Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2017

Exercise 4: Introduction to Weighted Least Squares

Prof. Dr. Moritz Diehl, Robin Verschueren, Alexander Resch

In this exercise you get to know how weighted linear least squares work and deepen you knowledge of covariance matrices.

Exercise Tasks

1. Let's consider the 2-norm squared of the residuals $r \in \mathbb{R}^n$

$$\|r\|_2^2 = \|y - \Phi\theta\|_2^2 = \left\| \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} - \begin{bmatrix} \phi^{\mathrm{T}}(1) \\ \vdots \\ \phi^{\mathrm{T}}(N) \end{bmatrix} \theta \right\|_2^2$$

denoting the difference between a measurement vector $y \in \mathbb{R}^n$ and a linear model $\Phi\theta$, with $\Phi \in \mathbb{R}^{n\times d}$ and $\theta \in \mathbb{R}^d$. Furthermore, consider the following optimization problem:

$$\underset{\theta}{\text{minimize}} \quad \frac{1}{2} r^{\mathrm{T}} W r,$$

with W symmetric, PD and of appropriate size.

(a) Expand the objective as a quadratic function of θ, using the following notation:

$$f(\theta) = \frac{1}{2}\theta^{\mathrm{T}} H \theta + g^{\mathrm{T}}\theta + c.$$

Please identify H, g, c with the given matrices y and Φ in the objective after expansion.

(b) Calculate the the gradient and the Hessian of the objective.

(c) Analytically solve the optimization problem.

(d) Is it a convex problem? Prove it.

2. A weighted least squares problem (WLS) can be formulated as an unweighted least squares problem with rescaled measurements and rescaled regressor matrix. Please write the WLS optimization problem from task 1 as an unweighted LLS by rescaling the objective function with appropriate matrices.

Computer Exercise

3. Please consider the newly prepared diode data, which can be found on the course page. Plot the current on the y-axis vs. the voltage on the x-axis. Have a look at the plot. As you will see, there is noise on both the input voltage and the forward current. As you might also notice, the noise is not i.i.d. (5 points)

   (a) Distribute the data into 8 clusters. (1 point)

   (b) Please calculate the output variances $\sigma_1^2, ..., \sigma_8^2$ for all data clusters. Estimate the covariance matrix $\Sigma_{I_d} = W$. What does W look like? Assume that the noise on I_d is independently distributed. (1 point)

$$W = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

   What does the optimization problem for the weighted least squares estimator look like? Please use a 4^{th} order polynomial model for your fit.

   (c) Formulate the optimization problem on paper and in matlab code. (1 point)

   (d) Solve the optimization problem. (1 point)

   (e) Compare the fit obtained with θ_{WLS} to the fit obtained with θ_{LLS} (1 point)