Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2017

Exercise 7: Recursive Least Squares

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In this exercise you will get to know the Recursive Least Squares (RLS) estimator and implement it. Furthermore you will deepen your knowledge on the estimation of covariances.
For the the MATLAB exercises, create a MATLAB script called `main.m` with your code, possibly calling other functions/scripts. From running this script, all the necessary results and plots should be clearly visible.

Exercise Tasks

1. **Covariance Estimation**
   For the following estimators, describe how the covariance matrix of the estimator can be estimated or approximated. If a general formula exists please give it otherwise describe your approach. State all assumptions that you make.

   (a) Weighted Least Squares Estimator

   (b) Maximum A Posteriori estimator

   (c) Recursive least squares

   Hint for (b): First consider the case discussed in the script section 5.2.1 (p. 41), then consider the general case. Can you make any statement for the non-linear case?

2. **Recursive Least Squares (RLS)**
   During each iteration of the RLS algorithm the following optimization problem has to be solved:

$$\hat{\theta}_{\mathrm{ML}}(N+1) = \arg\min_{\theta \in \mathbb{R}^d} \left(\frac{1}{2} \|\theta - \hat{\theta}_{\mathrm{ML}}(N)\|_{Q_{\mathrm{N}}}^2 + \frac{1}{2} \cdot \|y(N+1) - \varphi(N+1)^\top \cdot \theta\|_2^2 \right) \tag{1}$$

   with $Q_{\mathrm{N+1}} = Q_{\mathrm{N}} + \varphi(N+1) \cdot \varphi(N+1)^\top$ and $\hat{\theta}_{\mathrm{ML}}(N)$ is the optimal estimator for

$$\hat{\theta}_{\mathrm{ML}}(N) = \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{2} \|y_{\mathrm{N}} - \Phi_{\mathrm{N}} \cdot \theta\|_2^2.$$

   Show that

$$\hat{\theta}_{\mathrm{ML}}(N+1) = \hat{\theta}_{\mathrm{ML}}(N) + Q_{\mathrm{N+1}}^{-1} \cdot \varphi(N+1) \cdot (y(N+1) - \varphi(N+1)^\top \cdot \hat{\theta}_{\mathrm{ML}}(N)) \tag{2}$$

   is the minimizer of (1).

3. **Recursive Least Squares (RLS) vs. Linear Least Squares (LLS)** Show that for a given finite number of measurements N the RLS estimator and the LLS estimator are equivalent.
   Hint: you can use the result from the previous task.

Computer Exercise

4. **Variance Estimation of average wind power** (5 points)

In the last exercise sheet the parameters of the Weibull distribution $\theta = [\lambda, k]$ were estimated. With them, the expected value of the wind energy power was computed by a map $E_{\text{Power}}(\theta)$ which approximated $\mathbb{E}\{P_{\text{power}}\}$. In real life applications, not only the expected value of the power, but also its variance is an important quantity in order to assess the quality of the location of the wind farm.

(a) The covariance Σ_θ of the parameter estimates for the Weibull distribution can not be easily estimated. Propose one approximation for it that could be obtained with the measurement data you have. (2 points)

(b) Assume now that the covariance Σ_θ is known. Propose an approximation of the variance of the power $\sigma^2_{E_{\text{Power}}(\theta)}$ and compute this value. State the power expectation obtained in the last sheet in the form '$X[\text{kW}] \pm Y[\text{kW}]$'. (3 points)
Hint 1: approximate the map $E_{\text{Power}}(\theta)$ by its first order Taylor expansion:

$$E_{\text{Power}}(\theta) = E_{\text{Power}}(\hat{\theta}) + \frac{dE_{\text{Power}}}{d\theta}(\hat{\theta})(\theta - \hat{\theta}). \tag{3}$$

Hint 2: any derivative can be computed by finite differences.