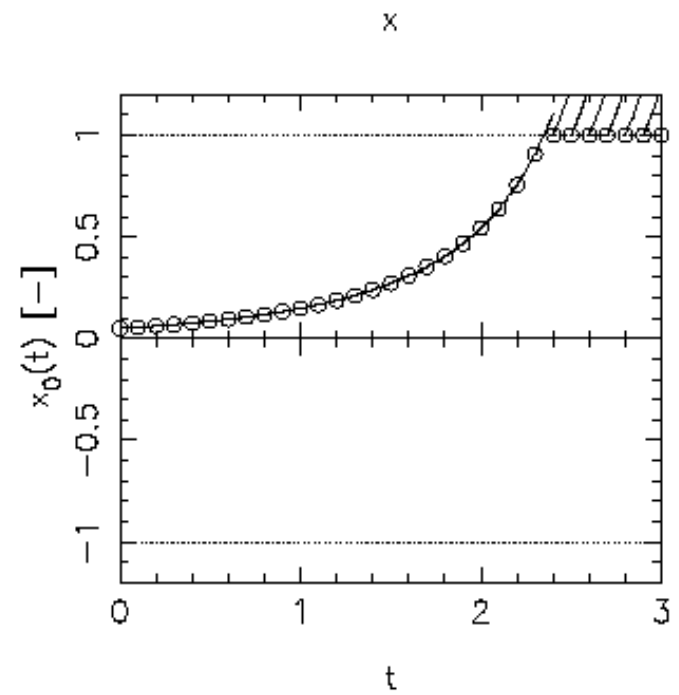
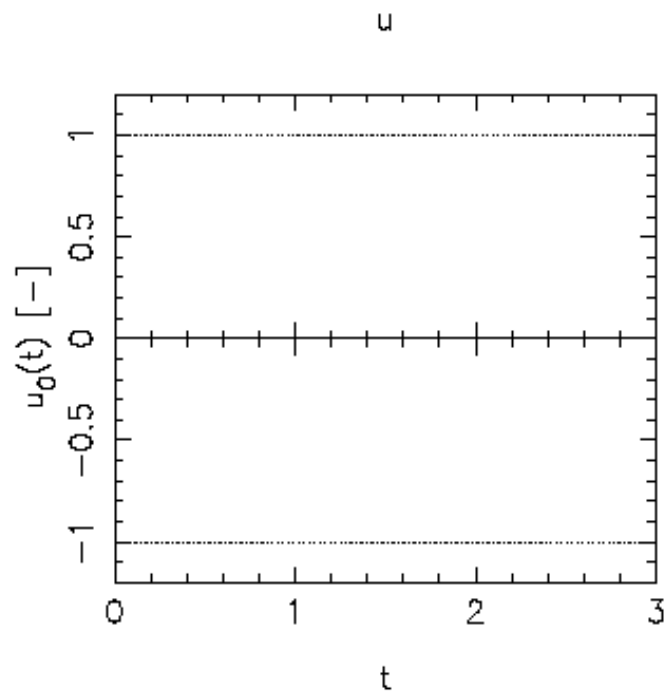
Unlifted: 6th Iteration



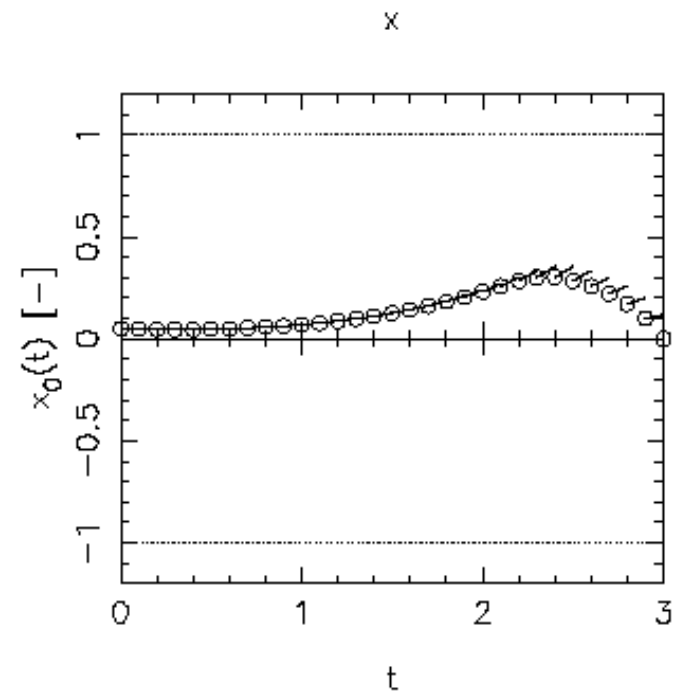
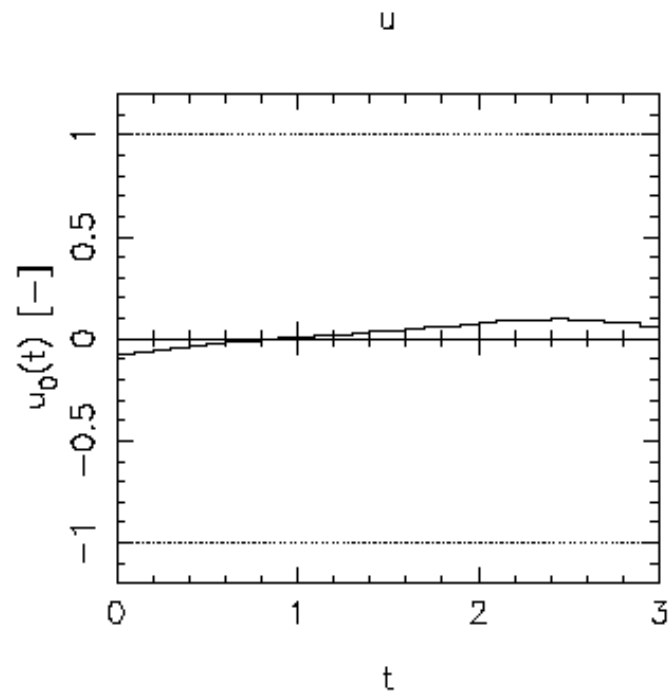
Unlifted: 7th Iteration (Solution)



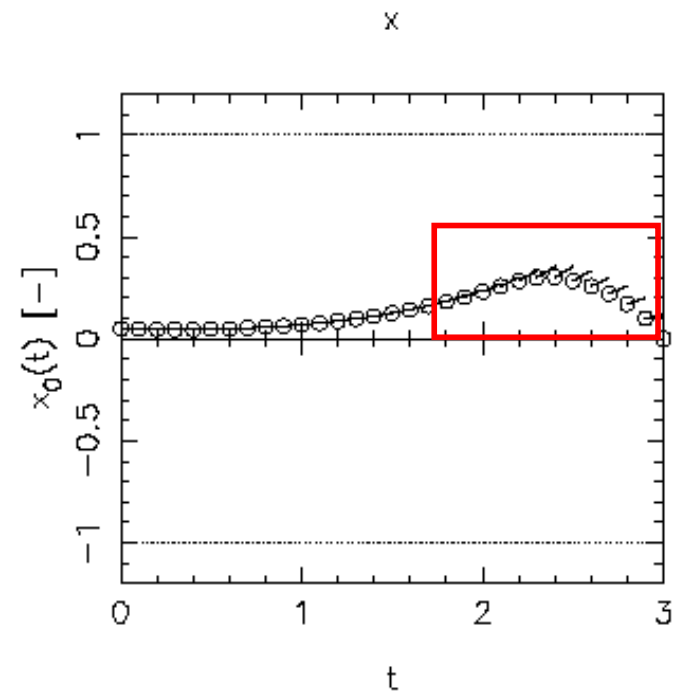
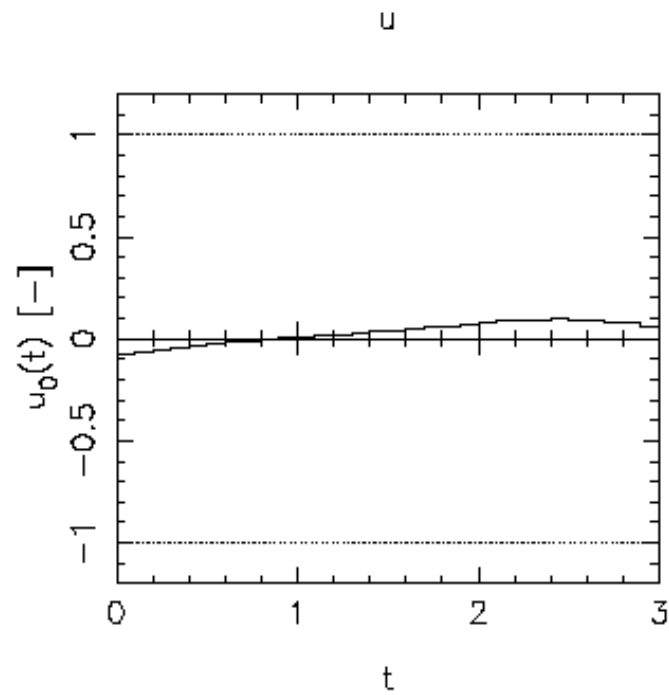
Toy Example: Lifted Initialization



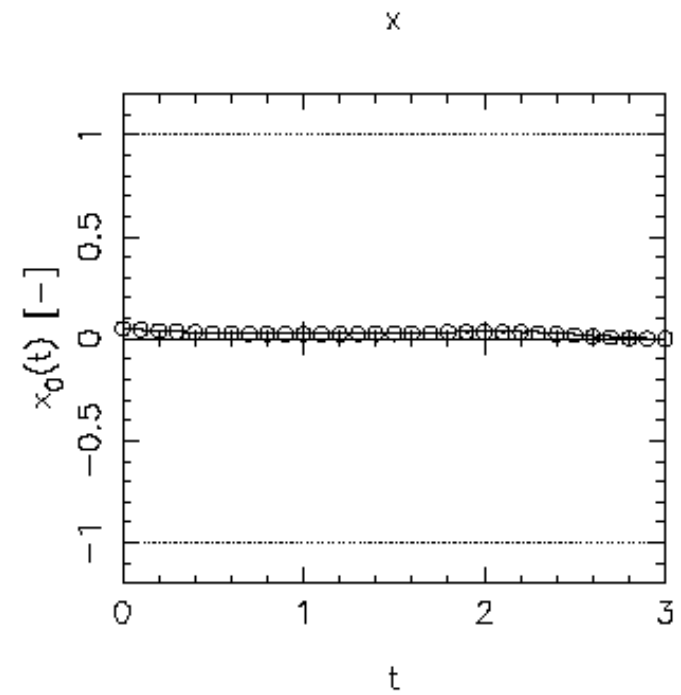
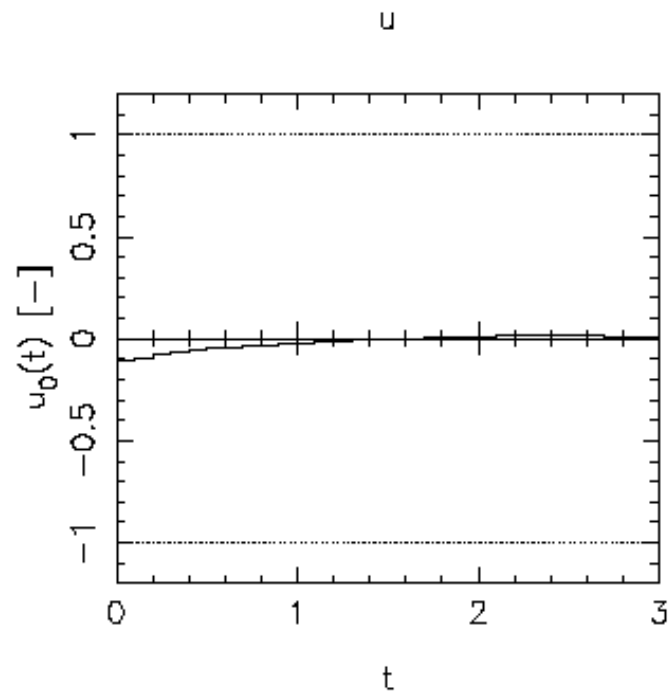
Lifted: First Iteration



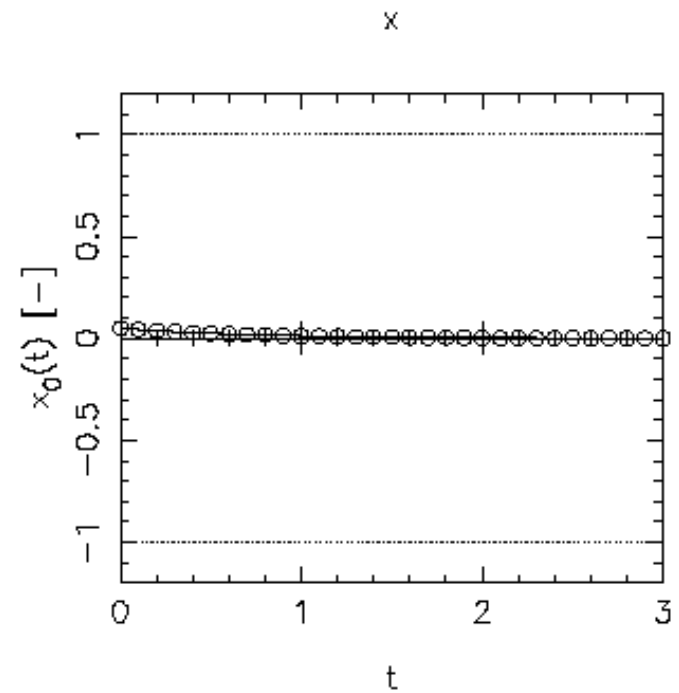
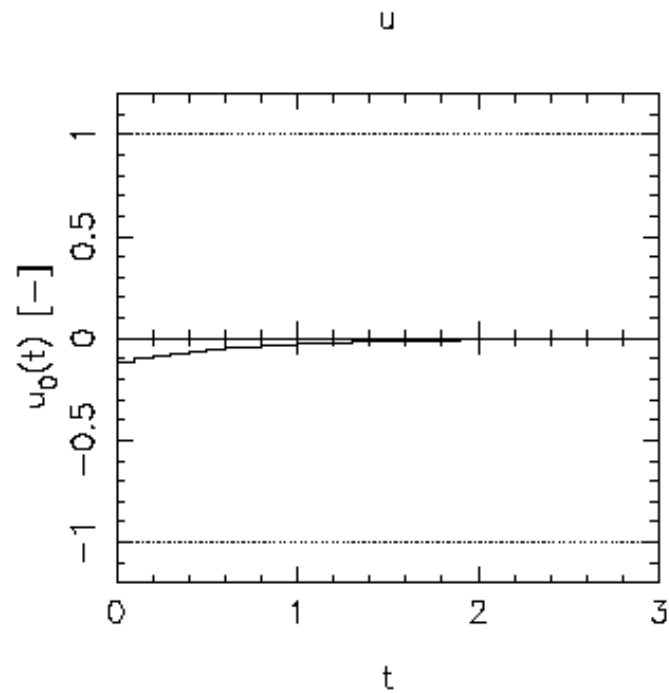
Lifted: First Iteration



Lifted: Second Iteration



Lifted: 3rd Iteration (already solution!)



Overview

- Idea of Lifted Newton Method
- An algorithmic trick to exploit structure "for free"
- Application to optimization
- Convergence analysis
- **Numerical Tests**

Large scale Gauss-Newton example 1

- Regard 2-D heat equation with 3 unknown diffusion parameters c

$$\Omega = (0, 1) \times (0, 1)$$

$$\partial_t u(t, x_1, x_2) - d(x_1, x_2) \Delta u(t, x_1, x_2) = 0, \quad (x_1, x_2) \in \Omega, t \in [0, 1]$$

$$d(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2$$

$$u(t, x_1, x_2) = 100, \quad (x_1, x_2) \in \{0\} \times [0, 1], t \in [0, 1]$$

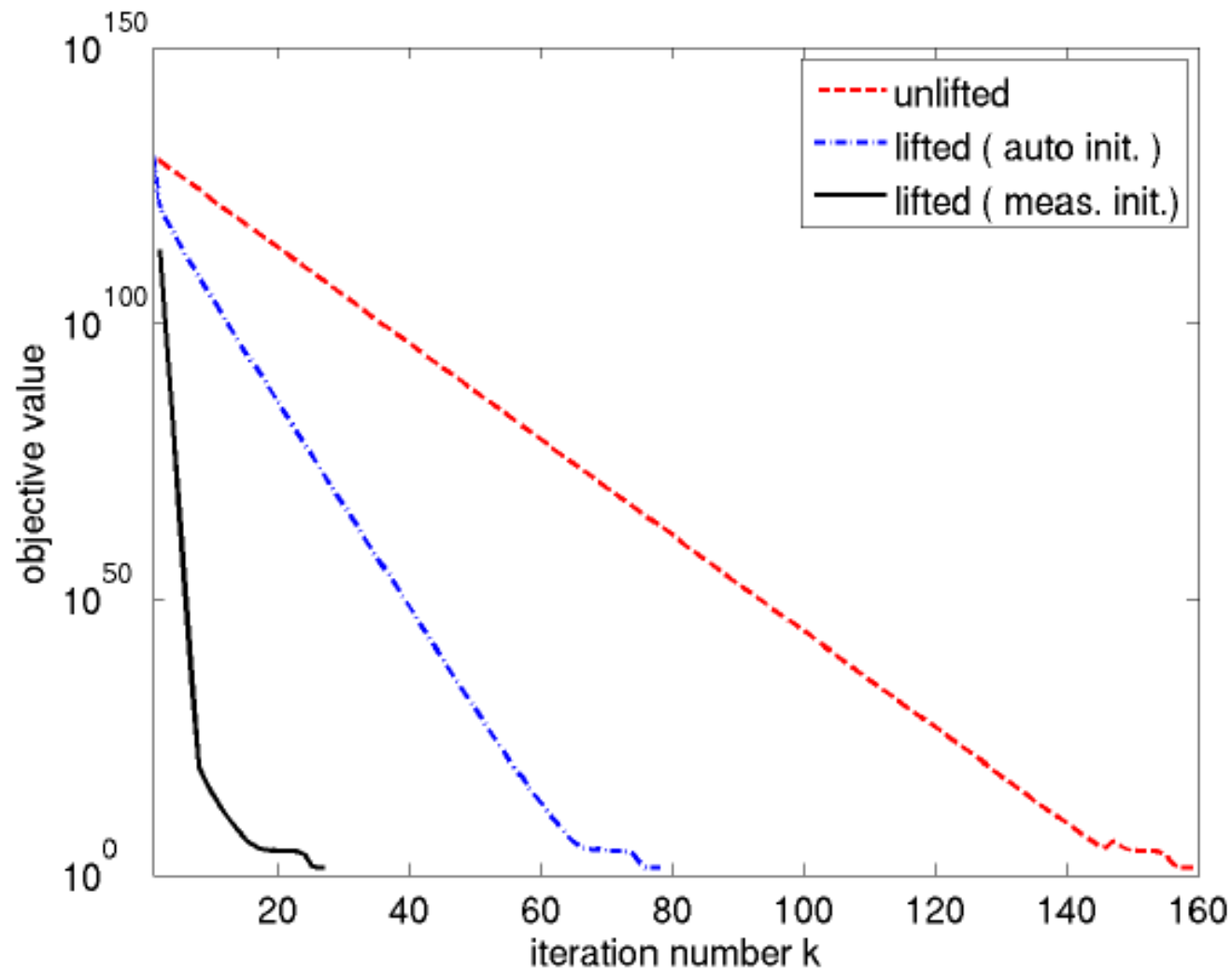
$$u(t, x_1, x_2) = 0, \quad (x_1, x_2) \in \partial\Omega \setminus (\{0\} \times [0, 1]), t \in [0, 1]$$

- Discretize on 9 x 9 grid and perform 200 timesteps
(= 16200 hidden variables)
- Measure state u every 10 time steps (= 1620 measurements)
- Aim: estimate unknown parameters

Compare three Gauss-Newton variants

- Non-lifted Newton with 3 variables only
- Lifted Newton (with 1620 lifted node variables), initialized like non-lifted variant via a forward simulation
- Lifted Newton, but all nodes initialized with measurements

Convergence for PDE parameter estimation example 1



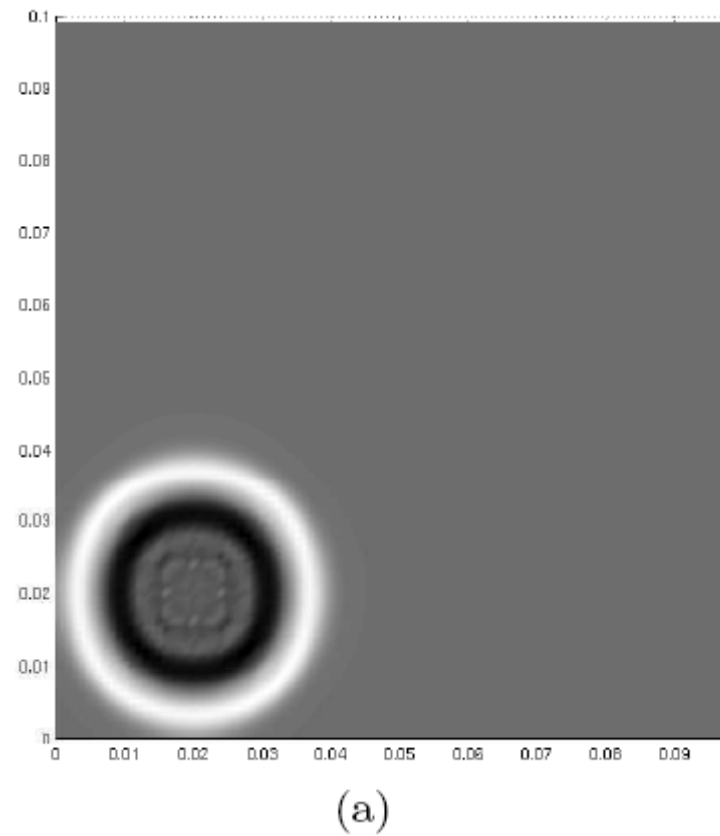
Large Scale Example 2

- Regard 2-D shallow water wave equation (with unknown parameters)

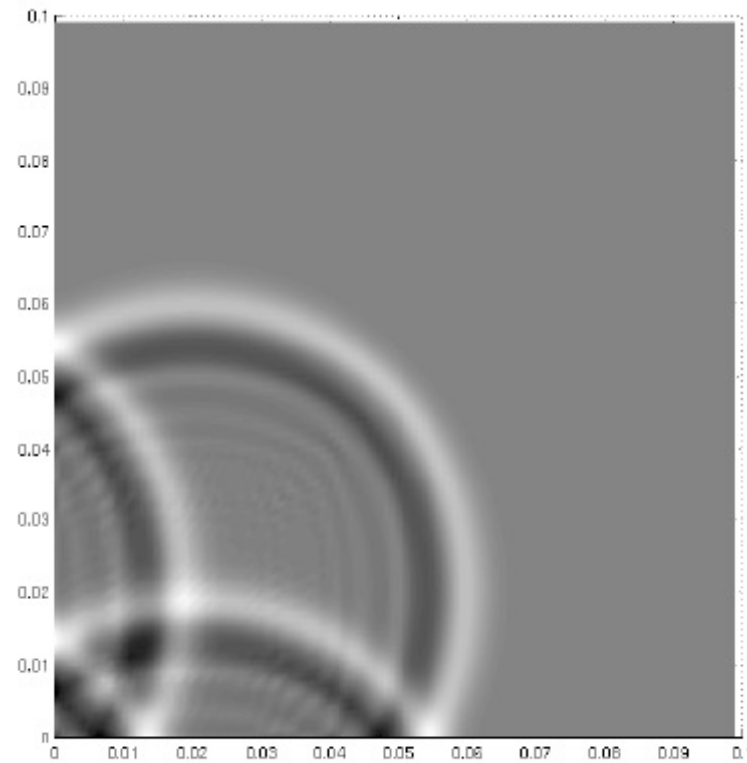
$$\begin{aligned}\partial_t u(t, x, y) &= -g \partial_x h(t, x, y) - b u(t, x, y) \\ \partial_t v(t, x, y) &= -g \partial_y h(t, x, y) - b v(t, x, y) \\ \partial_t h(t, x, y) &= -H [\partial_x u(t, x, y) + \partial_y v(t, x, y)]\end{aligned}$$

- Discretize all 3 states u , v , h on 30 x 30 grid and perform 10000 timesteps (= 27 million variables !)
- Measure **only height** h every 100 time steps (= 90 000 measurements)
- Aim: estimate 2 unknown parameters, water depth H and viscous friction coefficient b

Large Scale Example 2: Measurements

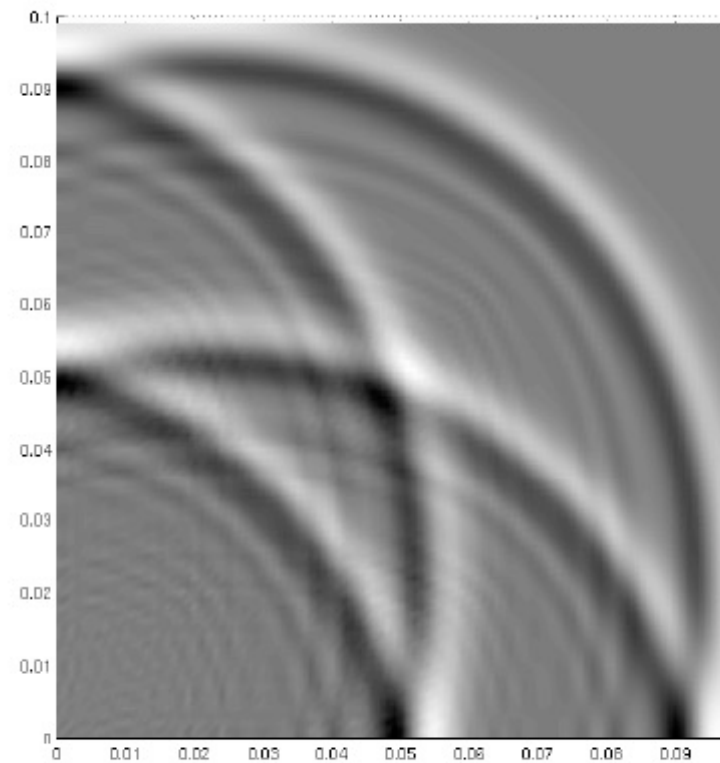


Large Scale Example 2: Measurements



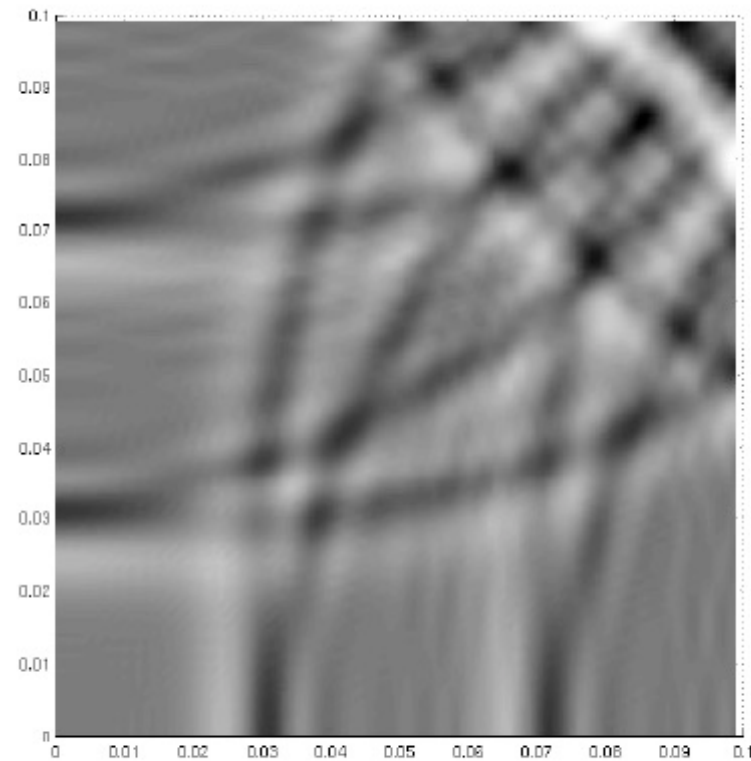
(b)

Large Scale Example 2: Measurements



(c)

Large Scale Example 2: Measurements



(d)

Compare three Gauss-Newton variants for example 2

- Non-lifted Newton with 2 variables only
- Lifted Newton (with 90 000 lifted variables), initialized like non-lifted variant via a forward simulation
- Lifted Newton, but initialize height with measurements

Performance of 3 variants for different initial guesses

b	H	# iterations unlifted	#iterations lifted (autom. init.)	#iterations lifted (meas init.)
0.5	0.01	5	5	4
5	0.01	6	5	4
15	0.01	17	7	6
30	0.01	27	7	6
2	0.005	31	9	5
2	0.02	38	12	5
2	0.1	44	13	8
0.2	0.001	33	12	7
1	0.005	47	10	5
4	0.02	56	10	5
1	0.02	44	9	6
20	0.001	24	10	6

⋮ true values $b = 2$ and $H = 0.01$

CPU time per iteration: 8.9 s for unlifted, 11.8 s for lifted (~1 min total)
(Note: formulation e.g. in AMPL would involve 27 million variables)

Summary

- Lifting offers advantages for Newton type optimization:
 - faster local convergence rate (→ proven in simplified setting)
 - more freedom for initialization
- Structure exploiting "Lifted Newton Methods" can easily be generated for **any given user functions** and **any Newton type method**:
 - only minor code additions
 - nearly no additional costs per iteration
 - compatible with SQP, BFGS, Gauss-Newton, ...
 - compatible with any linear solver for condensed systems e.g. GMRES, cf. [Shimizu, Ohtsuka & D. 2009]
- Code and examples available in C++ package LiftOpt [Albersmeyer&D. 2010]

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