

Theory of mathematical programs with complementarity constraints

Summer School on Nonlinear Optimization with Mixed Integer and Complementarity Constraints – NOMICC26

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Outline of the lecture

1. Motivating examples
2. Geometric optimality conditions
3. Algebraic optimality conditions
4. Summary

Refresher on the Karush-Kuhn-Tucker (KKT) conditions

Nonlinear Program (NLP)

$$\begin{array}{ll}\min_{x \in \mathbb{R}^n} & f(x) \\ \text{s.t.} & g(x) = 0, \\ & h(x) \geq 0.\end{array}$$

$$\mathcal{L}(x, \lambda, \mu) = f(x) - \lambda^\top g(x) - \mu^\top h(x)$$

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Definition (LICQ)

A point x satisfies LICQ if

$$[\nabla g(x), \nabla h_{\mathcal{A}}(x)]$$

is full column rank.

Active set $\mathcal{A}(x) = \{i \mid h_i(x) = 0\}$

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Theorem (KKT conditions)

Let f, g, h be $\mathcal{C}^1$. If x^* is a (local) *minimizer* and *satisfies LICQ*, then there are *unique vectors* λ^* *and* μ^* *such that* (x^*, λ^*, μ^*) *satisfies*:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0, \quad \mu^* \geq 0,$$

$$g(x^*) = 0, \quad h(x^*) \geq 0$$

$$\mu_i^* h_i(x^*) = 0, \quad \forall i$$

dual feasibility

primal feasibility

complementary slackness

Constraint qualifications: LICQ and MFCQ

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A point x satisfies the Mangasarian–Fromovitz Constraint Qualification (MFCQ), if $\nabla g(x)$ has full column rank, and if there exist a direction $d \in \mathbb{R}^n$ such that:

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- ▶ Both require full rank of $\nabla g(x)$.
- ▶ Example MFCQ holds, LICQ does not at $x = 0$:

$$x_2 - x_1^2 \geq 0,$$

$$x_2 - x_1^4 \geq 0.$$

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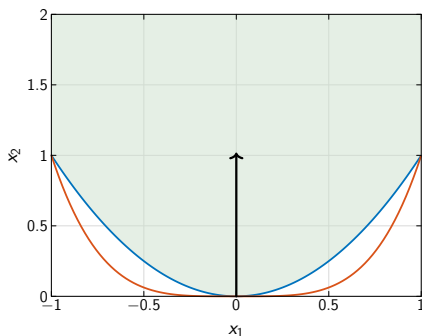
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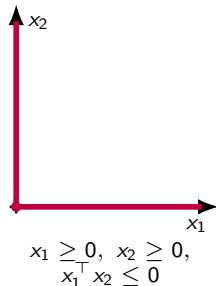
View the MPCC as a nonlinear program (NLP)

Replace the complementarity constraint by

$$0 \leq x_1 \perp x_2 \geq 0 \iff x_1 \geq 0, \quad x_2 \geq 0, \quad x_1^\top x_2 \leq 0.$$

Smooth NLP reformulation

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0, \\ & h(x) \geq 0, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & x_1^\top x_2 \leq 0. \end{aligned}$$



- **Advantage:** a smooth NLP that can be passed to your favorite NLP solver.
- **Problem:** the reformulation violates standard constraint qualifications.

Applying KKT conditions to MPCCs

[Scheel and Scholtes, 2000, Example 3].

Example linear MPCC

$$\begin{array}{ll} \min_{x \in \mathbb{R}^3} & x_1 + x_2 - x_3 \\ \text{s.t.} & h_1(x) = 4x_1 - x_3 \geq 0 \quad | \quad \mu_1, \\ & h_2(x) = 4x_2 - x_3 \geq 0 \quad | \quad \mu_2, \\ & x_1 \geq 0 \quad | \quad \mu_G, \\ & x_2 \geq 0 \quad | \quad \mu_H, \\ & x_1 x_2 \leq 0 \quad | \quad \mu_{GH} \end{array}$$

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- ▶ Global optimizer $x^* = (0, 0, 0)$.
- ▶ All constraints are active at the solution.
- ▶ Use the necessary KKT conditions to compute the optimal Lagrange multipliers.

Could anything possibly go wrong?

Example from [Scheel and Scholtes, 2000].

KKT conditions applied to a MPCC

$$0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \mu_1^* \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} - \mu_2^* \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} - \mu_G^* \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \mu_H^* \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mu_{GH}^* \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the non-negativity of the multipliers and this condition we obtain an inconsistent system:

$$\mu_G^* = 1 - 4\mu_1^*, \mu_G^*, \mu_1^* \geq 0 \implies \mu_1^* \in [0, 0.25],$$

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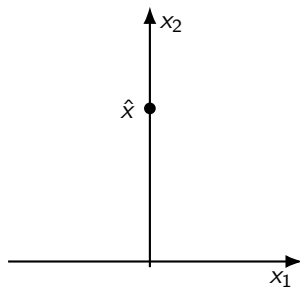
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Global optimum not a KKT point.

Reason: constraint qualifications violated.

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Consider a feasible complementarity pair $\hat{x} = (0, \hat{x}_2)$ with $\hat{x}_2 > 0$.

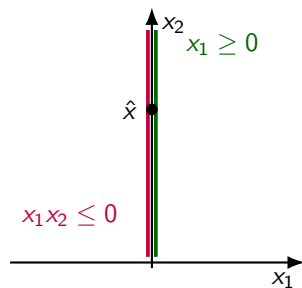


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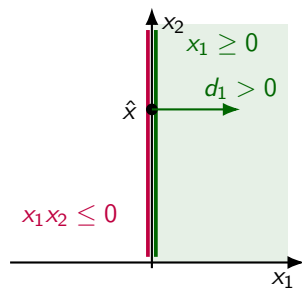
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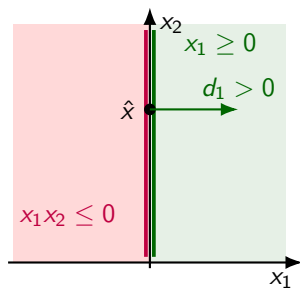
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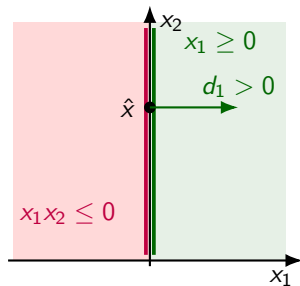
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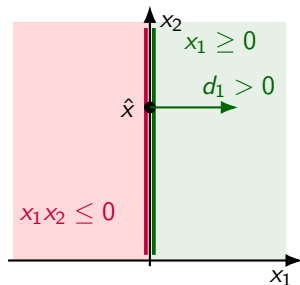
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MFCQ is a basic stability condition for NLPs. [Fletcher et al., 2006]

Its failure implies:

1. an unbounded (or empty set) of Lagrange multipliers, so $\nabla^2 \mathcal{L}$ may blow up,
2. linearly dependent active constraint gradients and ill-conditioned steps,
3. the absence of a central path, so interior-point methods may fail.

Dependent constraints and unbounded multiplier sets

Example from [Fletcher et al., 2006, Sec. 2.1.]

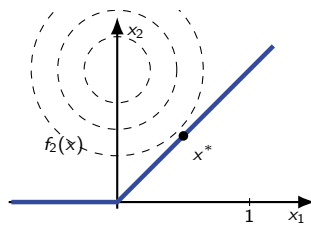
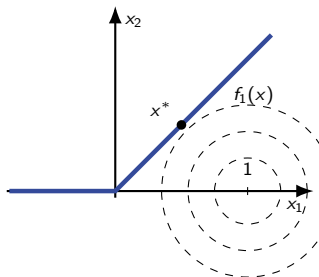
Consider the two quadratic MPCCs

Two MPCCs with the same feasible set

$$\min_{x \in \mathbb{R}^2} f_i(x) \quad \text{s.t.} \quad 0 \leq x_2 \perp x_2 - x_1 \geq 0,$$

$$f_1(x) = (x_1 - 1)^2 + x_2^2, \quad f_2(x) = x_1^2 + (x_2 - 1)^2.$$

Both problems have the solution $x^* = (1/2, 1/2)$.



Dependent constraints and unbounded multiplier sets

Example from [Fletcher et al., 2006, Sec. 2.1.]

Equivalent NLP reformulation and multipliers

$$\begin{array}{ll} \min_{x \in \mathbb{R}^2} & f_i(x) \\ \text{s.t.} & G(x) = x_2 \geq 0 \quad | \quad \mu_G \geq 0, \\ & H(x) = x_2 - x_1 \geq 0 \quad | \quad \mu_H \geq 0, \\ & G(x)H(x) = x_2(x_2 - x_1) \leq 0 \quad | \quad \mu_{GH} \geq 0 \end{array}$$

At $x^* = (1/2, 1/2)$, $G(x^*) > 0$ implies $\mu_G^* = 0$, and stationarity becomes

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mu_H^* \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \mu_{GH}^* \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

The active constraint gradients are linearly dependent.

Dependent constraints and unbounded multiplier sets

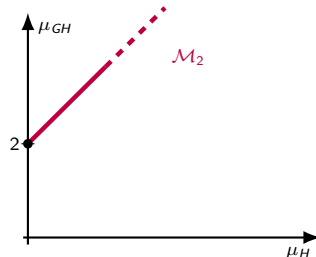
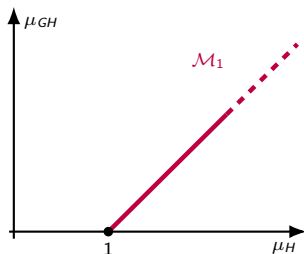
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$$\mathcal{M}_2 = \{(\mu_H, \mu_{GH}) \mid \mu_H \geq 0, -\mu_H + \frac{1}{2}\mu_{GH} = 1\}.$$

Both multiplier sets are unbounded.



Here MFCQ is also violated, but multipliers exist. Let us explore why.

Recall that if LICQ holds, the set of optimal multipliers is a singleton. The set of optimal multipliers is compact and convex if and only if MFCQ holds.

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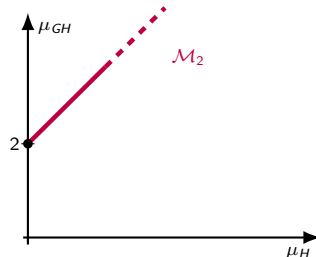
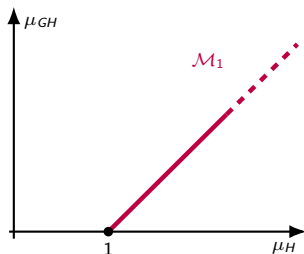
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Outline of MPCC optimality theory

Problem:

- ▶ LICQ and MFCQ violated at all feasible points of an NLP formulation of an MPCC
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Solution:

- ▶ Tailored first-order necessary optimality conditions for MPCCs, that hold under weaker, MPCC-specific constraint qualifications.
- ▶ We show primal (no multipliers) and dual (with Lagrange multipliers) optimality conditions for MPCCs, and when they are equal.

Tangent cone definition

Georges Louis Bouligand (1889 – 1979), a French mathematician, introduced this tangent cone definition in 1932.

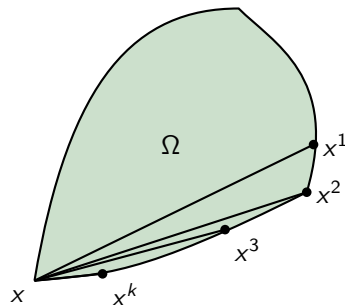
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$$\min_{x \in \mathbb{R}^n} f(x) \quad (1a)$$

$$\text{s.t.} \quad g(x) = 0, \quad (1b)$$

$$h(x) \geq 0 \quad (1c)$$

$$\Omega = \{x \in \mathbb{R}^n \mid g(x) = 0, h(x) \geq 0\}$$



Definition (Bouligand tangent cone)

The (Bouligand) tangent cone at $x \in \Omega$ is defined as the set:

$$\mathcal{T}_{\Omega}(x) = \{d \in \mathbb{R}^n \mid \exists \{x^k\} \subset \Omega, \{t^k\} \subset \mathbb{R}_{\geq 0} : \lim_{k \rightarrow \infty} t^k = 0, \lim_{k \rightarrow \infty} x^k = x, \lim_{k \rightarrow \infty} \frac{x^k - x}{t^k} = d\}$$

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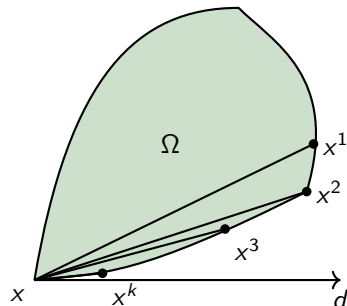
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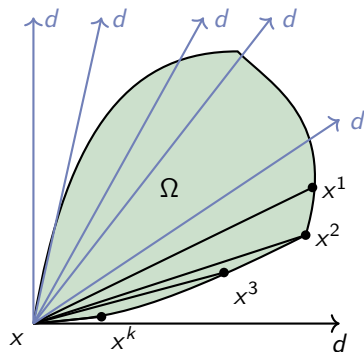
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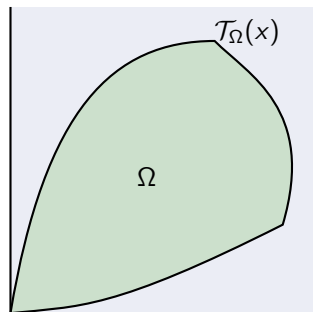
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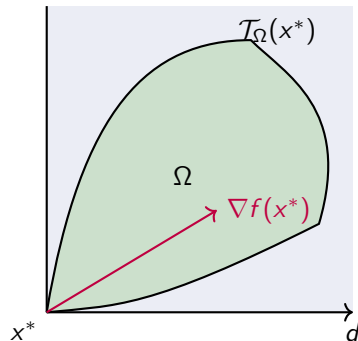
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First-order necessary conditions for optimality

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Theorem (First-Order Necessary Conditions)

If $x^* \in \Omega$ is a local minimizer of the NLP (1) then it holds that

$$\nabla f(x^*)^\top d \geq 0 \text{ for all tangents } d \in \mathcal{T}_\Omega(x^*). \quad (2)$$

The definition of B-stationarity

Definition (B-stationarity)

A point $x^* \in \Omega$ satisfying (2) is called a Bouligand stationary (B-stationary) point.

Definition (Linearized feasible cone)

Let $x \in \Omega$ and $\mathcal{A}(x) = \{i \mid h_i(x) = 0\}$. The linearized feasible cone is defined as the set:

$$\mathcal{T}_{\Omega}^{\text{lin}}(x) = \{d \in \mathbb{R}^n \mid \nabla g(x)^{\top} d = 0, \nabla h_i(x)^{\top} d \geq 0, i \in \mathcal{A}(x)\}$$

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$$\mathcal{T}_{\Omega}^{\text{lin}}(x) = \{d \in \mathbb{R}^n \mid \nabla g(x)^{\top} d = 0, \nabla h_i(x)^{\top} d \geq 0, i \in \mathcal{A}(x)\}$$

Standard nonlinear programming approach:

1. If a constraint qualification holds (e.g., MFCQ), $\mathcal{T}_{\Omega}(x) = \mathcal{T}_{\Omega}^{\text{lin}}(x)$.

The definition of B-stationarity

Definition (B-stationarity)

A point $x^* \in \Omega$ satisfying (2) is called a Bouligand stationary (B-stationary) point.

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Standard nonlinear programming approach:

1. If a constraint qualification holds (e.g., MFCQ), $\mathcal{T}_{\Omega}(x) = \mathcal{T}_{\Omega}^{\text{lin}}(x)$.
2. Use Farkas' lemma, obtain from the last Theorem the Karush–Kuhn–Tucker (KKT) conditions.

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Standard nonlinear programming approach:

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2. Use Farkas' lemma, obtain from the last Theorem the Karush–Kuhn–Tucker (KKT) conditions.
3. Solve a sequence of approximations of the KKT conditions to find a solution candidate (e.g., with IPOPT [Wächter and Biegler, 2006], SNOPT [Gill et al., 2005], MadNLP [Pacaud et al., 2024], uno [Vanaret and Leyffer, 2026], ...).

Cones and MPCCs

MPCC

$$\min_{x \in \mathbb{R}^n} f(x) \quad (3a)$$

$$\text{s.t.} \quad g(x) = 0, \quad (3b)$$

$$h(x) \geq 0, \quad (3c)$$

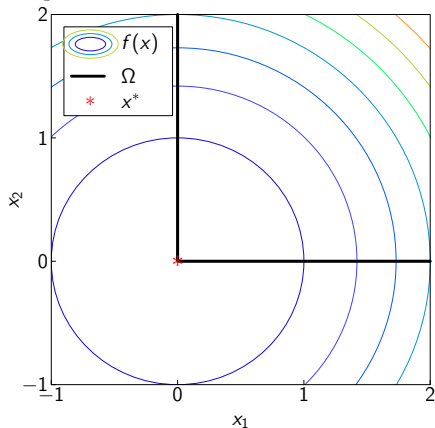
$$0 \leq x_1 \perp x_2 \geq 0. \quad (3d)$$

$$x = (x_0, x_1, x_2) \in \mathbb{R}^n, \quad x_0 \in \mathbb{R}^p, \quad x_1, x_2 \in \mathbb{R}^m,$$

$$\Omega = \{x \in \mathbb{R}^n \mid g(x) = 0, h(x) \geq 0, \\ 0 \leq x_1 \perp x_2 \leq 0\},$$

Running example

$$\min_{x \in \mathbb{R}^2} x_1^2 + x_2^2 \text{ s.t. } 0 \leq x_1 \perp x_2 \geq 0.$$



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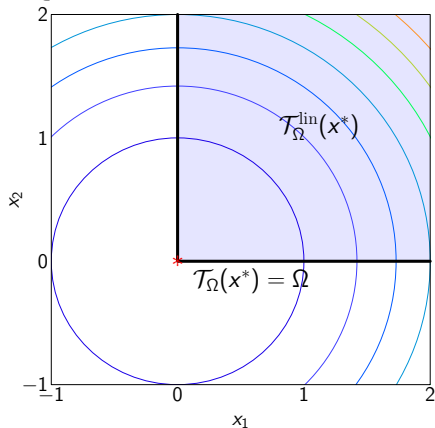
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In most interesting cases: $\mathcal{T}_\Omega(x) \neq \mathcal{T}_\Omega^{\text{lin}}(x)$.

Running example

$$\min_{x \in \mathbb{R}^2} x_1^2 + x_2^2 \text{ s.t. } 0 \leq x_1 \perp x_2 \geq 0.$$



MPCC active sets

Standard active set

$$\mathcal{A}(x) = \{i \in \{1, \dots, n_h\} \mid h_i(x) = 0\}.$$

MPCC active sets

$$\mathcal{I}_{+0}(x) = \{i \in \{1, \dots, m\} \mid x_{1,i} > 0, x_{2,i} = 0\},$$

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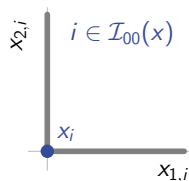
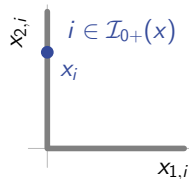
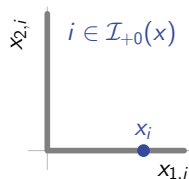
Partition of the degenerate set

$$\mathcal{D}_1(x) \cup \mathcal{D}_2(x) = \mathcal{I}_{00}(x),$$

$$\mathcal{D}_1(x) \cap \mathcal{D}_2(x) = \emptyset,$$

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Definition (MPCC lin. feasible cone)

Let $x \in \Omega$ and $d = (d_0, d_1, d_2) \in \mathbb{R}^{p+2m}$. The MPCC-linearized feasible cone is the set

$$\begin{aligned}\mathcal{T}_{\Omega}^{\text{MPCC}}(x) = \{d \in \mathbb{R}^n \mid & \nabla g(x)^{\top} d = 0, \\ & \nabla h_i(x)^{\top} d \geq 0, \forall i \in \mathcal{A}(x), \\ & d_{1,i} = 0, \forall i \in \mathcal{I}_{0+}(x), \\ & d_{2,i} = 0, \forall i \in \mathcal{I}_{+0}(x), \\ & 0 \leq d_{1,i} \perp d_{2,i} \geq 0, \forall i \in \mathcal{I}_{00}(x)\}.\end{aligned}$$

MPCC–tailored linearized feasible cone [Flegel and Kanzow, 2005, Pang and Fukushima, 1999]

Definition (MPCC lin. feasible cone)

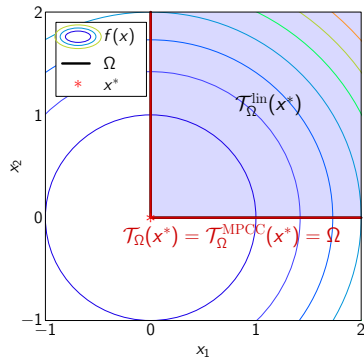
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Lemma (Lemma 3.2 in [Flegel and Kanzow, 2005])

Let $x \in \Omega$, then for the MPCC (3) it holds that:

$$\mathcal{T}_{\Omega}(x) \subseteq \mathcal{T}_{\Omega}^{\text{MPCC}}(x) \subseteq \mathcal{T}_{\Omega}^{\text{lin}}(x).$$



Primal stationarity conditions for MPECs

First order necessary optimality conditions – revisited

Theorem (Theorem 6 extended, [Luo et al., 1996])

Let $\mathcal{T}_\Omega(x^*) = \mathcal{T}_\Omega^{\text{MPCC}}(x^*)$, and $x^* \in \Omega$ be a minimizer of the MPCC (3), then it holds that

$$\nabla f(x^*)^\top d \geq 0 \text{ for all } d \in \mathcal{T}_\Omega^{\text{MPCC}}(x^*), \quad (4)$$

or equivalent to (4), $d = 0$ is a minimizer of the following optimization problem:

$$\min_{d \in \mathbb{R}^n} \nabla f(x^*)^\top d \quad \text{s.t.} \quad d \in \mathcal{T}_\Omega^{\text{MPCC}}(x^*). \quad (5)$$

► In interesting cases: $\mathcal{T}_\Omega(x) = \mathcal{T}_\Omega^{\text{MPCC}}(x) \neq \mathcal{T}_\Omega^{\text{lin}}(x)$.

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- ▶ In interesting cases: $\mathcal{T}_\Omega(x) = \mathcal{T}_\Omega^{\text{MPCC}}(x) \neq \mathcal{T}_\Omega^{\text{lin}}(x)$.
- ▶ Use the linear program with complementarity constraints (LPCC) (5) instead of the KKT conditions **to verify B-stationarity**.
- ▶ If $\mathcal{I}_{00}(x) \neq \emptyset$, LPCC is nonconvex, problem combinatorial in nature.
- ▶ If $\mathcal{I}_{00}(x) = \emptyset$, LPCC reduces to LP, at $d = 0$ LP KKT conditions = NLP KKT conditions.

Outline of the lecture

1. Motivating examples
2. Geometric optimality conditions
3. Algebraic optimality conditions
4. Summary

Pieces of the MPCC: the Tight NLP (TNLP)

Regular NLPs, used to define MPCC-specific concepts.

MPCC active sets:

$$\mathcal{I}_{+0}(x) = \{i \in \{1, \dots, m\} \mid x_{1,i} > 0, x_{2,i} = 0\},$$

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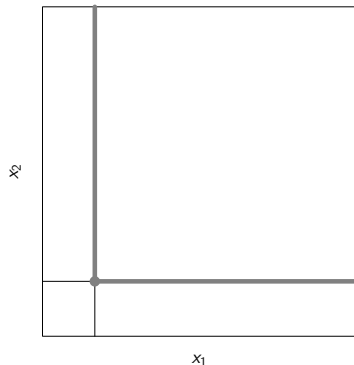
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Tight NLP (TNLP) at x^*

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0, \\ & h(x) \geq 0, \\ & x_{1,i} = 0, \quad x_{2,i} \geq 0, \quad i \in \mathcal{I}_{0+}(x^*), \\ & x_{1,i} \geq 0, \quad x_{2,i} = 0, \quad i \in \mathcal{I}_{+0}(x^*), \\ & x_{1,i} = 0, \quad x_{2,i} = 0, \quad i \in \mathcal{I}_{00}(x^*). \end{aligned}$$



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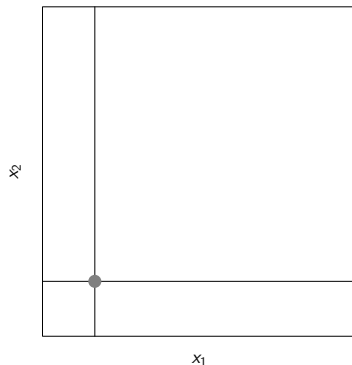
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Pieces of the MPCC: the Relaxed NLP (RNLP)

Regular NLPs, used to define MPCC-specific concepts.

MPCC active sets

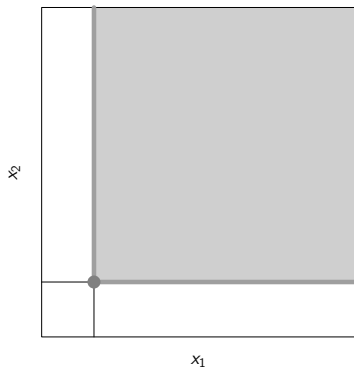
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Relaxed NLP (RNLP) at x^*

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) = 0, \\ & h(x) \geq 0, \\ & x_{1,i} = 0, \quad x_{2,i} \geq 0, \quad i \in \mathcal{I}_{0+}(x^*), \\ & x_{1,i} \geq 0, \quad x_{2,i} = 0, \quad i \in \mathcal{I}_{+0}(x^*), \\ & x_{1,i} \geq 0, \quad x_{2,i} \geq 0, \quad i \in \mathcal{I}_{00}(x^*). \end{aligned}$$



Pieces of the MPCC: the Branch NLP (BNLP)

Regular NLPs, used to define MPCC-specific concepts.

MPCC active sets

$$\mathcal{I}_{+0}(x) = \{i \in \{1, \dots, m\} \mid x_{1,i} > 0, x_{2,i} = 0\},$$

$$\mathcal{I}_{0+}(x) = \{i \in \{1, \dots, m\} \mid x_{1,i} = 0, x_{2,i} > 0\},$$

$$\mathcal{I}_{00}(x) = \{i \in \{1, \dots, m\} \mid x_{1,i} = 0, x_{2,i} = 0\}.$$

Partitioning of $\mathcal{I}_{00}(x)$

$$\mathcal{D}_1(x) \cup \mathcal{D}_2(x) = \mathcal{I}_{00}(x),$$

$$\mathcal{D}_1(x) \cap \mathcal{D}_2(x) = \emptyset,$$

$$\mathcal{I}_1(x) := \mathcal{I}_{0+}(x) \cup \mathcal{D}_1(x),$$

$$\mathcal{I}_2(x) := \mathcal{I}_{+0}(x) \cup \mathcal{D}_2(x),$$

Branch NLP (BNLP) at x^*

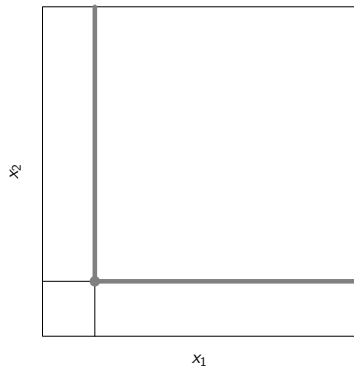
$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t.} \quad g(x) = 0,$$

$$h(x) \geq 0,$$

$$x_{1,i} = 0, \quad x_{2,i} \geq 0, \quad i \in \mathcal{I}_1(x^*),$$

$$x_{1,i} \geq 0, \quad x_{2,i} = 0, \quad i \in \mathcal{I}_2(x^*).$$



Pieces of the MPCC: the Branch NLP (BNLP)

Regular NLPs, used to define MPCC-specific concepts.

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Partitioning of $\mathcal{I}_{00}(x)$

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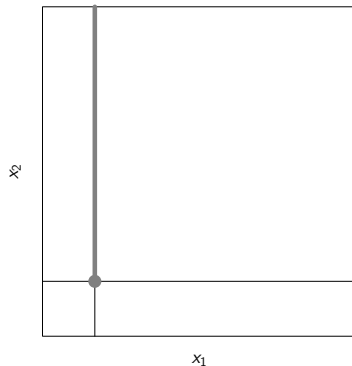
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Pieces of the MPCC: the Branch NLP (BNLP)

Regular NLPs, used to define MPCC-specific concepts.

MPCC active sets

$$\mathcal{I}_{+0}(x) = \{i \in \{1, \dots, m\} \mid x_{1,i} > 0, x_{2,i} = 0\},$$

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Branch NLP (BNLP) at x^*

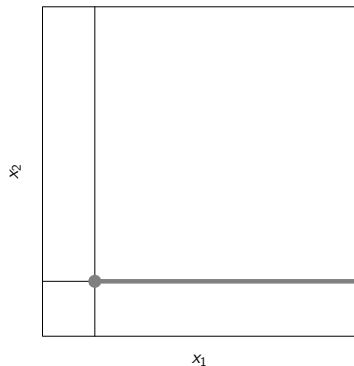
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MPCC-specific notions

- ▶ The tight NLP (TNLP), branch NLPs (BNLPs), and relaxed NLPs (RNLPs) are standard NLPs associated with an MPCC.
- ▶ Applying the KKT conditions to these auxiliary problems yields different stationarity notions for MPCCs.

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Definition (MPCC constraint qualifications)

The MPCC (3) satisfies *MPCC-LICQ* [*MPCC-MFCQ*] at a feasible point x^* if the corresponding tight NLP at x^* satisfies LICQ [MFCQ] at the same point.

Definition (MPCC Lagrangian)

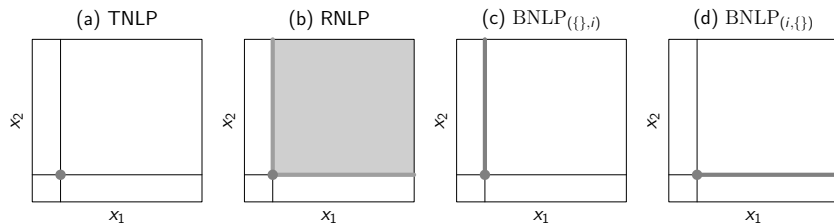
The MPCC Lagrangian is the standard Lagrangian associated with the tight NLP:

$$\mathcal{L}^{\text{MPCC}}(x, \lambda, \mu, \nu, \xi) := f(x) - \lambda^\top g(x) - \mu^\top h(x) - \nu^\top x_1 - \xi^\top x_2.$$

Relation between MPCC pieces

MPCC feasible set relations

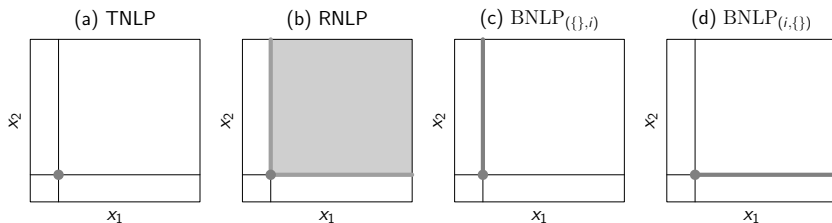
$$\Omega_{\text{TNLP}} = \bigcap_{(\mathcal{I}_1, \mathcal{I}_2)} \Omega_{\text{BNLP}(\mathcal{I}_1, \mathcal{I}_2)} \subset \Omega = \bigcup_{(\mathcal{I}_1, \mathcal{I}_2)} \Omega_{\text{BNLP}(\mathcal{I}_1, \mathcal{I}_2)}.$$



Relation between MPCC pieces

MPCC feasible set relations

$$\Omega_{\text{TNLP}} = \bigcap_{(\mathcal{I}_1, \mathcal{I}_2)} \Omega_{\text{BNLP}(\mathcal{I}_1, \mathcal{I}_2)} \subset \Omega = \bigcup_{(\mathcal{I}_1, \mathcal{I}_2)} \Omega_{\text{BNLP}(\mathcal{I}_1, \mathcal{I}_2)}.$$

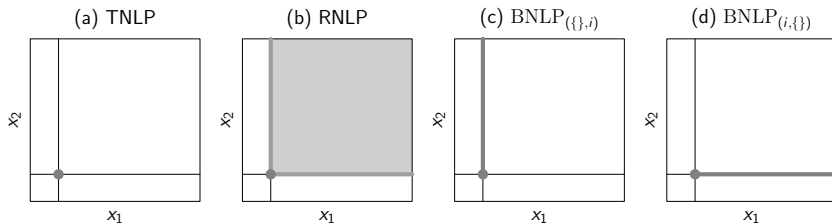


On the solutions:

- ▶ If x^* is a local minimizer of the RNLP, then it is a local minimizer of the MPCC. The converse is not true.
- ▶ If x^* is a local minimizer of the MPCC then it is a local minimizer of the TNLP.
- ▶ The point x^* is a local minimizer of the MPCC if and only if it is a local minimizer of every BNLP.

More facts on MPCC auxiliary NLPs

- ▶ If we know the right TNLP/BNLP, we can just solve a regular NLP to solve the MPCC.
- ▶ There are $2^{|\mathcal{I}_{00}|}$ BNLPs – highlighting the combinatorial nature.
- ▶ MPCC-specific second-order conditions are likewise formulated using these auxiliary NLPs.



Multiplier-based stationarity conditions for MPCCs

Definition (Stationarity conditions for MPCCs)

- ▶ Weak Stationarity (W-stationarity): A point $x^* \in \Omega$ is called W-stationary if the corresponding **Tight NLP** admits the satisfaction of the KKT conditions, i.e., there exist Lagrange multipliers λ^*, μ^*, ν^* and ξ^* such that:

$$\nabla_x f(x^*) - \nabla_x g(x^*)\lambda^* - \nabla_x h(x^*)\mu^* - (\nabla_x x_1)\nu^* - (\nabla_x x_2)\xi^* = 0,$$

$$g(x^*) = 0,$$

$$0 \leq \mu^* \perp h(x^*) \geq 0,$$

$$x_{1,i}^* \geq 0, \nu_i^* = 0, \text{ for all } i \in \mathcal{I}_{+0}(x^*),$$

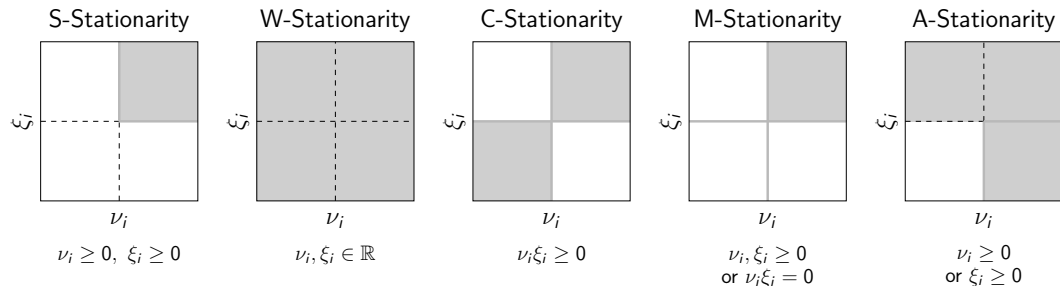
$$x_{2,i}^* \geq 0, \xi_i^* = 0, \text{ for all } i \in \mathcal{I}_{0+}(x^*),$$

$$x_{1,i}^* = 0, \nu_i^* \in \mathbb{R}, \text{ for all } i \in \mathcal{I}_{0+}(x^*) \cup \mathcal{I}_{00}(x^*),$$

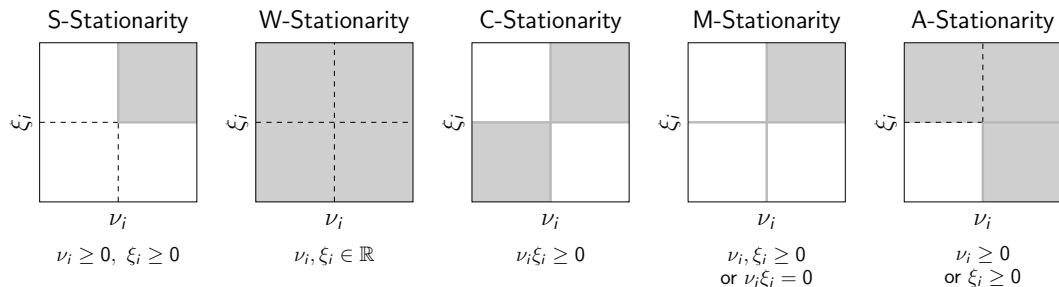
$$x_{2,i}^* = 0, \xi_i^* \in \mathbb{R}, \text{ for all } i \in \mathcal{I}_{+0}(x^*) \cup \mathcal{I}_{00}(x^*).$$

- ▶ Strong Stationarity (S-stationarity): A point $x^* \in \Omega$ is called S-stationary if it is weakly stationary and $\nu_i^* \geq 0, \xi_i^* \geq 0$ for all $i \in \mathcal{I}_{00}(x^*)$.

Relation between primal and dual stationarity for MPCCs



Relation between primal and dual stationarity for MPCCs



Theorem (Theorem 4 in [Scheel and Scholtes, 2000])

If x^ is a S-stationary point of the MPCC (3), then it is also B-stationarity. If in addition the MPCC-LICQ holds at x^* , then the reverse implication is also true.*

- ▶ MPCC-LICQ cannot be relaxed to MPCC-MFCQ. Next weaker concept, M stationary points can be B, but they do not have to.
- ▶ Our example from the beginning satisfied MPCC-MFCQ and was M-stationary.

Exercise: one MPCC, four outcomes

At $x^* = (0, 0)$, consider

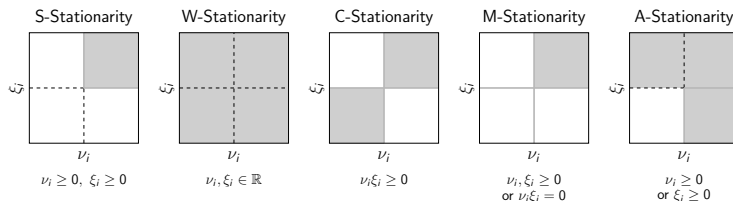
$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & ax_1 + bx_2 \\ \text{s.t.} \quad & 0 \leq x_1 \perp x_2 \geq 0. \end{aligned}$$

For $(a, b) = (1, 1), (-1, 0), (-1, -1), (-1, 1)$:

1. Is x^* a minimizer? If not, identify a feasible descent direction.
2. Which of W-, A-, C-, M-, S-, and B-stationarity hold?
3. Do MPCC-LICQ and MPCC-MFCQ hold at x^* ?

Bonus: For $(a, b) = (0, 1)$, is the minimizer unique?

Hint: Use the MPCC-Lagrangian: $\mathcal{L}^{\text{MPCC}}(x, \nu, \xi) := f(x) - \nu^\top x_1 - \xi^\top x_2$.



Solution: One MPCC, four outcomes

$$\mathcal{L}^{\text{MPCC}}(x, \nu, \xi) = ax_1 + bx_2 - \nu x_1 - \xi x_2,$$

$$\nabla_x \mathcal{L}^{\text{MPCC}}(x^*, \nu, \xi) = \begin{pmatrix} a - \nu \\ b - \xi \end{pmatrix} = 0 \implies \nu = a, \quad \xi = b.$$

At $x^* = 0$, $\mathcal{T}_\Omega(0) = \mathbb{R}_{\geq 0}e_1 \cup \mathbb{R}_{\geq 0}e_2$, $(\nu, \xi) = (a, b)$. Thus x^* is optimal (and S- and B-stationary) exactly when $a, b \geq 0$.

(a, b)	Optimal?	Descent direction	Stationarity at x^*
$(1, 1)$	Yes	None	W, A, C, M, S, B
$(-1, 0)$	No	e_1	W, A, C, M
$(-1, -1)$	No	e_1, e_2	W, C
$(-1, 1)$	No	e_1	W, A

Constraint qualifications: The tight NLP fixes $x_1 = x_2 = 0$; its equality gradients e_1, e_2 are independent. Hence MPCC-LICQ and MPCC-MFCQ hold for every (a, b) .

Bonus: For $(0, 1)$, every $(t, 0)$ with $t \geq 0$ is optimal. In general, the minimizer is unique exactly when $a > 0$ and $b > 0$.

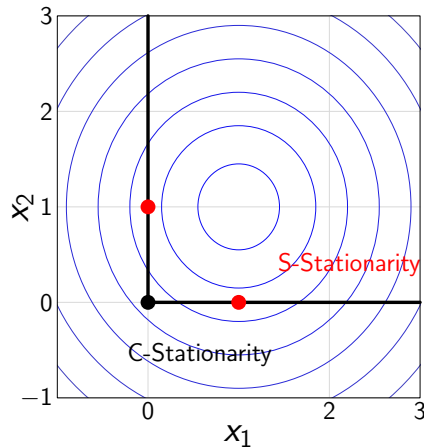
C-stationarity allows first-order descent directions

Example from [Leyffer and Munson, 2007]

Consider the two-dimensional MPCC:

$$\begin{array}{ll}\min_{x \in \mathbb{R}^2} & (x_1 - 1)^2 + (x_2 - 1)^2 \\ \text{s.t.} & 0 \leq x_1 \perp x_2 \geq 0.\end{array}$$

- The origin $x^* = 0$ is an C-stationary point with the optimal multipliers $\nu = -2$, $\xi = -2$.



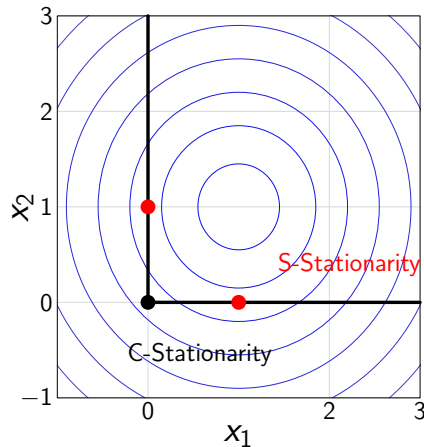
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- ▶ The origin $x^* = 0$ is an C-stationary point with the optimal multipliers $\nu = -2$, $\xi = -2$.
- ▶ There exists two descent direction $d = (1, 0)$ and $d = (0, 1)$.
- ▶ The origin is not B-stationary, $x = (1, 0)$ and $x = (0, 1)$ are S-stationary.



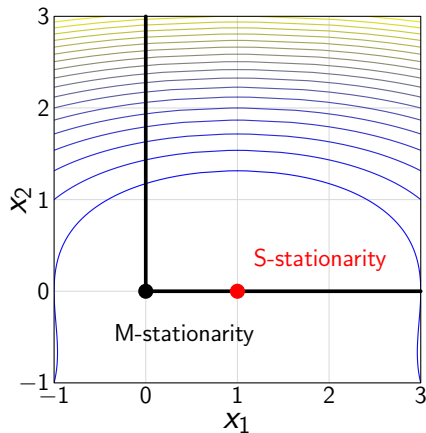
M-stationarity allows first-order descent directions

Example from [Leyffer and Munson, 2007]

Consider the two-dimensional MPCC:

$$\begin{array}{ll}\min_{x \in \mathbb{R}^2} & (x_1 - 1)^2 + x_2^2 + x_2^3 \\ \text{s.t.} & 0 \leq x_1 \perp x_2 \geq 0.\end{array}$$

- ▶ The origin $x^* = 0$ is an M-stationary point with the optimal multipliers $\nu = -2$, $\xi = 0$.



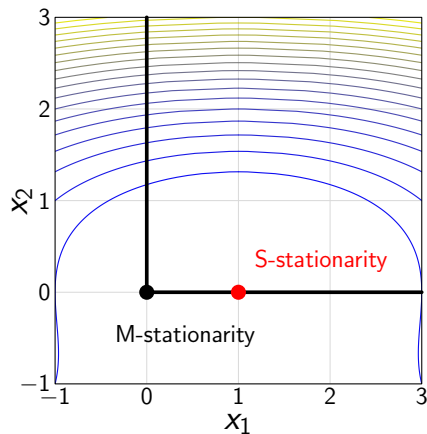
M-stationarity allows first-order descent directions

Example from [Leyffer and Munson, 2007]

Consider the two-dimensional MPCC:

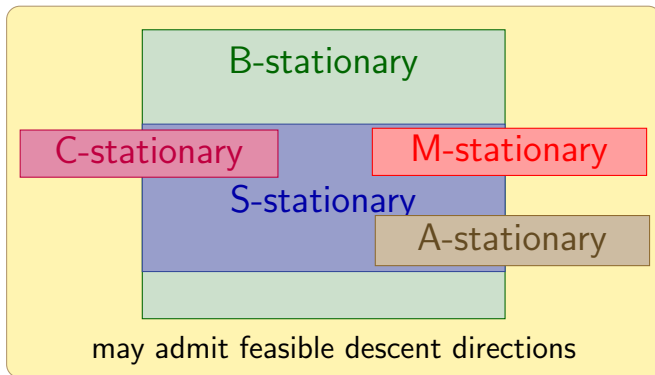
$$\begin{array}{ll}\min_{x \in \mathbb{R}^2} & (x_1 - 1)^2 + x_2^2 + x_2^3 \\ \text{s.t.} & 0 \leq x_1 \perp x_2 \geq 0.\end{array}$$

- ▶ The origin $x^* = 0$ is an M-stationary point with the optimal multipliers $\nu = -2$, $\xi = 0$.
- ▶ There exists a descent direction $d = (1, 0)$ with $\nabla f(x^*)^\top d = -2 < 0$.
- ▶ The origin is not B-stationary.



Alphabet soup of stationarity

Image adapted from [Leyffer and Munson, 2007]



A-, B-, C-, M-, and S-stationarity coincide if $\mathcal{I}_{00}(x^*) = \emptyset$.

Outline of the lecture

1. Motivating examples
2. Geometric optimality conditions
3. Algebraic optimality conditions
4. Summary

Summary

- ▶ NLP reformulations of MPCCs violate standard constraint qualifications at all feasible points.
- ▶ KKT conditions are not the right tool to identify solution candidates and build algorithm upon.
- ▶ Tailored MPCC theory (and algorithms) exploit the piecewise structure.
- ▶ Only S-stationarity can verify B-stationarity, all others may allow descent directions!

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