Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2017

Exercise 3: Linear Least Squares

Prof. Dr. Moritz Diehl, Robin Verschueren, Alexander Resch

In this exercise you deepen your knowledge of linear least squares and covariance matrices.
Please remember that each group slot in the presentation form can only be filled by students of the same group (with the same code on the same computer).

Exercise Tasks

1. Given a sequence of i.i.d. scalar random variables $x(1), \dots, x(N)$, each with mean μ and variance σ^2, what is the expected value and variance of their sample mean, i.e. of the random variable y_N defined by $y_N = \frac{1}{N}\sum_{k=1}^{N} x(k)$.

2. In your lecture notes, the sample variance S^2 is defined as

$$S^2 = \frac{1}{N-1}\sum_{n=1}^{N}(Y(n) - M(Y(N)))^2,$$

where $M(Y(N))$ is the sample mean. Explain, why the division by $N-1$ is preferable over N. (Hint: Formally calculate the expected value of the sample variance and compare it to the expected value of the mean squared deviations estimator.)

3. Recall the experimental setup to estimate the value of a resistor from exercise 1. We assumed that the measurements were perturbed by additive noise $n_i(k)$ and $n_u(k)$: $i(k) = i_0 + n_i(k)$ and $u(k) = u_0 + n_u(k)$. Given these assumptions, derive why the estimator for the resistance given by

$$\hat{R}_{\text{LS}}(N) = \frac{\frac{1}{N}\sum_{k=1}^{N} u(k)i(k)}{\frac{1}{N}\sum_{k=1}^{n} i(k)^2}$$

is a least squares estimator. Please give the full derivation, pointing to the script is not sufficient.

4. Consider the diode model given in exercise 2. As you have already found out, polynomial fitting over exponential relationships might not necessarily lead to satisfactory performance. Better performance can be achieved by directly fitting an exponential function to the data. (Remember that for LLS to work, the model must only be linear in the parameters (**LIP**), while the regressor functions can still be nonlinear.) The corresponding dataset `Data.mat` can be found on the course homepage.
(Hint: Each of the $m = 1000$ measurement series consists of 8 voltage-current pairs ($N = 8000$).)

After first inspecting the data, you decide to estimate both the reverse leakage current $I_s = \theta_1$ and the measurement offset $I_{\text{Offset}} = \theta_2$ for the model

$$I_d(k) = \theta_1 \left(e^{\frac{qU(k)}{mk_BT}} - 1 \right) + \theta_2 + \epsilon(k),$$

where $U(k)$ is the applied voltage, $I_d(k)$ is the measured forward current, $m = 1.5$ is the ideality factor, $q = 1.6022 \cdot 10^{-19}$ C is the elementary charge, $k_B = 1.3806e \cdot 10^{-23}$ J/K is the Boltzmann constant and $T = 293$ K is the temperature. (5 points)

(a) Before implementing the LLS estimator in your code, schematically write down the objective function for the optimization problem (1 point)

$$\underset{\theta_1,\theta_2}{\text{minimize}} \quad \|I_d - \Phi\theta\|_2^2.$$

How do I_d , Φ and θ look?

$$\Phi = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \quad I_d = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}, \quad \theta = \begin{bmatrix} \\ \end{bmatrix}$$

(b) Estimate the reverse leakage current I_s and the offset current I_{Offset} by solving the above optimization problem. (1 point)

(c) Use the backslash operator, don't invert matrices and have a look at the coefficients. (1 point)

(d) In order to assess the performance of your parameter estimation, one can look at the covariance $\text{cov}(\hat{\theta}_N)$ of the estimated parameters. It is given by

$$\text{cov}(\hat{\theta}_N) = \Phi_N^T \sigma_{\epsilon_N}^2 \Phi_N,$$

with σ_ϵ^2 being the variance of the noise $\epsilon_N = [\epsilon_1, ..., \epsilon_N]^T$, which in turn can be estimated by its sample variance. (1 point)

(e) Calculate the covariance $\text{cov}(\hat{\theta}_N)$ of the estimated parameters for $N = 4000$ and for $N = 8000$ experiments and compare them. Do you notice any tendencies? Which implications can you draw from that insight? (Hint: Remember you have to readjust your regression matrix Φ_N for different N.) (1 point)