Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2017

Exercise 2: Linear Least Squares Introduction

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In this exercise, you will discover basic properties of the linear least squares estimation method, as well as the covariance operator.

Exercise Tasks

1. The covariance matrix of a vector-valued random variable X in $\mathbb{R}^n$ with mean $\mathbb{E}\{X\} = \mu_X$ is defined by
$$\operatorname{cov}(X) \coloneqq \mathbb{E}\left\{(X - \mu_X)(X - \mu_X)^{\mathrm{T}}\right\}.$$
Prove that the covariance matrix of a vector-valued variable $Y = AX + b$ with constant $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is given by
$$\operatorname{cov}(Y) = A \operatorname{cov}(X) A^{\mathrm{T}}.$$

2. Suppose you are measuring a constant $x \in \mathbb{R}$ perturbed by Gaussian, zero-mean noise $\epsilon \in \mathbb{R}$ ($\epsilon \sim \mathcal{N}(0, \sigma^2)$). Prove that the LLS estimate for this constant is the sample mean.

3. A semiconductor diode is a crystalline piece of semiconductor material with a p-n junction connected to two electrical terminals. In the most common form it consists of a p-n doped silicon substrate. The Shockley equation
$$I_d = I_s\left(e^{\dfrac{qU}{mk_BT}} - 1\right),$$
where U is the applied voltage in forward direction, m is the ideality factor, q is the elementary charge, k_B is the Boltzmann constant and T is the temperature, is used to describe the diode current I_d flowing through the diode in forward direction.
Let's consider current-voltage data of a common silicon diode following the Shockley equation. This data is available on the course page in a matlab data file called `Data.mat`. (5 points)

   (a) Plot the diode current I_d [mA] (second column) vs. the applied voltage U [mV] (first column) data, with voltage along the x-axis and current along the y-axis. When looking at the data for current as a function of voltage, what is the lowest order polynomial relation that you would expect to give a meaningful linear least squares fit? What form does this relation take? (Hint: remember that linear least squares does not necessarily require a linear relationship between I_d and U.) (1 point)

   (b) Which properties should the data fulfil, for linear least squares to be an appropriate estimation method? Make those assumptions for the following questions. (0.5 points)

   (c) Choose $y \in \mathbb{R}^N$, $\phi_i \in \mathbb{R}^{1\times d}$ and $\mathbf{\Phi} \in \mathbb{R}^{N\times d}$, to correspond to the linear least squares problem form. (1.5 points)

   (d) Is $\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi}$ invertible? Why? What could lead to non-invertability? Does Matlab return a warning, if you attempt to invert this matrix? Why would it? (1 point)

(e) In general, it is not advised to use a linear system solve, using the pseudo-inverse, for problems that have this (see 3d) behavior. Try it anyways. Plot the estimated currents as a function of voltage in your data-plot, and consider the coefficients of the fitted polynomial. Do these coefficients appear reasonable? (Hint: use the backslash operator in the linear system solve.) (0.5 points)

(f) Reconsider your fitting problem, using units of [A] for I_d and [V] for U. Apply a linear system solve with the pseudo-inverse to this problem. Add this estimate to the data-plot, and consider the coefficients of the fitted polynomial. Does scaling the measurements improve the performance of linear least squares? Why? (0.5 points)