Time-freezing for optimal control of systems with state jumps

Moritz Diehl

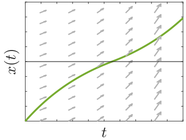
Systems Control and Optimization Laboratory
Department of Microsystems Engineering and Department of Mathematics
University of Freiburg, Germany

based on joint work with
Armin Nurkanović, Sebastian Albrecht, Bernard Brogliato

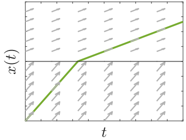
9th Annual Symposium, Toulouse
LAAS – CNRS, Toulouse, France
20th – 22nd September, 2022

Nonsmooth Dynamics (NSD) - a classification

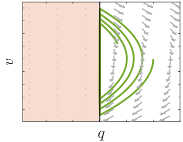
Regard ordinary differential equation (ODE) with a **nonsmooth** right-hand side (RHS).
Distinguish three cases:



NSD1: non-differentiable RHS, e.g., $\dot{x} = 1 + |x|$

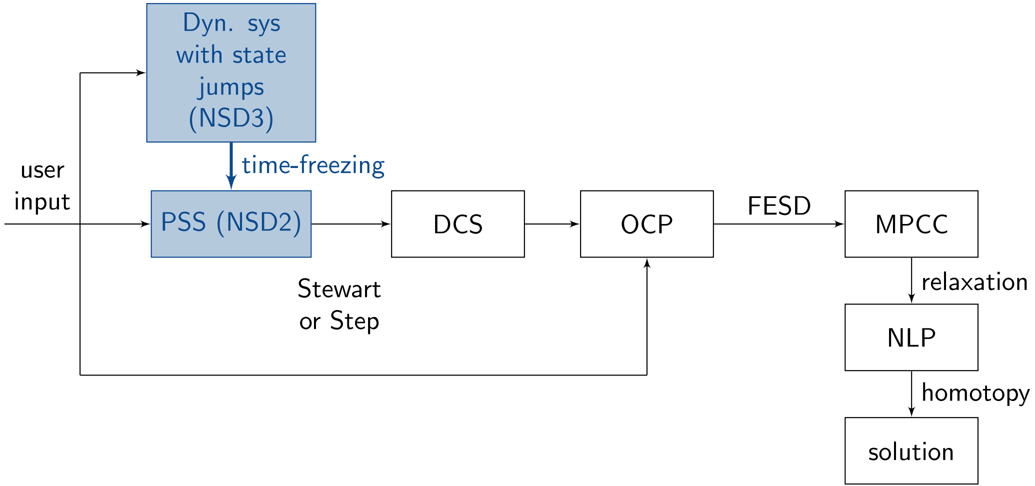


NSD2: state dependent switch of RHS, e.g., $\dot{x} = 2 - \mathrm{sign}(x)$



NSD3: **state dependent jump**, e.g., bouncing ball, $v(t_+) = -0.9\, v(t_-)$

Aim of time-freezing: transform NSD3 to NSD2 (and then use the rest of the toolchain)



PSS - piecewise smooth systems; DCS - dynamic complementarity system; OCP - optimal control problem; FESD - finite elements with switch detection; MPCC - mathematical program with complementarity constraints ; NLP - nonlinear program

Overview

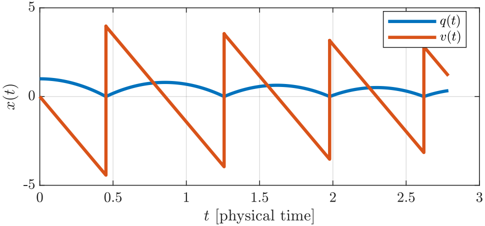
- The time-freezing reformulation
- Elastic impacts
- Inelastic impacts
- Hybrid systems with hysteresis
- Conclusions and outlook

NSD3 state jump example: bouncing ball

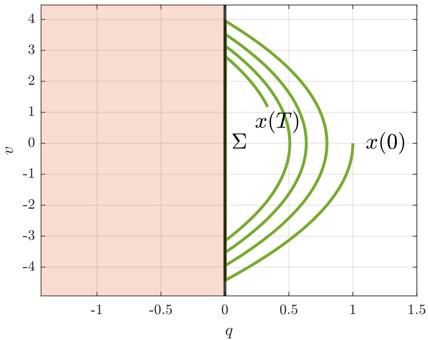
Bouncing ball with state $x = (q, v)$:

$$\dot{q} = v,\ m\dot{v} = -mg, \qquad \text{if } q > 0$$
$$v(t^+) = -0.9\, v(t^-), \qquad \text{if } q(t^-) = 0 \text{ and } v(t^-) < 0$$

Time plot of bouncing ball trajectory:



Phase plot of bouncing ball trajectory:



Question: could we transform NSD3 systems into (easier) NSD2 systems?

Three ideas:

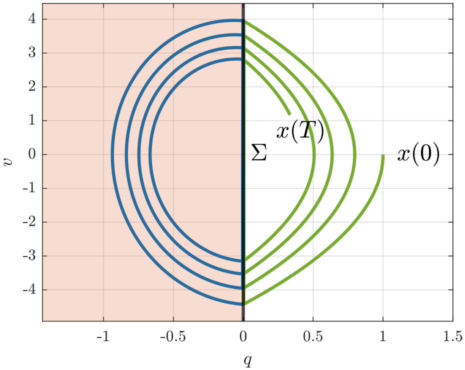
1. mimic state jump by **auxiliary dynamic system** $\dot{x} = f_{\mathrm{aux}}(x)$ on prohibited region
2. introduce a **clock state** $t(\tau)$ that stops counting when the auxiliary system is active
3. adapt speed of time, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = s$ with $s \geq 1$, and **impose terminal constraint** $t(T) = T$

The time-freezing reformulation

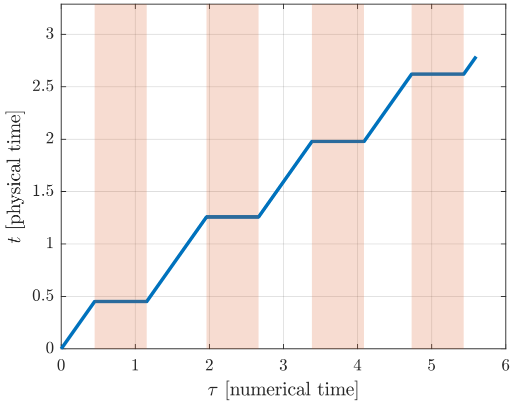
Augmented state $(x, t) \in \mathbb{R}^{n+1}$ evolves in **numerical time** τ. Augmented system is nonsmooth, of NSD2 type:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} x \\ t \end{bmatrix} = \begin{cases} s \begin{bmatrix} f(x) \\ 1 \end{bmatrix}, & \text{if } c(x) \geq 0 \\ \begin{bmatrix} s f_{\mathrm{aux}}(x) \\ 0 \end{bmatrix}, & \text{if } c(x) < 0 \end{cases}$$

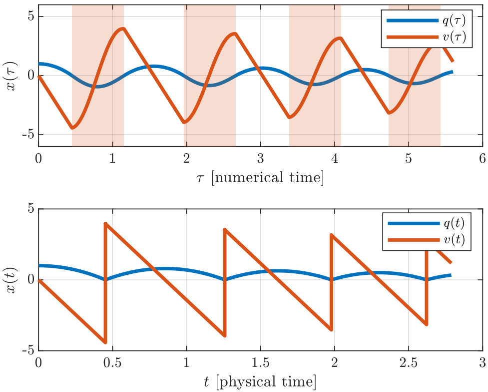
- During normal times, system and clock state evolve with adapted speed $s \geq 1$.
- Auxiliary system $\frac{\mathrm{d}x}{\mathrm{d}\tau} = f_{\mathrm{aux}}(x)$ mimics state jump while time is frozen, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = 0$.



Time-freezing for bouncing ball example



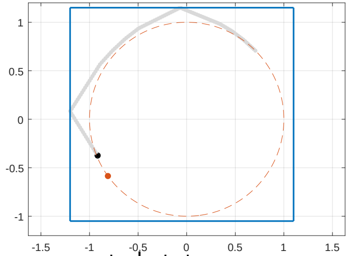
Evolution of physical time (clock state) during augmented system simulation ($s = 1$).



We can recover the true solution by plotting $x(\tau)$ vs. $t(\tau)$ and disregarding "frozen pieces".

A tracking OCP example with Time-Freezing and FESD in NOSNOC

Regard bouncing ball in two dimensions driven by bounded force: $\boxed{\ddot{q} = u}$



- augmented state $x = (q, \dot{q}, t) \in \mathbb{R}^5$
- $n_f = 9$ regions (8 with auxiliary dynamics for state jumps)

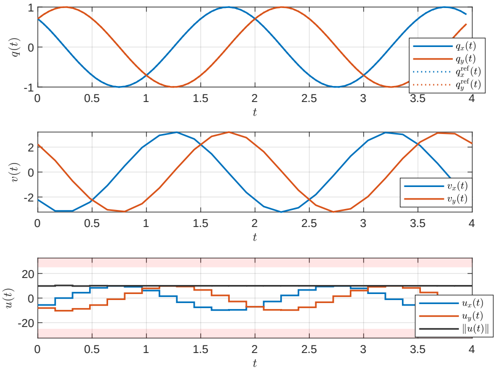
$$
\begin{aligned}
\min_{\substack{x(.),u(.),s(.),\\ \theta(.),\lambda(.),\mu(.)}} \quad & \int_0^T (q - q_{\mathrm{ref}}(\tau))^\top (q - q_{\mathrm{ref}}(\tau))\, s(\tau)\,\mathrm{d}\tau \\
\text{s.t.} \quad & x(0) = x_0, \quad t(T) = T, \\
& x'(\tau) = \sum_{i=1}^{n_f} \theta_i(\tau) f_i(x(\tau), u(\tau), s(\tau)), \\
& 0 = g(x(\tau)) - \lambda(\tau) - \mu(\tau) e, \\
& 0 \leq \lambda(\tau) \perp \theta(\tau) \geq 0, \\
& 1 = e^\top \theta(\tau), \\
& \|u(\tau)\|_2^2 \leq u_{\max}^2, \\
& 1 \leq s(\tau) \leq s_{\max}, \ \tau \in [0, T].
\end{aligned}
$$

$$q_{\mathrm{ref}}(\tau) = (R\cos(\omega\, t(\tau)), R\sin(\omega\, t(\tau))).$$

Results with slowly moving reference

For $\omega = \pi$, tracking is easy: no jumps occur in optimal solution.

- Regard time horizon of two periods
- $N = 25$ equidistant control intervals
- use FESD with $N_{\mathrm{FE}} = 3$ finite elements with Radau 3 on each control interval
- each FESD interval has one constant control u and one speed of time s
- MPCC solved via ℓ_∞ penalty reformulation and homotopy
- For homotopy convergence: in total 4 NLPs solved with `IPOPT` via `CasADi`



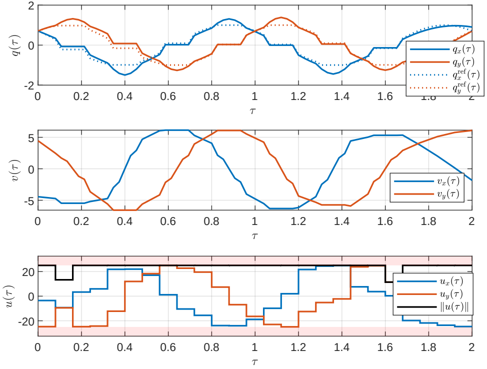
States and controls in physical time.

Results with slowly moving reference - movie

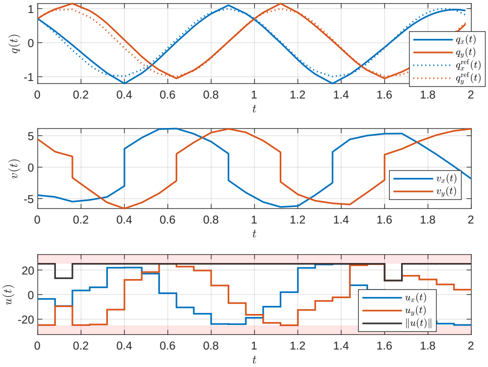
For $\omega = \pi$, tracking is easy: no jumps occur in optimal solution.

Results with fast reference

For $\omega = 2\pi$, tracking is only possible if ball bounces against walls.



States and controls in numerical time.



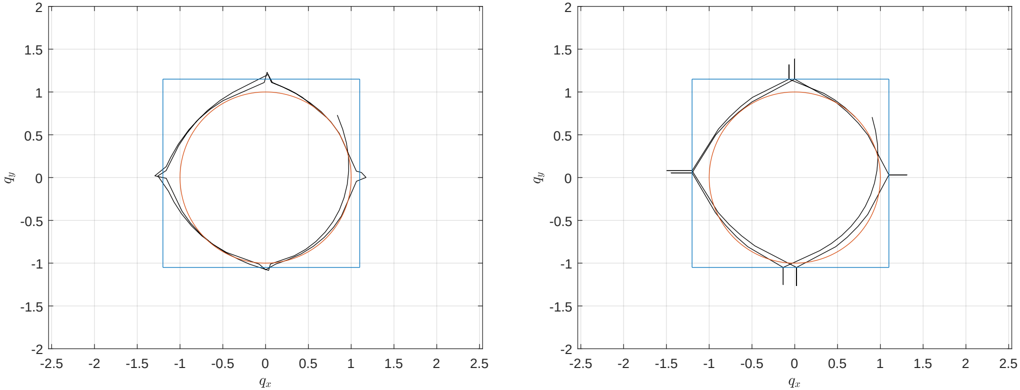
States and controls in physical time.

Results with fast reference - movie

For $\omega = 2\pi$, tracking is only possible if ball bounces against walls.

Homotopy: first iteration vs converged solution

Geometric trajectory

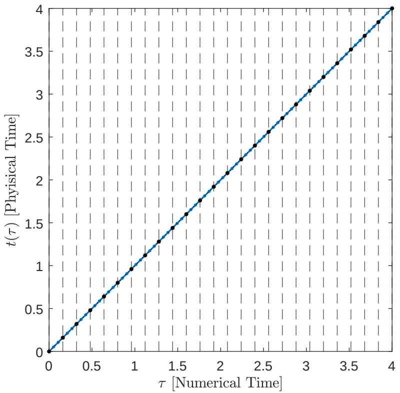


After the first homotopy iteration

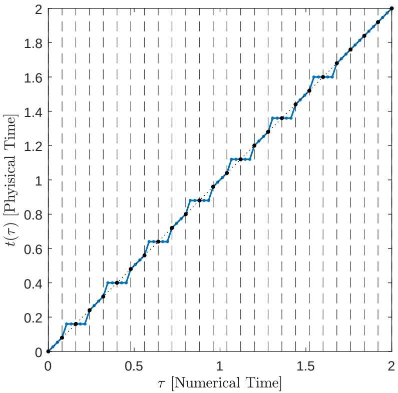
The solution trajectory after convergence

Physical vs. Numerical Time

for $\omega = \pi$



for $\omega = 2\pi$



Overview

- The time-freezing reformulation
- Elastic impacts
- **Inelastic impacts**
- Hybrid systems with hysteresis
- Conclusions and outlook

Complementarity Lagrangian systems with impacts and friction

Complementarity Lagrangian Systems (CLS)

$$
\begin{aligned}
\dot{q} &= v, \\
M(q)\dot{v} &= \tilde{f}_{\mathrm{v}}(q, v, u) + \nabla_q f_c(q)\lambda_{\mathrm{n}} + B(q)\lambda_{\mathrm{t}}, \\
0 &\leq \lambda_{\mathrm{n}} \perp f_c(q) \geq 0, \\
0 &= n(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^+), \ \text{if } f_c(q(t_{\mathrm{s}})) = 0 \text{ and } n(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^-) < 0, \\
\lambda_{\mathrm{t}} &\in \arg \min_{\tilde{\lambda}_{\mathrm{t}} \in \mathbb{R}^{n_{\mathrm{t}}}} \quad -v^\top B(q)\tilde{\lambda}_{\mathrm{t}} \\
& \qquad \text{s.t.} \quad \|\tilde{\lambda}_{\mathrm{t}}\|_2 \leq \mu\lambda_{\mathrm{n}}.
\end{aligned}
$$

- we regard a single unilateral constraint $f_c(q) \geq 0$
- $n(q) := \nabla_q f_c(q)$ is the normal vector of the contact manifold $\{q \in \mathbb{R}^{n_q} \mid f_c(q) = 0\}$
- $B(q) \in \mathbb{R}^{n_q \times n_t}$, $n_{\mathrm{t}} = n_q - 1$ spans the tangent plane
- For a moment let us ignore tangential friction (red terms)

Complementarity Lagrangian systems with impacts and friction

Complementarity Lagrangian Systems (CLS)

$$
\begin{aligned}
\dot{q} &= v, \\
M(q)\dot{v} &= \tilde{f}_{\mathrm{v}}(q,v,u) + \nabla_q f_c(q)\lambda_{\mathrm{n}} \\
0 &\leq \lambda_{\mathrm{n}} \perp f_c(q) \geq 0, \\
0 &= n(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^+), \text{ if } f_c(q(t_{\mathrm{s}})) = 0 \text{ and } n(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^-) < 0,
\end{aligned}
$$

- we regard a single unilateral constraint $f_c(q) \geq 0$
- $n(q) := \nabla_q f_c(q)$ is the normal vector of the contact manifold $\{q \in \mathbb{R}^{n_q} \mid f_c(q) = 0\}$
-
- For a moment let us ignore tangential friction

Notation and basic definitions

CLS modes and contact LCP

unconstrained ODE mode

$$\begin{aligned}
\dot{q} &= v, \\
\dot{v} &= \underbrace{M(q)^{-1}\tilde{f}_{\mathrm{v}}(q,v,u)}_{=:f_{\mathrm{v}}(q,v,u)},
\end{aligned}$$

contact mode - DAE of index 3

$$\begin{aligned}
\dot{q} &= v \\
\dot{v} &= M(q)^{-1}\left(\tilde{f}_{\mathrm{v}}(q,v,u) + \nabla_q f_c(q)\lambda_{\mathrm{n}}\right), \\
0 &= f_c(q).
\end{aligned}$$

Notation and basic definitions

CLS modes and contact LCP

unconstrained ODE mode

$$\dot{q} = v,$$
$$\dot{v} = \underbrace{M(q)^{-1}\tilde{f}_{\mathrm{v}}(q,v,u)}_{=: f_{\mathrm{v}}(q,v,u)},$$

contact mode - DAE of index 3

$$\dot{q} = v$$
$$\dot{v} = M(q)^{-1}\left(\tilde{f}_{\mathrm{v}}(q,v,u) + \nabla_q f_c(q)\lambda_{\mathrm{n}}\right),$$
$$0 = f_c(q).$$

The *contact LCP* tells us if the system will stay in contact mode or switch to the ODE mode:

$$0 \leq \frac{\mathrm{d}^2}{\mathrm{d}t^2} f_c(q(t)) \perp \lambda_{\mathrm{n}}(t) \geq 0 \iff$$
$$0 \leq D(q)\lambda_{\mathrm{n}} + \varphi(x) \perp \lambda_{\mathrm{n}} \geq 0, \text{ solution: } \quad \lambda_{\mathrm{n}} = \max(0, -D(q)^{-1}\varphi(x))$$

Notation and basic definitions

CLS modes and contact LCP

unconstrained ODE mode

$$\dot{q} = v,$$
$$\dot{v} = \underbrace{M(q)^{-1}\tilde{f}_{\mathrm{v}}(q,v,u)}_{=:f_{\mathrm{v}}(q,v,u)},$$

contact mode - DAE of index 3

$$\dot{q} = v$$
$$\dot{v} = M(q)^{-1}\left(\tilde{f}_{\mathrm{v}}(q,v,u) + \nabla_q f_c(q)\lambda_{\mathrm{n}}\right),$$
$$0 = f_c(q).$$

The *contact LCP* tells us if the system will stay in contact mode or switch to the ODE mode:

$$0 \leq \frac{\mathrm{d}^2}{\mathrm{d}t^2} f_c(q(t)) \perp \lambda_{\mathrm{n}}(t) \geq 0 \iff$$
$$0 \leq D(q)\lambda_{\mathrm{n}} + \varphi(x) \perp \lambda_{\mathrm{n}} \geq 0, \text{ solution: } \quad \lambda_{\mathrm{n}} = \max(0, -D(q)^{-1}\varphi(x))$$

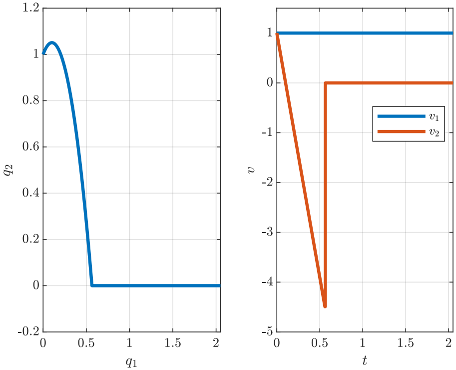
where $D(q)$ is the Delassus' matrix (scalar in this case) and

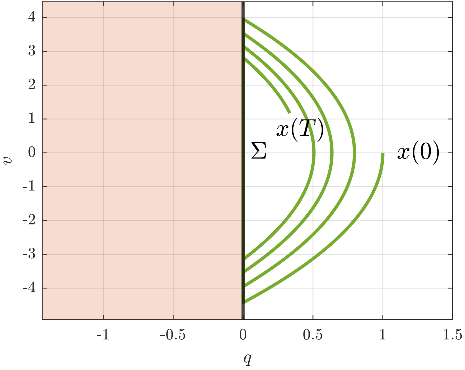
$$D(q) \coloneqq \nabla_q f_c(q)^\top M(q)^{-1} \nabla_q f_c(q) \succ 0, \quad \varphi(x) \coloneqq \nabla_q f_c(q)^\top f_{\mathrm{v}}(q,v,u) + \nabla_q(\nabla_q f_c(q)^\top v)^\top v.$$

2D frictionless particle with an inelastic impact

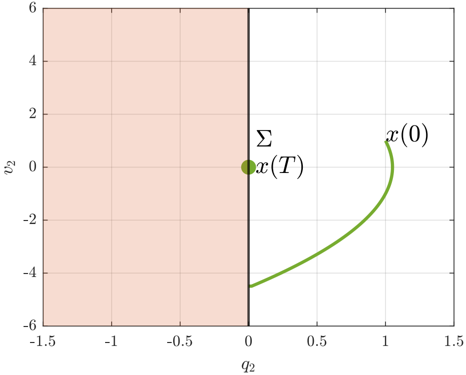
$$
\begin{aligned}
\dot{q} &= v,\\
m\dot{v} &= \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \lambda_{\mathrm{n}} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\\
0 &\leq \lambda_{\mathrm{n}} \perp q_2 \geq 0,\\
v_2(t_{\mathrm{s}}^{+}) &= 0, \text{ if } q_2(t_{\mathrm{s}}) = 0 \text{ and } v_2(t_{\mathrm{s}}^{-}) < 0.
\end{aligned}
$$

Trajectory with $u(t) = 0$:





elastic impact



inelastic impact

Time-freezing for inelastic impacts

Back to the more general setting

- State space in numerical time τ: $y = (q, v, t) \in \mathbb{R}^{n_y}$, $n_y = n_x + 1$ and $x = (q, v)$

Switching functions

$$c_1(y) := f_c(q)$$

$$c_2(y) := \nabla_q f_c(q)^\top \quad \left(= \frac{\mathrm{d} f_c}{\mathrm{d}t}(q) \right)$$

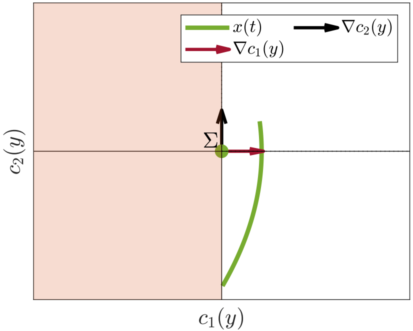
Regions

$$R_1^a = \{y \in \mathbb{R}^{n_y} \mid c_1(y) > 0\}$$

$$R_1^b = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) > 0\}$$

$$R_1 = R_1^a \cup {R_2}^b$$

$$R_2 = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$



- R_1 - unconstrained dynamics
- R_2 - auxiliary dynamics
- **After impact**: $c_1(y) = c_2(y) = 0$
- sliding mode on $\Sigma = \{y \mid c_1(y) = 0, c_2(y) = 0\}$

Time-freezing for inelastic impacts

Back to the more general setting

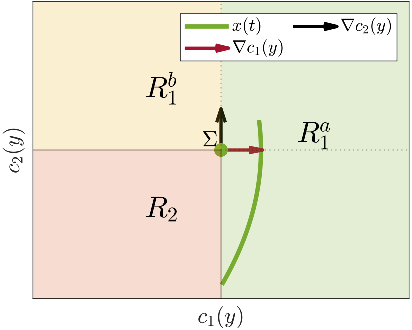
- State space in numerical time τ: $y = (q, v, t) \in \mathbb{R}^{n_y}$, $n_y = n_x + 1$ and $x = (q, v)$

Switching functions

$$c_1(y) := f_c(q)$$
$$c_2(y) := \nabla_q f_c(q)^\top \quad \left(= \frac{\mathrm{d} f_c}{\mathrm{d}t}(q) \right)$$

Regions

$$R_1^a = \{ y \in \mathbb{R}^{n_y} \mid c_1(y) > 0 \}$$
$$R_1^b = \{ y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) > 0 \}$$
$$R_1 = R_1^a \cup {R_2}^b$$
$$R_2 = \{ y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0 \}$$



- R_1 - unconstrained dynamics
- R_2 - auxiliary dynamics
- **After impact**: $c_1(y) = c_2(y) = 0$
- sliding mode on $\Sigma = \{ y \mid c_1(y) = 0, c_2(y) = 0 \}$

Unconstrained and auxiliary dynamics

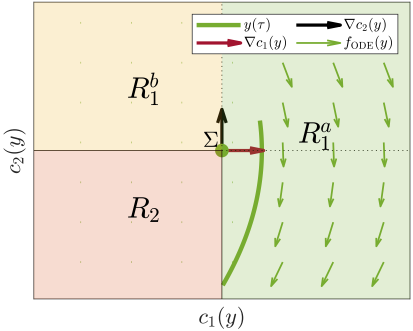
Unconstrained free-flight ODE in R_1

$$y' = f_{\mathrm{ODE}}(y,u) := \begin{bmatrix} v \\ f_{\mathrm{v}}(q,v,u) \\ 1 \end{bmatrix}$$

Auxiliary ODE in R_2

$$y'(\tau) = f_{\mathrm{aux,n}}(y) := \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1} n(q) a_{\mathrm{n}} \\ 0 \end{bmatrix}$$

with $a_{\mathrm{n}} > 0$.



- $f_{\mathrm{ODE}}(y,u)$ stops $y(\tau)$ on Σ
- dynamics on Σ is $y' \in \overline{\mathrm{conv}}\{f_{\mathrm{ODE}}(y) f_{\mathrm{aux,n}}(y)\}$

Unconstrained and auxiliary dynamics

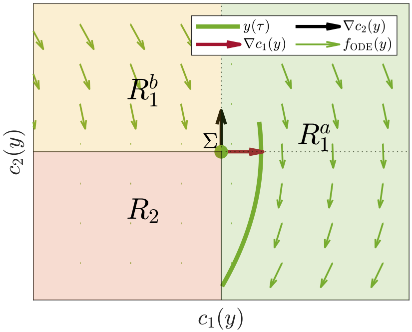
Unconstrained free-flight ODE in R_1

$$y' = f_{\mathrm{ODE}}(y,u) := \begin{bmatrix} v \\ f_{\mathrm{v}}(q,v,u) \\ 1 \end{bmatrix}$$

Auxiliary ODE in R_2

$$y'(\tau) = f_{\mathrm{aux,n}}(y) := \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1}n(q)a_{\mathrm{n}} \\ 0 \end{bmatrix}$$

with $a_{\mathrm{n}} > 0$.



- $f_{\mathrm{ODE}}(y,u)$ stops $y(\tau)$ on Σ
- dynamics on Σ is $y' \in \overline{\mathrm{conv}}\{f_{\mathrm{ODE}}(y) f_{\mathrm{aux,n}}(y)\}$

Unconstrained and auxiliary dynamics

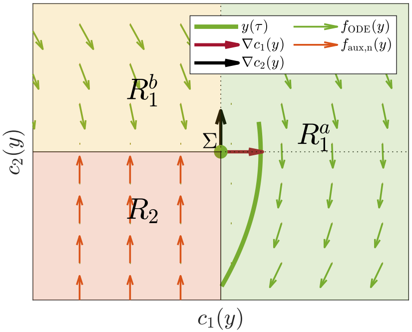
Unconstrained free-flight ODE in R_1

$$y' = f_{\mathrm{ODE}}(y,u) := \begin{bmatrix} v \\ f_{\mathrm{v}}(q,v,u) \\ 1 \end{bmatrix}$$

Auxiliary ODE in R_2

$$y'(\tau) = f_{\mathrm{aux,n}}(y) := \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1}n(q)a_{\mathrm{n}} \\ 0 \end{bmatrix}$$

with $a_{\mathrm{n}} > 0$.

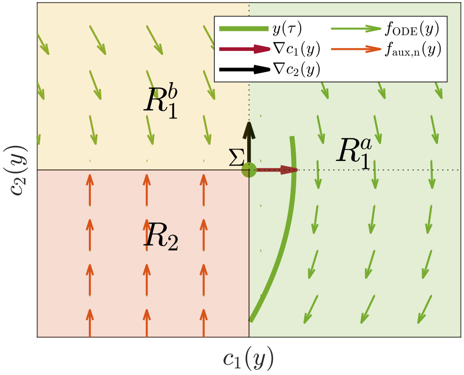


- $f_{\mathrm{ODE}}(y,u)$ stops $y(\tau)$ on Σ
- dynamics on Σ is $y' \in \overline{\mathrm{conv}}\{f_{\mathrm{ODE}}(y) f_{\mathrm{aux,n}}(y)\}$

Contact breaking

The contact LCP function $\varphi(x)$ tells us about the vector field in R_1

- $\varphi(x)$ determines stability of Σ (remember the contact LCP)
- staying in sliding mode (persistent contact) or leaving sliding mode (contact breaking) is possible

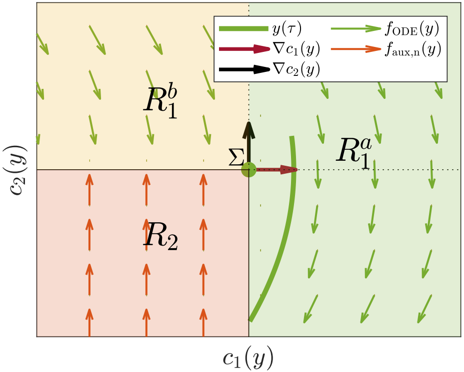


Sliding mode if $\varphi(x) \leq 0$

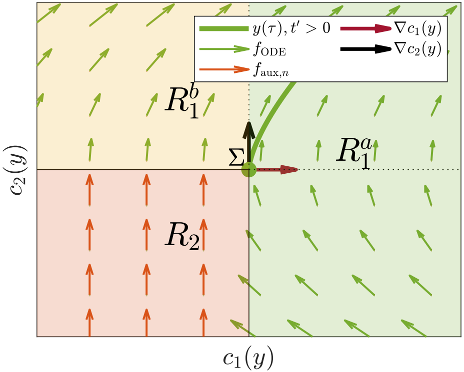
Contact breaking

The contact LCP function $\varphi(x)$ tells us about the vector field in R_1

- $\varphi(x)$ determines stability of Σ (remember the contact LCP)
- staying in sliding mode (persistent contact) or leaving sliding mode (contact breaking) is possible



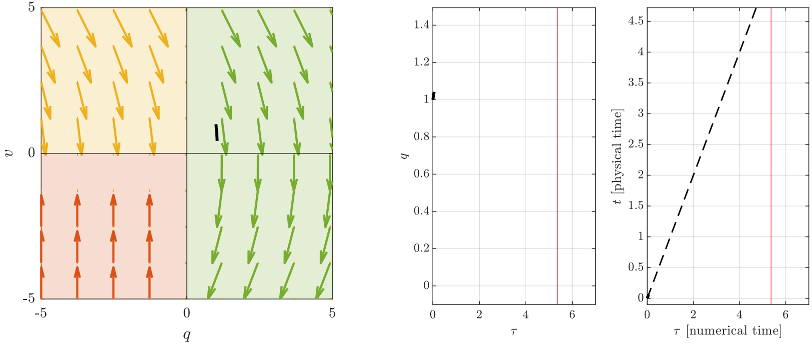
Sliding mode if $\varphi(x) \leq 0$



Breaking contact if $\varphi(x) > 0$

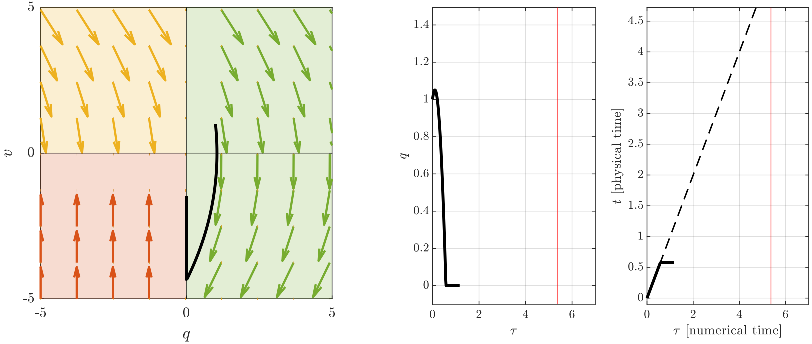
Animation of leaving of sliding mode

Warm up example: a linearly increasing vertical force beats gravity



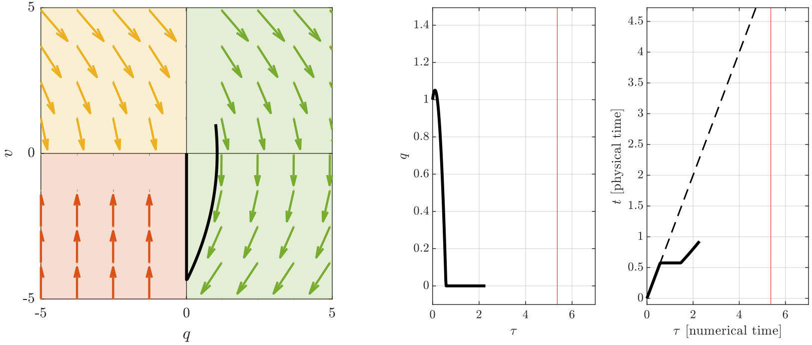
Animation of leaving of sliding mode

Warm up example: a linearly increasing vertical force beats gravity



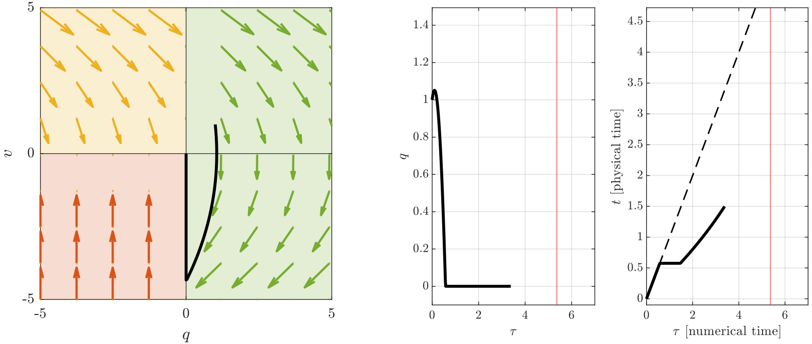
Animation of leaving of sliding mode

Warm up example: a linearly increasing vertical force beats gravity



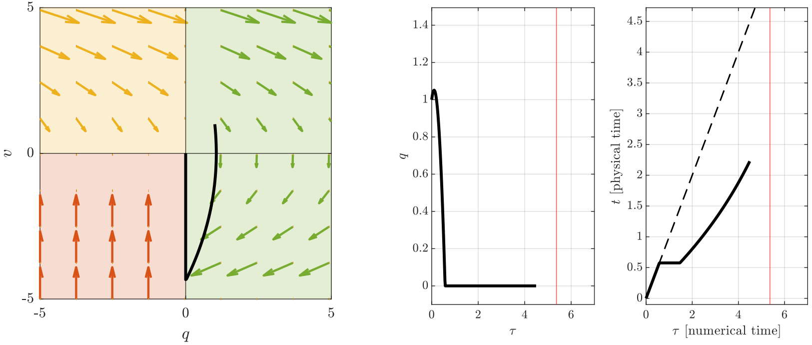
Animation of leaving of sliding mode

Warm up example: a linearly increasing vertical force beats gravity



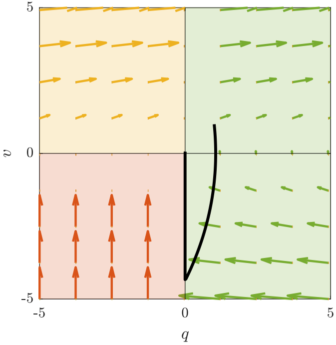
Animation of leaving of sliding mode

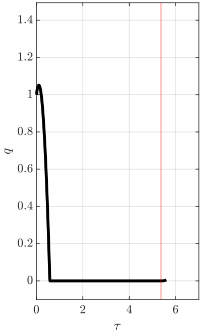
Warm up example: a linearly increasing vertical force beats gravity

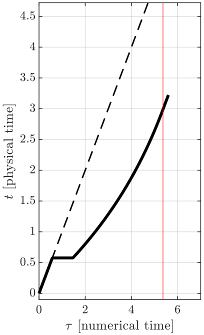


Animation of leaving of sliding mode

Warm up example: a linearly increasing vertical force beats gravity

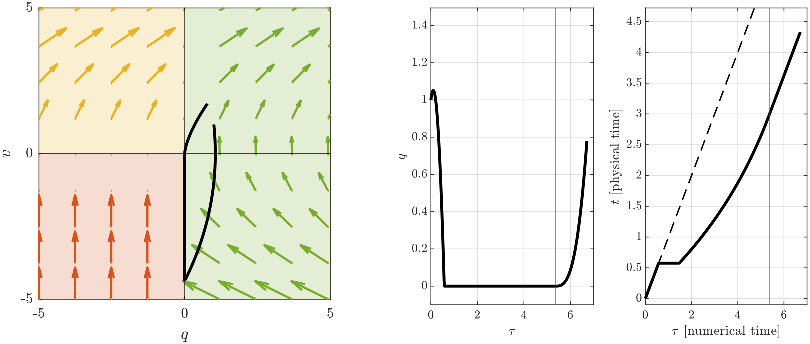






Animation of leaving of sliding mode

Warm up example: a linearly increasing vertical force beats gravity

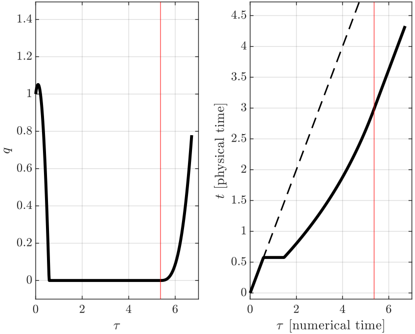


Why is the time slowed-down?

Time-freezing system

$$y' \in F_{\mathrm{TF}}(y,u) = \{\theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_{\mathrm{aux},n}(y) \mid \theta^\top e = 1,\ \theta \geq 0\}$$

- fractional $\theta_1, \theta_2 \in (0,1)$ ensures sliding on Σ
- speed of time $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \theta_1 \cdot 1 + \theta_2 \cdot 0 < 1$ - slow down
- resulting dynamics equal to *reduced* DAE index 3 dynamics $f_{\mathrm{DAE}}(x,u)$ (contact mode)
- auxiliary dynamics plays role of contact force (keeps $v = 0$ and avoids penetration)



The sliding mode is unique

Time-freezing system

$$y' \in F_{\mathrm{TF}}(y,u) = \{\theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_{\mathrm{aux},n}(y) \mid \theta^\top e = 1,\ \theta \geq 0\} \tag{1}$$

Theorem

Let $y(\tau)$ be a solution of the dyn. system (1) with $y(0) \in \Sigma = \{y \in \mathbb{R}^{n_y} \mid c_1(y) = 0, c_2(y) = 0\}$ and $\tau \in [0, \tau_{\mathrm{f}}]$. Suppose that $\varphi(x(\tau), u(\tau)) \leq 0$ for all $\tau \in [0, \tau_{\mathrm{f}}]$ (persistent contact), then the following statements are true:

(i) *the convex multipliers $\theta_1, \theta_2 \geq 0$ are unique,*

(ii) *the dynamics of the sliding mode are given by $y' = \gamma(x,u) \begin{bmatrix} f_{\mathrm{DAE}}(x,u) \\ 1 \end{bmatrix}$, where*

$\gamma(x,u) \in (0,1]$ is a time-rescaling factor given by

$$\gamma(x,u) := \frac{D(q)a_{\mathrm{n}}}{D(q)a_{\mathrm{n}} - \varphi(x,u)}. \tag{2}$$

Time-freezing with friction

Complementarity Lagrangian systems with impact and friction

Complementarity Lagrangian Systems (CLS)

$$
\begin{aligned}
\dot{q} &= v, \\
M(q)\dot{v} &= \tilde{f}_{\mathrm{v}}(q,v,u) + \nabla_q f_c(q)\lambda_{\mathrm{n}} + B(q)\lambda_{\mathrm{t}}, \\
0 &\leq \lambda_{\mathrm{n}} \perp f_c(q) \geq 0, \\
0 &= n(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^+), \text{ if } f_c(q(t_{\mathrm{s}})) = 0 \text{ and } n(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^-) < 0, \\
\lambda_{\mathrm{t}} &\in \arg \min_{\tilde{\lambda}_{\mathrm{t}} \in \mathbb{R}^{n_{\mathrm{t}}}} \quad -v^\top B(q)\tilde{\lambda}_{\mathrm{t}} \\
&\quad \text{s.t.} \quad \|\tilde{\lambda}_{\mathrm{t}}\|_2 \leq \mu\lambda_{\mathrm{n}}.
\end{aligned}
$$

- we regard $f_c(x) \in \mathbb{R}$ (single unilateral constraint)
- $B(q) \in \mathbb{R}^{n_q \times n_{\mathrm{t}}}$ spans the tangent plane at contact points $C(q) := \{q \in \mathbb{R}^{n_q} \mid f_c(q) = 0\}$, $n_{\mathrm{t}} \in \{1,2\}$, tang. velocity $v_{\mathrm{t}} = B(q)v$
- We derive time-freezing for the red terms

Coulomb's friction

Solution map for a given λ_n

Coulomb's friction law

$$\lambda_t \in \arg \min_{\tilde{\lambda}_t \in \mathbb{R}^{n_t}} \quad -v_t^\top \tilde{\lambda}_t$$
$$\text{s.t.} \quad \|\tilde{\lambda}_t\|_2 \leq \mu \lambda_n.$$

Friction solution map

$$\lambda_t \in \begin{cases} \{-\mu\lambda_n \frac{v_t}{\|v_t\|_2}\}, & \text{if } \|v_t\|_2 > 0, \\ \{\tilde{\lambda}_t \mid \|\tilde{\lambda}_t\|_2 \leq \mu\lambda_n\}, & \text{if } \|v_t\|_2 = 0. \end{cases}$$

Coulomb's friction

Solution map for a given λ_n

Coulomb's friction law

$$\lambda_\mathrm{t} \in \arg \min_{\tilde{\lambda}_\mathrm{t} \in \mathbb{R}^{n_\mathrm{t}}} \quad -v_\mathrm{t}^\top \tilde{\lambda}_\mathrm{t}$$

$$\text{s.t.} \quad \|\tilde{\lambda}_\mathrm{t}\|_2 \leq \mu \lambda_\mathrm{n}.$$

Friction solution map

$$\lambda_\mathrm{t} \in \begin{cases} \{-\mu \lambda_\mathrm{n} \frac{v_\mathrm{t}}{\|v_\mathrm{t}\|_2}\}, & \text{if } \|v_\mathrm{t}\|_2 > 0, \\ \{\tilde{\lambda}_\mathrm{t} \mid \|\tilde{\lambda}_\mathrm{t}\|_2 \leq \mu \lambda_\mathrm{n}\}, & \text{if } \|v_\mathrm{t}\|_2 = 0. \end{cases}$$

- reduces to $\lambda_\mathrm{t} \in -\lambda_\mathrm{n} \mathrm{sign}(v_\mathrm{t})$ in planar case
- the normal impulse is $a_\mathrm{n} \tau_\mathrm{jump} \implies$ the tangential impulse should be $-\mu a_\mathrm{n} \tau_\mathrm{jump} \mathrm{sign}(v_\mathrm{t})$
- trivially, tangential impulse happens at same time as the normal impulse
- **Conclusion**: make aux. dyn. in tangential directions $B(q)$ "proportional" to $f_\mathrm{aux,n}$ and let them evolve simultaneously

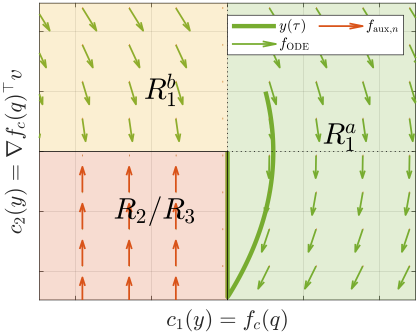
Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_{t}

New additional switching function $c_3(y) = v_{\mathrm{t}}$

Regions

$$Q = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$
$$R_1 = R_1^a \cup R_1^b$$
$$R_2 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) > 0\}$$
$$R_3 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) < 0\}$$



Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\} \tag{3}$$

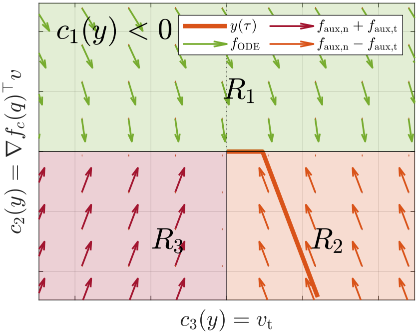
Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_{t}

New additional switching function $c_3(y) = v_{\mathrm{t}}$

Regions

$$Q = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$
$$R_1 = R_1^a \cup R_1^b$$
$$R_2 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) > 0\}$$
$$R_3 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) < 0\}$$



Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\} \qquad (3)$$

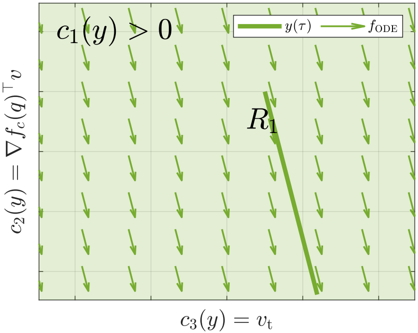
Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_t

New additional switching function $c_3(y) = v_\mathrm{t}$

Regions

$$Q = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$
$$R_1 = R_1^a \cup R_1^b$$
$$R_2 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) > 0\}$$
$$R_3 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) < 0\}$$



Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\} \tag{3}$$

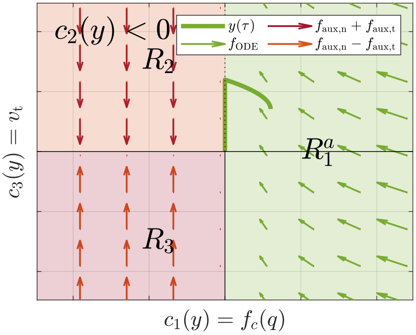
Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_{t}

New additional switching function $c_3(y) = v_{\mathrm{t}}$

Regions

$$
\begin{aligned}
Q &= \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\} \\
R_1 &= R_1^a \cup R_1^b \\
R_2 &= Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) > 0\} \\
R_3 &= Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) < 0\}
\end{aligned}
$$



Time-freezing system with friction

$$
y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\} \tag{3}
$$

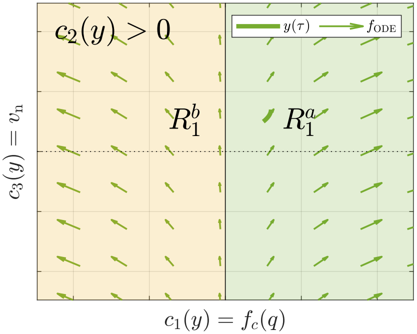
Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_{t}

New additional switching function $c_3(y) = v_{\mathrm{t}}$

Regions

$$Q = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$
$$R_1 = R_1^a \cup R_1^b$$
$$R_2 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) > 0\}$$
$$R_3 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) < 0\}$$



Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^{\top}\theta = 1 \right\} \tag{3}$$

Time-freezing with friction in the planar case

PSS modes

$$
\begin{aligned}
f_1(y,u) &= (f_{\mathrm{ODE}}(x,u),1)\\
f_2(y) &= f_{\mathrm{aux,n}}(y) - f_{\mathrm{aux,t}}(y)\\
f_3(y) &= f_{\mathrm{aux,n}}(y) + f_{\mathrm{aux,t}}(y)
\end{aligned}
$$

Auxiliary ODE for tangential directions

$$
f_{\mathrm{aux,t}}(y) := \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1} B(q)\mu\, a_{\mathrm{n}} \\ 0 \end{bmatrix}
$$

$$
f_{\mathrm{aux,n}}(y) := \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1} n(q) a_{\mathrm{n}} \\ 0 \end{bmatrix}
$$

- Simply sum the auxiliary dynamics in normal and tangential directions (recall that $B(q) \in \mathbb{R}^{n_q \times 1}$ and $n(q) \perp B(q)$)
- State jump is over when $n(q)^\top v = 0$, friction to slow down
- With $v_{\mathrm{t}} = 0$ sliding mode on $\Gamma = \{y \mid c_1(y) = 0, c_2(y) = 0, c_3(y) = 0\}$

Time-freezing with friction - sliding mode dynamics

- $\dot{x} = f_{\mathrm{Slip}}(x, u)$ reduced DAE in slip mode, $v_{\mathrm{t}} \neq 0$
- $\dot{x} = f_{\mathrm{Stick}}(x, u)$ reduced DAE in stick mode, $v_{\mathrm{t}} = 0$

Theorem (Slip-stick sliding mode)

Let $y(\tau)$ be a solution of time freezing system (3) *with $y(0) \in \Sigma$ and $\tau \in [0, \tau_{\mathrm{f}}]$. Suppose that $\varphi(x(\tau), u(\tau)) \leq 0$ for all $\tau \in [0, \tau_{\mathrm{f}}]$ (persistent contact), then the following statements are true:*

(i) *If $v_{\mathrm{t}} \neq 0$ (slip motion), then the sliding mode dynamics are given by*

$$y' = \gamma(x, u) \begin{bmatrix} f_{\mathrm{Slip}}(x, u) \\ 1 \end{bmatrix}$$

(ii) *If $v_{\mathrm{t}} = 0$ (stick motion), then the sliding mode dynamics are given by*

$$y' = \gamma(x, u) \begin{bmatrix} f_{\mathrm{Stick}}(x, u) \\ 1 \end{bmatrix}$$

where $\gamma(x, u) \in (0, 1]$ is a time-rescaling factor defined in Eq. (2).

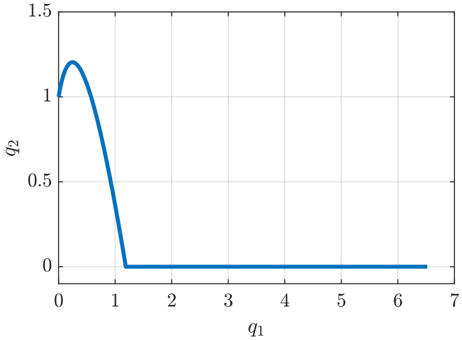
Simulation example - slip/stick

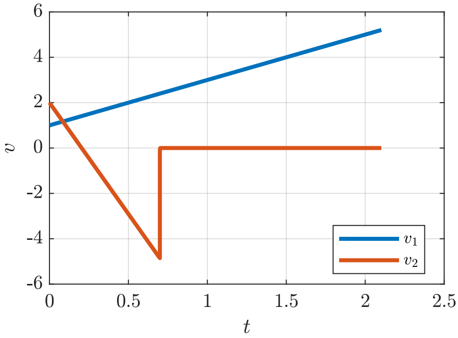
Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0$

No friction





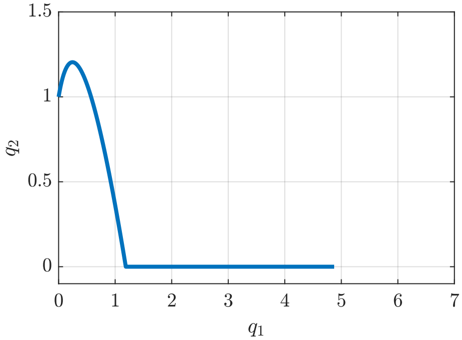
Simulation example - slip/stick

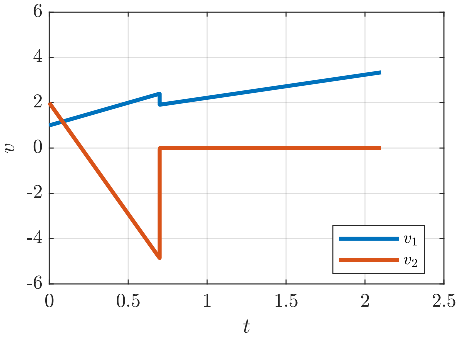
Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.1$

External force stronger than friction





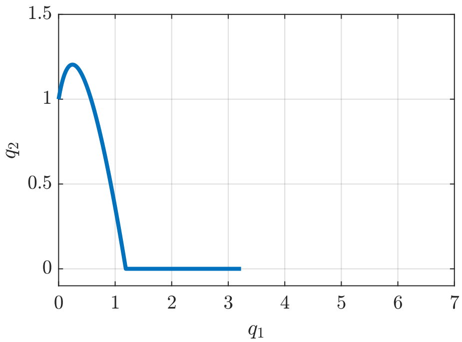
Simulation example - slip/stick

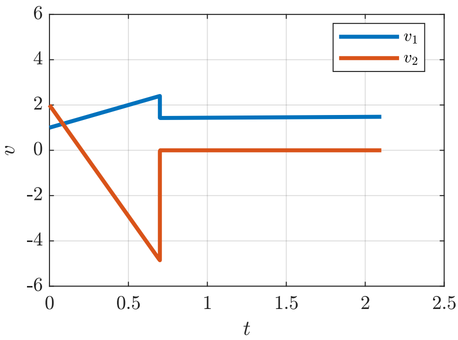
Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.2$

External force equal to friction





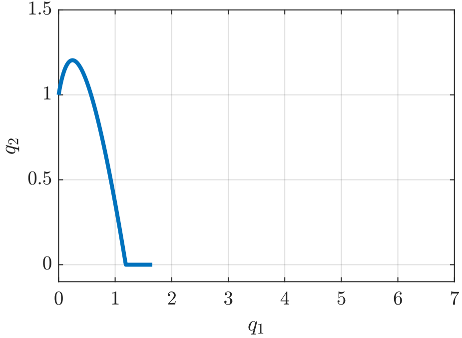
Simulation example - slip/stick

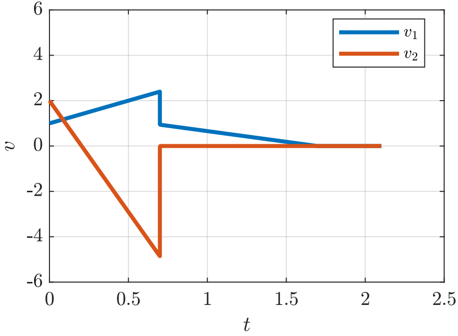
Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.3$

External force weaker than friction





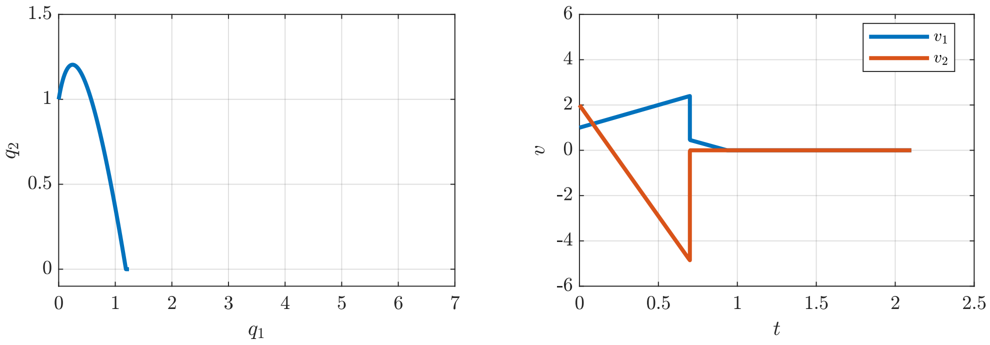
Simulation example - slip/stick

Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.4$

External force weaker than friction



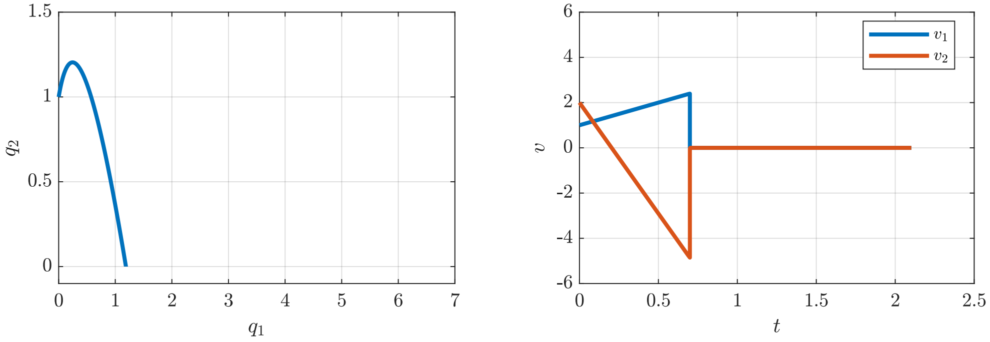
Simulation example - slip/stick

Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.5$

Tangential velocity zero after impact

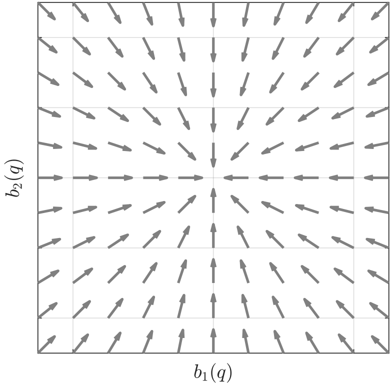


Friction for 3D contacts

Friction solution map

$$\lambda_{\mathrm{t}} \in \begin{cases} \{-\mu\lambda_{\mathrm{n}} \frac{v_{\mathrm{t}}}{\|v_{\mathrm{t}}\|_2}\}, & \text{if } \|v_{\mathrm{t}}\|_2 > 0, \\ \{\tilde{\lambda}_{\mathrm{t}} \mid \|\tilde{\lambda}_{\mathrm{t}}\|_2 \leq \mu\lambda_{\mathrm{n}}\}, & \text{if } \|v_{\mathrm{t}}\|_2 = 0. \end{cases}$$

- The set $\{v_{\mathrm{t}} \mid v_{\mathrm{t}} = 0\}$ has an empty interior
- Problematic for defining Filippov system via θ multipliers
- Problem not present with polyhedral approximations

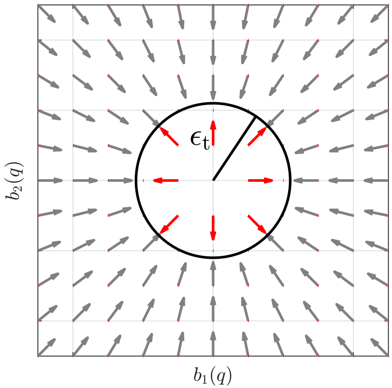


Friction for 3D contacts - relaxed solution

Relaxed riction solution map

$$\lambda_{\mathrm{t}} = \begin{cases} -\mu\lambda_{\mathrm{n}}\frac{v_{\mathrm{t}}}{\|v_{\mathrm{t}}\|_2}, & \text{if } \|v_{\mathrm{t}}\|_2 > \epsilon_{\mathrm{t}}, \\ v_{\mathrm{t}}, & \text{if } \|v_{\mathrm{t}}\|_2 < \epsilon_{\mathrm{t}}, \end{cases}$$

- $\epsilon_{\mathrm{t}} > 0$ can be arbitrarily small
- Obtain set with nonempty interior
- Slip mode: approximation is exact
- Stick mode: sliding mode on $\|v_{\mathrm{t}}\|_2 = \epsilon_{\mathrm{t}}$
- Approximation can be made arbitrarily good



Friction for 3D contacts - the time-freezing system

Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\}$$

PSS modes

$$\begin{aligned} f_1(y,u) &= (f_{\mathrm{ODE}}(x,u), 1) \\ f_2(y) &= f_{\mathrm{aux,n}}(y) - f_{\mathrm{aux,t,2}}(y) \\ f_3(y) &= f_{\mathrm{aux,n}}(y) + f_{\mathrm{aux,t,3}}(y) \end{aligned}$$

- Use same definition of regions R_1, R_2 and R_3
- Switching function $c_3(y) = \|v_\mathrm{t}\|_2 - \epsilon_\mathrm{t}$

Auxiliary ODEs for 3D friction

$$f_{\mathrm{aux,t,2}}(y) = \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1} B(q) \mu a_\mathrm{n} \frac{v_\mathrm{t}}{\|v_\mathrm{t}\|} \\ 0 \end{bmatrix}$$

$$f_{\mathrm{aux,t,3}}(y) = \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1} B(q) v_\mathrm{t} \\ 0 \end{bmatrix}$$

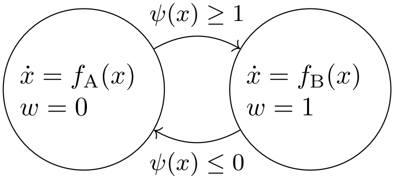
Hopping robot - move with minimal effort from start to end position

Homotopy initialized with start position everywhere. Optimizer finds creative solution.

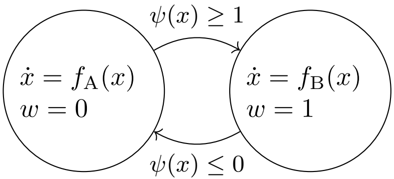
Overview

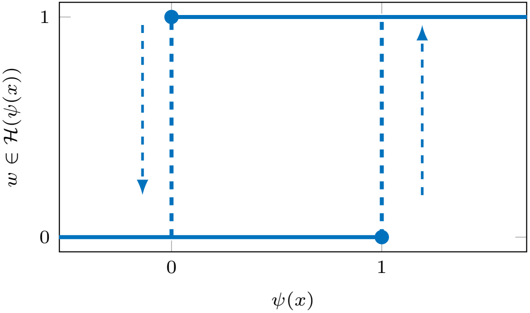
- The time-freezing reformulation
- Elastic impacts
- Inelastic impacts
- **Hybrid systems with hysteresis**
- Conclusions and outlook

Hybrid systems and finite automaton



Hybrid systems and finite automaton

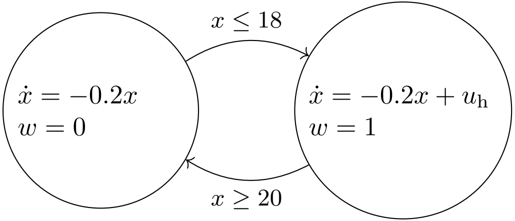




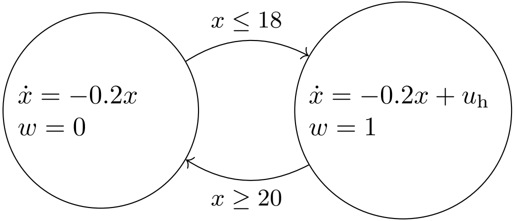
Hybrid system with hysteresis (*incomplete description*)

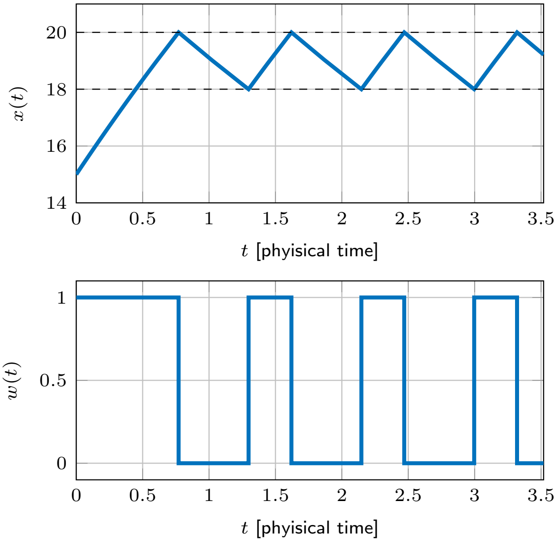
$$\dot{x} = f(x, w) = (1 - w) f_{\mathrm{A}}(x) + w f_{\mathrm{B}}(x)$$

Tutorial example: thermostat with hysteresis



Tutorial example: thermostat with hysteresis

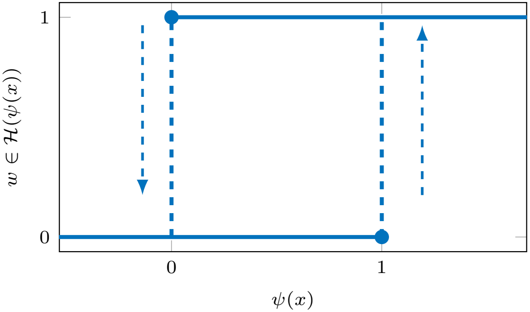




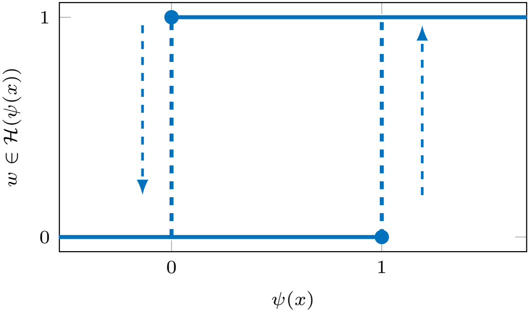
Hysteresis: a system with state jumps

Hybrid system with hysteresis

$$\dot{x} = f(x, w) = (1 - w) f_{\mathrm{A}}(x) + w f_{\mathrm{B}}(x)$$
$$\dot{w} = 0$$



Hysteresis: a system with state jumps



Hybrid system with hysteresis

$$\dot{x} = f(x, w) = (1 - w) f_{\mathrm{A}}(x) + w f_{\mathrm{B}}(x)$$
$$\dot{w} = 0$$

The State Jump Law

1. if $w(t_{\mathrm{s}}^{-}) = 0$ and $\psi(x(t_{\mathrm{s}}^{-})) = 1$, then $x(t_{\mathrm{s}}^{+}) = x(t_{\mathrm{s}}^{-})$ and $w(t_{\mathrm{s}}^{+}) = 1$
2. if $w(t_{\mathrm{s}}^{-}) = 1$ and $\psi(x(t_{\mathrm{s}}^{-})) = 0$, then $x(t_{\mathrm{s}}^{+}) = x(t_{\mathrm{s}}^{-})$ and $w(t_{\mathrm{s}}^{+}) = 0$

Remember: $w(t)$ is now a discontinuous differential state!

Recap: principles of time-freezing

1. mimic state jump by **auxiliary dynamical system** on prohibited region
2. introduce a **clock state** $t(\tau)$ that stops counting when the auxiliary system is active

Recap: principles of time-freezing

1. mimic state jump by **auxiliary dynamical system** on prohibited region
2. introduce a **clock state** $t(\tau)$ that stops counting when the auxiliary system is active
3. time-freezing system (a PSS) evolves in **numerical time** τ, initial system (with state jumps) in **physical time** $t(\tau)$
4. adapt speed of time, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = s$ with $s \geq 1$, and **impose terminal constraint** $t(T) = T$

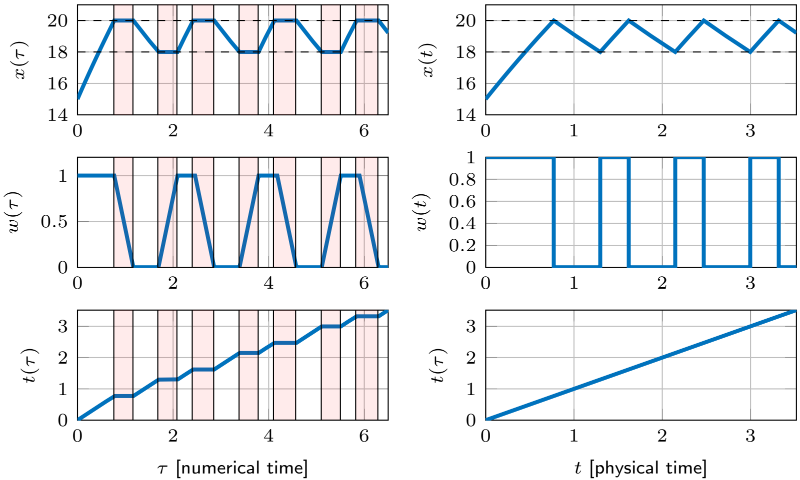
Recap: principles of time-freezing

1. mimic state jump by **auxiliary dynamical system** on prohibited region
2. introduce a **clock state** $t(\tau)$ that stops counting when the auxiliary system is active
3. time-freezing system (a PSS) evolves in **numerical time** τ, initial system (with state jumps) in **physical time** $t(\tau)$
4. adapt speed of time, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = s$ with $s \geq 1$, and **impose terminal constraint** $t(T) = T$
5. if the state dimension reduces after a state jump, construct an appropriate **sliding mode**

Recap: principles of time-freezing

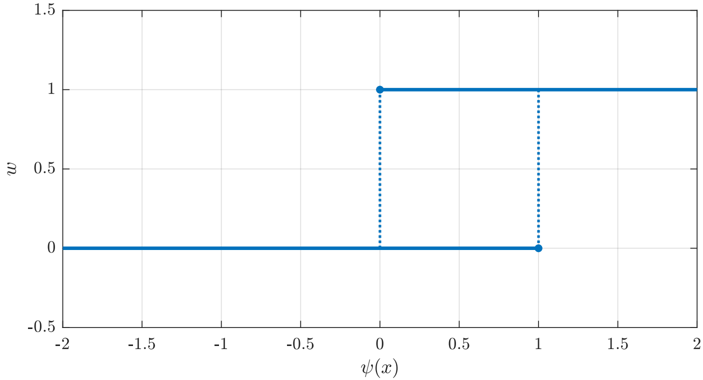
1. mimic state jump by **auxiliary dynamical system** on prohibited region
2. introduce a **clock state** $t(\tau)$ that stops counting when the auxiliary system is active
3. time-freezing system (a PSS) evolves in **numerical time** τ, initial system (with state jumps) in **physical time** $t(\tau)$
4. adapt speed of time, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = s$ with $s \geq 1$, and **impose terminal constraint** $t(T) = T$
5. if the state dimension reduces after a state jump, construct an appropriate **sliding mode**
6. take $x(t(\tau))$ instead of $x(\tau)$ to recover the **original solution** with **state jumps**
7. ...

Tutorial example: thermostat and time-freezing



Time-freezing: the state space

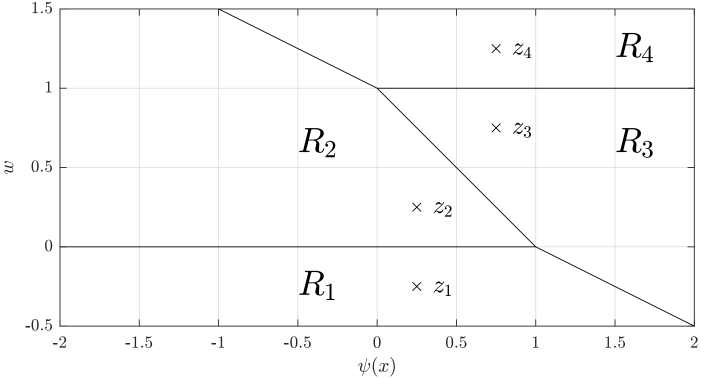
A look at the $(\psi(x), w)-$plane



- Everything except the blue solid curve is prohibited in the $(\psi, w)-$ space (use 1^{st} principle of time-freezing)
- The evolution happens in a lower-dimensional space $\implies$ *sliding mode* (use 4^{th} principle of time-freezing)

Time-freezing: partitioning of the space

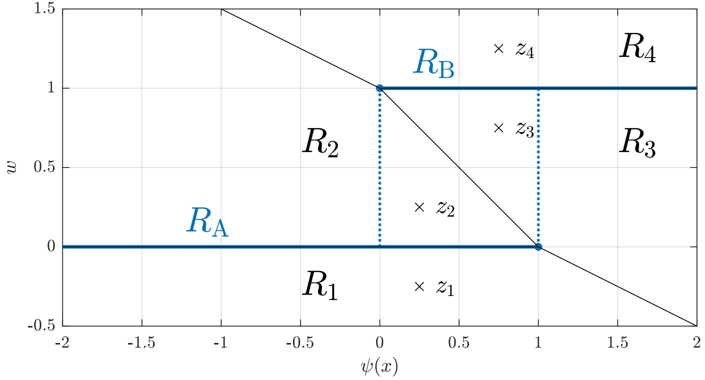
An efficient partition leads to less variables in FESD



- Partition the state space into *Voronoi regions*:
  $R_i = \{z \mid \|z - z_i\|^2 < \|z - z_j\|^2,\ j = 1, \dots, 4, j \neq i\},\ z = (\psi(x), w)$

Time-freezing: partitioning of the space

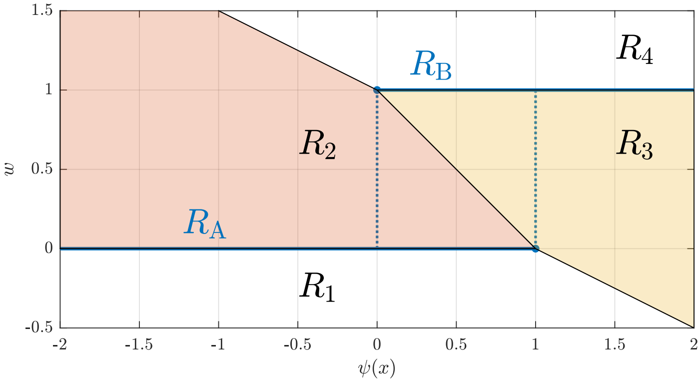
An efficient partition leads to less variables in FESD



- Partition the state space into *Voronoi regions*:
  $R_i = \{z \mid \|z - z_i\|^2 < \|z - z_j\|^2,\ j = 1, \ldots, 4, j \neq i\},\ z = (\psi(x), w)$
- Feasible region for initial *hybrid system with hysteresis* on the region boundaries

Time-freezing: auxiliary dynamics

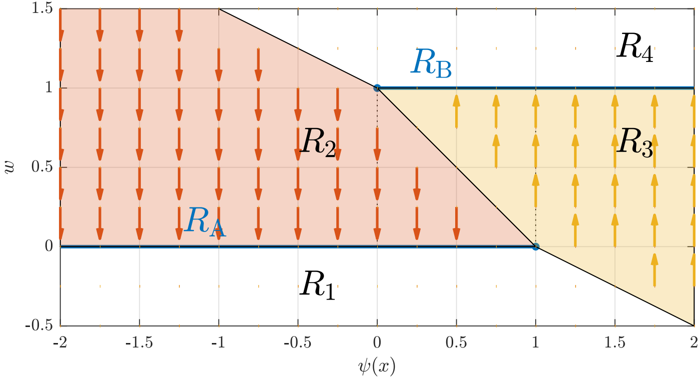
To mimic state jumps in finite numerical time



- Use regions R_2 and R_3 to define auxiliary dynamics for the state jumps of $w(\cdot)$

Time-freezing: auxiliary dynamics

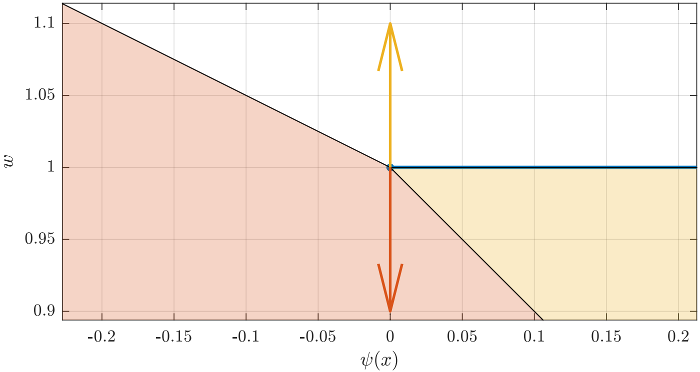
To mimic state jumps in finite numerical time



- Use regions R_2 and R_3 to define auxiliary dynamics for the state jumps of $w(\cdot)$
- Evolution in $w-$direction happens only for $\psi \in \{0, 1\}$

Time-freezing: auxiliary dynamics

To mimic state jumps in finite numerical time



- Use regions R_2 and R_3 to define auxiliary dynamics for the state jumps of $w(\cdot)$
- Evolution in $w-$direction happens only for $\psi \in \{0, 1\}$
- Zoom in: with a naive approach one has locally nonunique solutions

Time-freezing: auxiliary dynamics

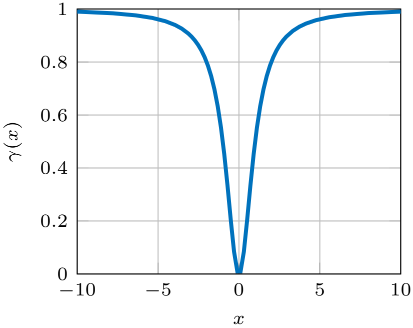
The new state space of the system is $y = (x, w, t) \in \mathbb{R}^{n_x+2}$

Auxiliary dynamics

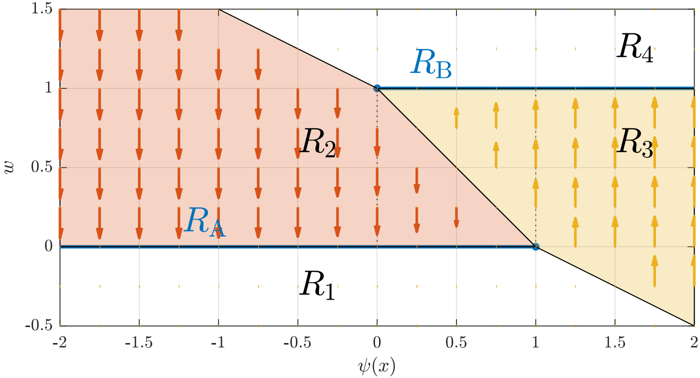
$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{aux,A}}(y) := \begin{bmatrix} 0 \\ -\gamma(\psi(x)) \\ 0 \end{bmatrix}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{aux,B}}(y) := \begin{bmatrix} 0 \\ \gamma(\psi(x) - 1) \\ 0 \end{bmatrix}$$

$$\gamma(x) = \frac{ax^2}{1+x^2}$$

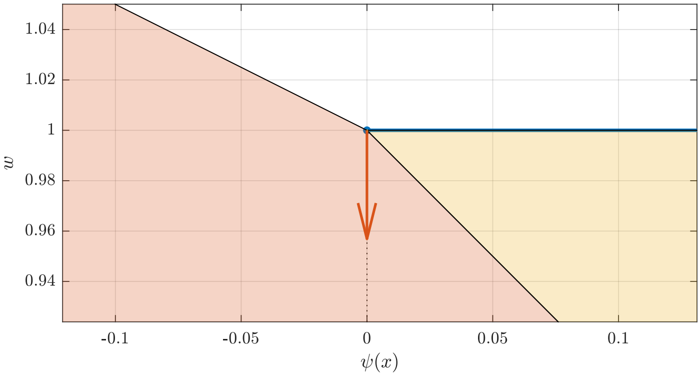


Time-freezing: auxiliary dynamics



- Smart choice of auxiliary dynamics resolves the nonuniqueness issue

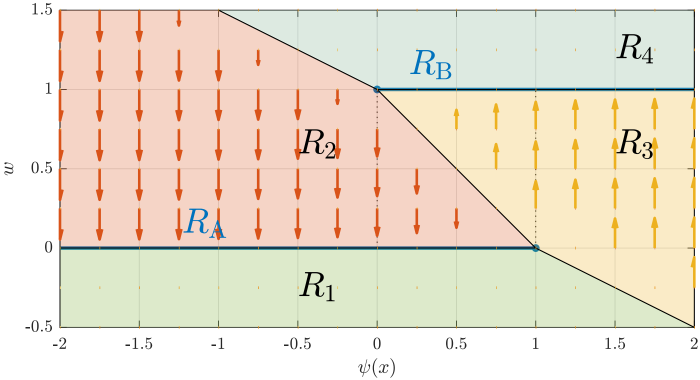
Time-freezing: auxiliary dynamics



- Smart choice of auxiliary dynamics resolves the nonuniqueness issue
- Zoom in: escape only in one direction possible

Time-freezing: DAE forming dynamics

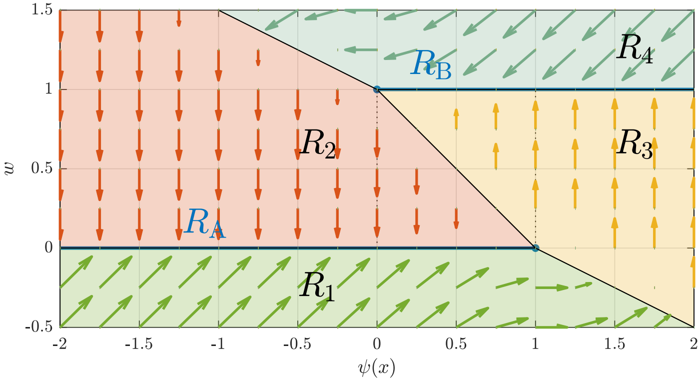
Stop the state jump and construct suitable sliding mode



- Dynamics in R_1 and R_4 stops evolution of auxiliary ODE - similar to inelastic impacts

Time-freezing: DAE forming dynamics

Stop the state jump and construct suitable sliding mode



- Dynamics in R_1 and R_4 stops evolution of auxiliary ODE - similar to inelastic impacts
- Sliding modes on $R_{\mathrm{A}} \coloneqq \partial R_1 \cap \partial R_2$ and $R_{\mathrm{B}} \coloneqq \partial R_3 \cap \partial R_4$ match $f_{\mathrm{A}}(y)$ and $f_{\mathrm{B}}(y)$, resp.

Time-freezing: summary

DAE-forming dynamics

$$y = (x, w, t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{df,A}}(y) := \begin{bmatrix} 2f_{\mathrm{A}}(x) \\ \gamma(\psi(x)) \\ 2 \end{bmatrix}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{df,B}}(y) := \begin{bmatrix} 2f_{\mathrm{B}}(x) \\ -\gamma(\psi(x) - 1) \\ 2 \end{bmatrix}$$

- In total four regions R_i , $i = 1, 2, 3, 4$ and evolution of original system is the **sliding mode**

Time-freezing: summary

DAE-forming dynamics

$$y = (x, w, t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{df,A}}(y) \coloneqq \begin{bmatrix} 2f_{\mathrm{A}}(x) \\ \gamma(\psi(x)) \\ 2 \end{bmatrix}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{df,B}}(y) \coloneqq \begin{bmatrix} 2f_{\mathrm{B}}(x) \\ -\gamma(\psi(x) - 1) \\ 2 \end{bmatrix}$$

- In total four regions R_i , $i = 1, 2, 3, 4$ and evolution of original system is the **sliding mode**
- Regions R_2 and R_3 equipped with aux. dynamics $y' = f_2(y) = f_{\mathrm{aux,A}}(y)$ and $y' = f_3(y) = f_{\mathrm{aux,B}}(y)$, resp., to mimic state jump

Time-freezing: summary

DAE-forming dynamics

$$y = (x, w, t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{df,A}}(y) := \begin{bmatrix} 2f_{\mathrm{A}}(x) \\ \gamma(\psi(x)) \\ 2 \end{bmatrix}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{df,B}}(y) := \begin{bmatrix} 2f_{\mathrm{B}}(x) \\ -\gamma(\psi(x) - 1) \\ 2 \end{bmatrix}$$

- In total four regions R_i , $i = 1, 2, 3, 4$ and evolution of original system is the **sliding mode**
- Regions R_2 and R_3 equipped with aux. dynamics $y' = f_2(y) = f_{\mathrm{aux,A}}(y)$ and $y' = f_3(y) = f_{\mathrm{aux,B}}(y)$, resp., to mimic state jump
- Regions R_1 and R_4 equipped with DAE-forming dynamics $y' = f_1(y) = f_{\mathrm{df,A}}(y)$ and $y' = f_4(y) = f_{\mathrm{df,B}}(y)$, resp., to recover original dynamics in sliding mode

Time-freezing: summary

DAE-forming dynamics

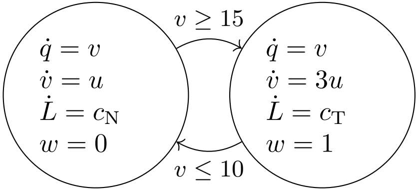
$$y = (x, w, t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{df,A}}(y) := \begin{bmatrix} 2f_{\mathrm{A}}(x) \\ \gamma(\psi(x)) \\ 2 \end{bmatrix}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = f_{\mathrm{df,B}}(y) := \begin{bmatrix} 2f_{\mathrm{B}}(x) \\ -\gamma(\psi(x)-1) \\ 2 \end{bmatrix}$$

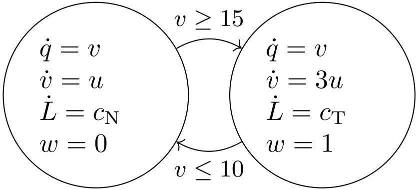
- In total four regions R_i , $i = 1, 2, 3, 4$ and evolution of original system is the **sliding mode**
- Regions R_2 and R_3 equipped with aux. dynamics $y' = f_2(y) = f_{\mathrm{aux,A}}(y)$ and $y' = f_3(y) = f_{\mathrm{aux,B}}(y)$, resp., to mimic state jump
- Regions R_1 and R_4 equipped with DAE-forming dynamics $y' = f_1(y) = f_{\mathrm{df,A}}(y)$ and $y' = f_4(y) = f_{\mathrm{df,B}}(y)$, resp., to recover original dynamics in sliding mode
- E.g., $w' = 0 \implies \theta_1 f_{\mathrm{df,A}}(y) + \theta_2 f_{\mathrm{aux,A}}(y) = f_{\mathrm{A}}(y)$ (sliding mode)
- Conclusion: we have a PSS and can treat it with FESD

Example from [Avraam, 2000] solved with NOSNOC



Time optimal control of a car with a turbo accelerator

Example from [Avraam, 2000] solved with NOSNOC

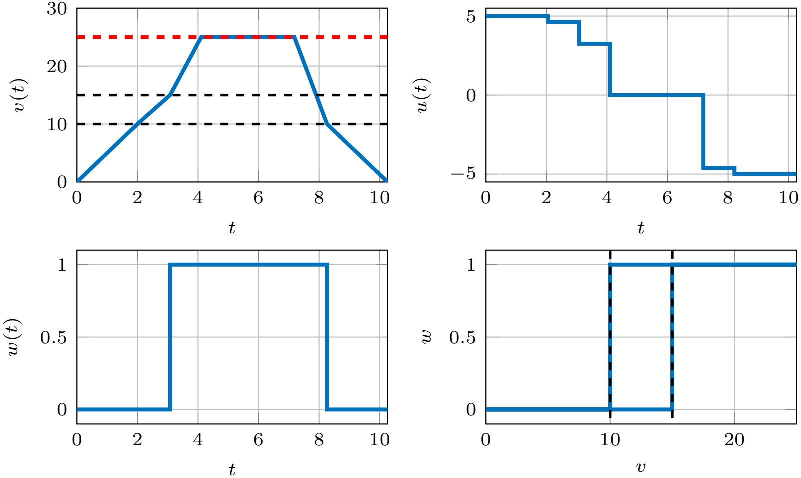


Time optimal control problem

$$
\begin{aligned}
\min_{y(\cdot),u(\cdot),s(\cdot)} \quad & t(\tau_{\mathrm{f}}) + L(\tau_{\mathrm{f}}) \\
\text{s.t.} \quad & y(0) = (z_0, 0) \\
& y'(\tau) \in s(\tau) F_{\mathrm{TF}}(y(\tau), u(\tau)) \\
& -\bar{u} \leq u(\tau) \leq \bar{u} \\
& \bar{s}^{-1} \leq s(\tau) \leq \bar{s} \\
& -\bar{v} \leq v(\tau) \leq \bar{v} \ \tau \in [0, \tau_{\mathrm{f}}] \\
& (q(\tau_{\mathrm{f}}), v(\tau_{\mathrm{f}})) = (q_{\mathrm{f}}, v_{\mathrm{f}})
\end{aligned}
$$

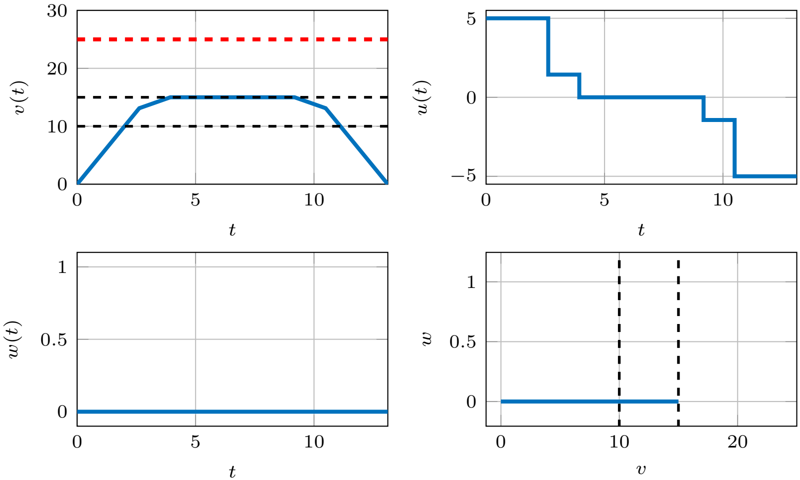
Scenario 1: turbo and nominal cost the same

$c_{\mathrm{N}} = c_{\mathrm{T}}$



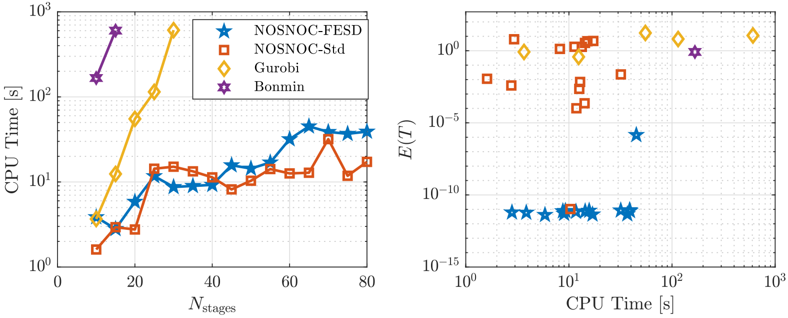
Scenario 2: Turbo is Expensive

$c_{\mathrm{N}} < c_{\mathrm{T}}$



NOSNOC vs MILP/MINLP formulations

Benchmark on time-optimal control problem of a car with turbo



- compare CPU time as function of number of control intervals N (left) and solution accuracy (right)
- MILP (Gurobi): solve problem with fixed T until indefeasibly happens with grid search in T
- MILP/MINLP and NOSNOC-Std no switch detection $=$ low accuracy

Conclusions and Outlook

Conclusions

- Mathematical Programs with Complementarity Constraints (MPCC) are a powerful tool to formulate and solve nonsmooth and nonconvex optimization problems.
- Time-freezing allows us to transform systems with state jumps of level NSD3 to the easier level NSD2.
- Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for switched systems of level NSD2.

Conclusions and Outlook

Conclusions

- Mathematical Programs with Complementarity Constraints (MPCC) are a powerful tool to formulate and solve nonsmooth and nonconvex optimization problems.
- Time-freezing allows us to transform systems with state jumps of level NSD3 to the easier level NSD2.
- Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for switched systems of level NSD2.

Outlook

- Time-freezing for multiple and simultaneous impacts with friction (work in progress)
- Time-freezing for more general hybrid automaton
- Do generic time-freezing principles, easily applicable to *any* system with state jumps, exist?

Related References

- A time-freezing approach for numerical optimal control of nonsmooth differential equations with state jumps.
  A. Nurkanović, T. Sartor, S. Albrecht, and M. Diehl, IEEE Cont. Sys. Lett., 2021.
- Continuous optimization for control of hybrid systems with hysteresis via time-freezing
  A. Nurkanović and M. Diehl, IEEE Cont. Sys. Lett., 2022.
- The Time-Freezing Reformulation for Numerical Optimal Control of Complementarity Lagrangian Systems with State Jumps.
  A. Nurkanović, S. Albrecht, B. Brogliato, and M. Diehl, arXiv preprint 2022
- Set-Valued Rigid Body Dynamics for Simultaneous Frictional Impact.
  Mathew Halm and Michael Posa, arXiv Preprint, 2021.

Thank you very much for your attention!