Robust Optimization

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Course on Numerical Methods for Nonlinear Optimal Control
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slides jointly developed with **Titus Quah**, Katrin Baumgärtner, Florian Messerer
based on joint work with B. Houska and some illustrations from his PhD thesis

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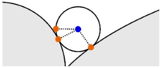
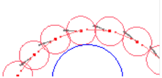
Overview

- Robust optimization problem as semi-infinite optimization problem and guiding example
- Five favourable cases (that can exactly be formulated as finite NLPs)
  1. Finite uncertainty
  2. Polytopic uncertainty, maximization function convex in uncertainty
  3. Affine, norm bounded uncertainty
  4. Concave maximization on convex uncertainty set
  5. Quadratic maximization on ℓ_2-ball
- Approximate NLP formulations (conservative in special cases)
  - Linearization (conservative in concave case)
  - Lagrangian relaxation (exact in concave and quadratic case)

Problem statement





$$\begin{array}{ll} \underset{u \in \mathbb{U}}{\text{minimize}} & F_0(u) \\ \text{subject to} & \underbrace{\max_{w \in \mathbb{W}} F_i(u,w)}_{=:\varphi_i(u)} \leq 0, \quad i = 1, \ldots, n_F \end{array}$$

- game interpretation: we choose control $u \in \mathbb{U} \subset \mathbb{R}^{n_u}$, then adverse player (nature) chooses uncertain disturbance $w \in \mathbb{W} \subset \mathbb{R}^{n_w}$
- bi-level interpretation: high-level, "outer" problem in u, with low-level, "inner" maximizations finding $w_1^*(u), \ldots, w_{n_F}^* \in \mathbb{R}^{n_w}$ such that $\varphi_i(u) = F_i(u, w_i^*(u))$.
- relevant dimensions: n_u, n_w, n_F
- often $\mathbb{W}$ is a unit ball (which can represent all ellipsoidal uncertainties)
- we assume no uncertainty in objective, without loss of generality (see next slide)

Epigraph slack reformulation in case of uncertain objective

If the objective function is also uncertain and given by $\max_{w \in \mathbb{W}} F_0(u, w)$, one could augment the control vector with one extra component $u_0 \in \mathbb{R}$ - a so-called slack variable - and move the uncertainty into an extra constraint, so that the objective is again fully certain

$$\begin{array}{ll} \underset{u \in \mathbb{U}, u_0 \in \mathbb{R}}{\text{minimize}} & u_0 \\ \text{subject to} & \max\limits_{w \in \mathbb{W}} F_0(u, w) - u_0 \leq 0, \\ & \max\limits_{w \in \mathbb{W}} F_i(u, w) \leq 0, \quad i = 1, \dots, n_F \end{array}$$

We can therefore restrict our attention to the previous problem formulation

$$\begin{array}{ll} \underset{u \in \mathbb{U}}{\text{minimize}} & F_0(u) \\ \text{subject to} & \max\limits_{w \in \mathbb{W}} F_i(u, w) \leq 0, \quad i = 1, \dots, n_F \end{array}$$

Two extreme but equivalent formulations

The robust optimization problem can either be formulated with one single constraint

$$\begin{array}{ll} \underset{u \in \mathbb{U}}{\text{minimize}} & F_0(u) \\ \text{subject to} & \max_{i \in \{1,\ldots,n_F\}} \max_{w \in \mathbb{W}} F_i(u,w) \leq 0 \end{array}$$

or it can be formulated with many constraints

$$\begin{array}{ll} \underset{u \in \mathbb{U}}{\text{minimize}} & F_0(u) \\ \text{subject to} & F_i(u,w) \leq 0 \quad \text{for all } i \in \{1,\ldots,n_F\} \text{ and all } w \in \mathbb{W} \end{array}$$

If the uncertainty set $\mathbb{W}$ is not a finite set, the second problem has infinitely many constraints, but finitely many variables $u \in \mathbb{R}^{n_u}$, and is therefore a **semi-infinite program (SIP)**

A guiding example with two variables and one uncertain constraint

Guiding Example

$$\begin{aligned} \underset{u \in \mathbb{R}^2}{\text{minimize}} \quad & u_2 \\ \text{subject to} \quad & -1 \le u_1 \le 1, \\ & \max_{w \in \mathbb{W}} F(u,w) \le 0 \end{aligned}$$

with unit ball $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\| \le 1\}$ and $F(u,w) := -x_2 + f(x_1)$ with $x = u + w \in \mathbb{R}^2$ and scalar $f(z) := -(1/2)z + (c_1/2)z^2 + (c_2/16)z^4$ with $c_1 = -1, c_2 = -1$

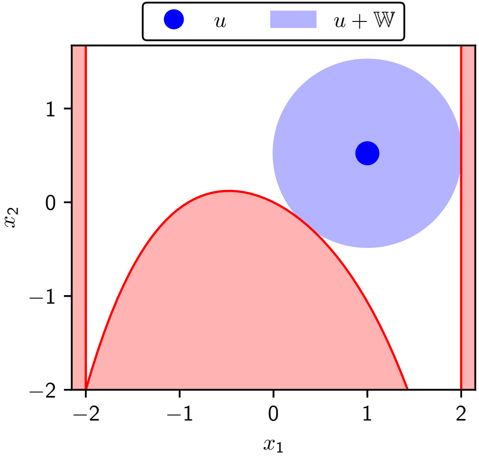
For visualization, regard set $\mathbb{B}(u) := \{x \in \mathbb{R}^2 \mid \exists w \in \mathbb{W} : x = u + w\}$, i.e., a movable unit ball. Minimize ball height but ensure whole ball remains above line described by $x_2 \ge f(x_1)$.

Note: for negative coefficients c_1, c_2, functions f and F are strictly concave in w, thus the inner maximization problem is strictly convex and the worst-case (contact) point unique.

In the sequel, we will sometimes change coefficients c_1, c_2 and uncertainty set $\mathbb{W}$.

Visualization of the Guiding Example

For convex inner maximization, with $c_1 = -1, c_2 = -1$ and Euclidean ball $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$



One ambitious and a more realistic aim

The ambitious aim - that is only achievable in some favourable cases - is to find a finite **nonlinear program (NLP)** that is equivalent to the original robust optimization problem:

Exact problem with $\boxed{\varphi_i(u) = \max_{w \in \mathbb{W}} \; F_i(u, w)}$

$$\begin{aligned} \underset{u \,\in\, \mathbb{U}}{\text{minimize}} \quad & F_0(u) \\ \text{subject to} \quad & \varphi_i(u) \leq 0, \quad i = 1, \ldots, n_F \end{aligned}$$

The realistic aim - that can be achieved in more cases - is to find an NLP formulation that is equivalent to an approximation of the original problem:

Approximate problem with $\tilde{\varphi}_i(u) \approx \varphi_i(u)$, ideally *conservative* with $\varphi_i(u) \leq \tilde{\varphi}_i(u)$ for all $u \in \mathbb{U}$

$$\begin{aligned} \underset{u \,\in\, \mathbb{U}}{\text{minimize}} \quad & F_0(u) \\ \text{subject to} \quad & \tilde{\varphi}_i(u) \leq 0, \quad i = 1, \ldots, n_F \end{aligned}$$

For a conservative ("pessimistic") approximation, every feasible point is also feasible for the exact problem. Thus, it allows one to find feasible, but suboptimal solutions.

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Favourable Case 1: Finite Uncertainty

If the disturbance set $\mathbb{W}$ is finite and is given by $\mathbb{W} = \{w_1, \ldots, w_M\}$ with each element $w_i \in \mathbb{R}^{n_w}$, and functions $F(u, w)$ are smooth w.r.t. u, an exact NLP formulation is given by

Exact reformulation for finite uncertainty

$$\begin{aligned} \underset{u \in \mathbb{U}}{\text{minimize}} \quad & F_0(u) \\ \text{subject to} \quad & F_i(u, w_j) \leq 0, \quad i = 1, \ldots, n_F, \; j = 1, \ldots, M \end{aligned}$$

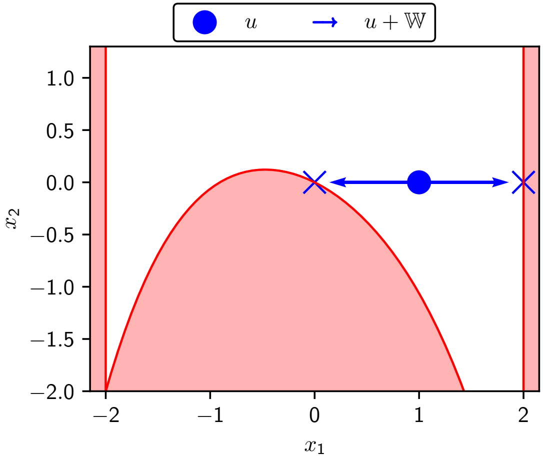
This NLP has $M \cdot n_F$ inequalities and n_u variables.

This case inspires the **sampling-approximation**, which just selects M samples $\{w_1, \ldots, w_M\} \subset \mathbb{W}$ from an infinite set $\mathbb{W}$. Sampling yields an outer approximation of the true feasible set, i.e., it is too optimistic, as

$$\tilde{\varphi}_i(u) = \max_{j=1,\ldots,M} F_i(u, w_j) \quad \leq \quad \max_{w \in \mathbb{W}} F_i(u, w) = \varphi_i(u)$$

Favourable Case 1: Visualization of Finite Uncertainty

with $c_1 = -1, c_2 = -1$ and finite uncertainty set $\mathbb{W} = \{(-1,0),(1,0)\}$



Favourable Case 2: Polytopic Uncertainty with F convex in w

If the disturbance set $\mathbb{W}$ is a polytope, i.e., the convex hull of a finite vertex set $\{w_1, \ldots, w_M\} \subset \mathbb{R}^{n_w}$

$$\mathbb{W} = \left\{ \sum_{j=1}^{M} \lambda_j w_j \;\middle|\; \sum_{j=1}^{M} \lambda_j = 1, \lambda \geq 0 \right\}$$

and if functions $F_i(u, w)$ are **convex** w.r.t. w, and smooth w.r.t. u, then sampling only the vertex set suffices to obtain an exact NLP formulation. In this case one can show that $\boxed{\tilde{\varphi}_i(u) := \max_{j=1,\ldots,M} F_i(u, w_j)}$ equals the exact $\varphi_i(u)$.

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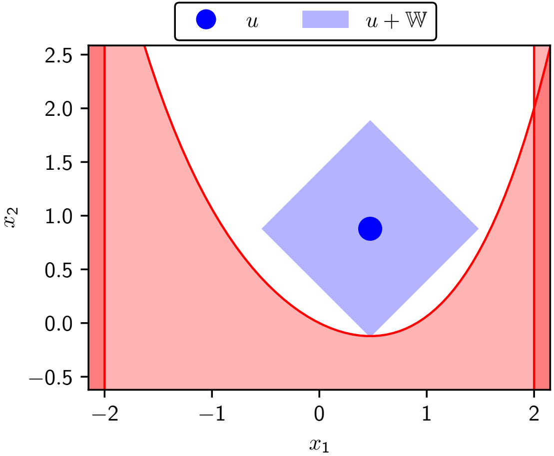
Proof of exactness. As before, we have $\tilde{\varphi}_i(u) \leq \varphi_i(u)$, so only need to show $\varphi_i(u) \leq \tilde{\varphi}_i(u)$. For this regard inner maximizing $w_i^*(u)$ with $\varphi_i(u) = F_i(u, w_i^*(u))$. As $w_i^*(u) \in \mathbb{W}$, there exist multipliers $\lambda \geq 0$ with $\sum_{j=1}^{M} \lambda_j = 1$ so that

$$w_i^*(u) = \sum_{j=1}^{M} \lambda_j w_j \quad \text{which due to convexity implies} \quad \underbrace{F(u, w_i^*(u))}_{=\varphi_i(u)} \leq \underbrace{\sum_{j=1}^{M} \lambda_j F(u, w_j)}_{\leq \tilde{\varphi}_i(u)} \qquad \square$$

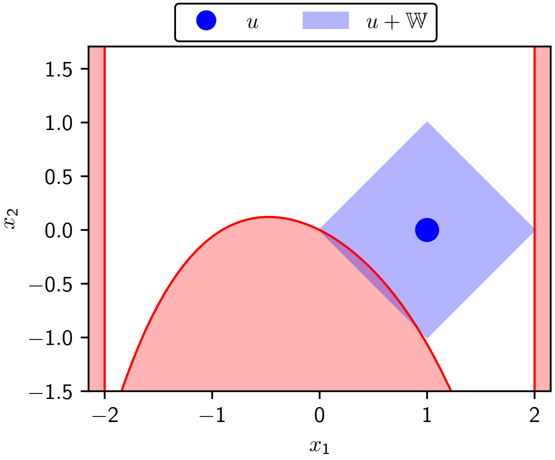
Case 2: Visualization of Polytopic Uncertainty with F convex in w

F convex in w, with $c_1 = 1, c_2 = 1$ and polytopic uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_1 \leq 1\}$



Case 2: Sampling the vertices looses exactness if F is **not** convex in w

F concave in w, with $c_1 = -1, c_2 = -1$ and polytopic uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_1 \leq 1\}$



Favourable Case 3: Affine, Norm Bounded Uncertainty

Assume that the uncertainty set is a norm ball

$$\mathbb{W} = \{w \in \mathbb{R}^{n_w} \mid \|w\| \leq 1\} \quad \text{for a given "primal" norm } \|\cdot\|$$

and that each F is smooth in u and affine in w, i.e., that it equals its first order Taylor series

$$\boxed{F(u, w) = F(u, 0) + \nabla_w F(u, 0)^\top w}$$

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$$\boxed{F(u,w) = F(u,0) + \nabla_w F(u,0)^\top w}$$

Using the dual norm $\|\cdot\|_*$ defined via $\boxed{\|v\|_* := \max_{\|w\|\leq 1} v^\top w}$ one can exactly compute

$$\max_{w \in \mathbb{W}} F(u,w) = F(u,0) + \max_{\|w\|\leq 1} \nabla_w F(u,0)^\top w = F(u,0) + \|\nabla_w F(u,0)\|_*$$

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Thus, an exact NLP formulation is given by

Exact reformulation for affine, norm bounded uncertainty

$$\begin{aligned} \underset{u \in \mathbb{U}}{\text{minimize}} \quad & F_0(u) \\ \text{subject to} \quad & F_i(u,0) + \|\nabla_w F_i(u,0)\|_* \leq 0, \quad i = 1, \ldots, n_F \end{aligned}$$

This NLP has n_F inequalities and n_u variables.

Favourable Case 3: Frequently used norms and their duals

We present some pairs of norms that are dual to each other.
Note that in $\mathbb{R}^n$, the dual of the dual equals the original norm.

ℓ_2-norm	$\|w\|_2 = \sqrt{w^\top w}$	is dual to	$\|v\|_2 = \sqrt{v^\top v}$	ℓ_2-norm
ℓ_∞-norm	$\|w\|_\infty = \max\{\|w_1\|, \ldots, \|w_n\|\}$	is dual to	$\|v\|_1 = \sum_{j=1}^{n} \|v_j\|$	ℓ_1-norm
ℓ_p-norm	$\|w\|_p = \sqrt[p]{\sum_{j=1}^{n} \|x_j\|^p}$	is dual to	$\|v\|_q$	ℓ_q-norm

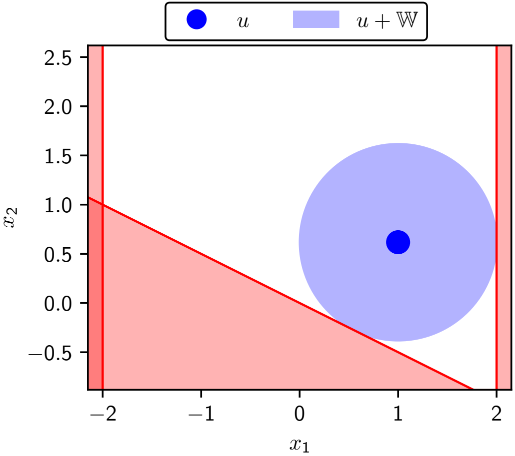
for $p, q > 1$ with $(1/p) + (1/q) = 1$

ℓ_∞-ℓ_p-norm $\quad \|(w_1, w_2)\|_{\infty,p} = \max\{\|w_1\|_p, \|w_2\|_p\}$

is dual to $\quad \|(v_1, v_2)\|_{1,q} = \|v_1\|_q + \|v_2\|_q \quad$ ℓ_1-ℓ_q-norm

Case 3: Visualization of Affine, Norm Bounded Uncertainty

F linear in w, with $c_1 = 0, c_2 = 0$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$



Very Favourable Case 3 - F doubly affine in u and w

One of the nicest - and most successfully used - cases occurs for norm-bounded uncertainty with doubly affine F_i, i.e., if

$$\boxed{F_i(u,w) = F_i(0,0) + \nabla_u F_i(0,0)^\top u + \nabla_w F_i(0,0)^\top w + w^\top \nabla^2_{w,u} F_i(0,0) u}$$

In this case, the exact, dual-norm involving constraints become

$$F_i(0,0) + \nabla_u F_i(0,0)^\top u + \left\| \nabla_w F_i(0,0) + \nabla^2_{w,u} F_i(0,0) u \right\|_* \leq 0$$

As concatenation of an outer convex function with an inner affine function, the function on the left side is convex in u, rendering the constraint convex and making it possible to find globally optimal solutions in many cases.

This doubly affine F is the basis of "affine disturbance feedback", sketched next (which is relevant for robust dynamic optimization).

Very Favourable Case 3 - Affine Disturbance Feedback

The doubly affine case is the basis of **affine disturbance feedback** for linear systems. Given a linear constraint function $\bar{F}(\bar{u}, w) = a + b^\top \bar{u} + c^\top w$ one searches for controls $\bar{u}$ that can be "affinely adjusted" after the observation of the uncertainty, using affine disturbance feedback of the form $\bar{u}(u, w) := u_1 + D(u_2)w$. Here, the optimization variables $u = (u_1, u_2)$ consist of "nominal controls" u_1 and feedback parameters u_2 that enter linearly in the matrix $D(u_2)$. Inserting this into the original constraint function gives

$$F(u, w) := \bar{F}(\bar{u}(u, w), w) = a + b^\top u_1 + c^\top w + b^\top D(u_2)w$$

which is doubly affine so a variant of the "very favourable case 3".

A bit of history:

- idea of "affinely adjustable robust counterparts" goes back to [Ben-Tal, Goryashko, Guslitzer, Nemirovski. Adjustable robust solutions of uncertain linear programs. Math. Prog.. 99(2), 351–376, 2004]
- convex "affine disturbance feedback" was shown to be equivalent to nonconvex state feedback in robust MPC by [Goulart, Kerrigan, Maciejowski. Optimization over state feedback policies for robust control with constraints. Automatica 42(4): 523–533, 2006]
- was recently rediscovered - and generalized - in framework of "system level synthesis" [Anderson, Doyle, Low, Matni. System level synthesis. Ann. Rev. in Contr., vol. 47, pp. 364–393, 2019.]

Favourable Case 4: Convex inner problems

Convex uncertainty set, maximization function F concave in w

Assume convex uncertainty set (with non-empty interior) described by smooth convex inequalites $H : \mathbb{R}^{n_w} \to \mathbb{R}^{n_H}$

$$\mathbb{W} = \{w \in \mathbb{R}^{n_w} \mid H(w) \leq 0\}$$

and that each $F_i(u, w)$ is concave in w and smooth.

In this case, the inner maximization problem is convex, and its KKT conditions are necessary and sufficient to characterize each function's worst-case point $w_i^*(u)$. We define corresponding dual variables $\lambda_i \in \mathbb{R}^{n_H}$ and inner Lagrangian functions

$$\mathcal{L}_i(u, w_i, \lambda_i) := F_i(u, w) - \lambda_i^\top H(w)$$

KKT-optimality conditions for inner maximization problem

If a pair (w_i, λ_i) satisfies:

$$\nabla_w \mathcal{L}_i(u, w_i, \lambda_i) = 0$$
$$0 \leq \lambda_i \perp H(w_i) \leq 0$$

then:

$$F_i(u, w_i) = \max_{w \in \mathbb{W}} F_i(u, w)$$

Favourable Case 4: exact but naive formulation as MPCC

Convex uncertainty set, maximization function F concave in w

Augment problem with n_F primal-dual inner variables $W := (w_1, \ldots, w_{n_w}) \in \mathbb{R}^{n_w \times n_F}$ and $\Lambda := (\lambda_1, \ldots, \lambda_{n_F}) \in \mathbb{R}^{n_H \times n_F}$, and n_F lower level KKT-conditions as constraints.

Exact reformulation for inner convex maximization problems

$$
\begin{aligned}
\underset{u, W, \Lambda}{\text{minimize}} \quad & F_0(u) \\
\text{subject to} \quad & F_i(u, w_i) \leq 0 \quad \text{for} \quad i = 1, \ldots, n_F, \\
& \nabla_w \mathcal{L}_i(u, w_i, \lambda_i) = 0, \\
& \lambda_i \geq 0, \\
& H(w_i) \leq 0, \\
& -\lambda_i^\top H(w_i) = 0
\end{aligned}
$$

NLP with $n_u + n_F(n_w + n_H)$ variables, $n_F(n_w + 1)$ equalities, $n_F(1 + 2n_H)$ inequalities. The nonsmooth complementarity conditions make it an MPCC, which is difficult to solve.

Favourable Case 4: exact but naive formulation as MPCC

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\begin{aligned}
\underset{u, W, \Lambda}{\text{minimize}} \quad & F_0(u) \\
\text{subject to} \quad & F_i(u, w_i) - \lambda_i^\top H(w_i) \leq 0 \quad \text{for} \quad i = 1, \ldots, n_F, \\
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Exact reformulation for inner convex maximization problems

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& \lambda_i \geq 0,
\end{aligned}$$

NLP with $n_u + n_F(n_w + n_H)$ variables, $n_F(n_w + 1)$ equalities, $n_F(1 + 2n_H)$ inequalities. The nonsmooth complementarity conditions make it an MPCC, which is difficult to solve. Fortunately, one can reformulate this problem as a smooth NLP, as follows.

Favourable Case 4: exact but naive formulation as MPCC

Convex uncertainty set, maximization function F concave in w

Augment problem with n_F primal-dual inner variables $W := (w_1, \ldots, w_{n_w}) \in \mathbb{R}^{n_w \times n_F}$ and $\Lambda := (\lambda_1, \ldots, \lambda_{n_F}) \in \mathbb{R}^{n_H \times n_F}$, and n_F lower level KKT-conditions as constraints.

Exact reformulation for inner convex maximization problems

$$
\begin{aligned}
\underset{u, W, \Lambda}{\text{minimize}} \quad & F_0(u) \\
\text{subject to} \quad & F_i(u, w_i) - \lambda_i^\top H(w_i) \leq 0 \quad \text{for} \quad i = 1, \ldots, n_F, \\
& \nabla_w \mathcal{L}_i(u, w_i, \lambda_i) = 0, \\
& \lambda_i \geq 0,
\end{aligned}
$$

NLP with $n_u + n_F(n_w + n_H)$ variables, $n_F(n_w + 1)$ equalities, $n_F(1 + 2n_H)$ inequalities. The nonsmooth complementarity conditions make it an MPCC, which is difficult to solve. Fortunately, one can reformulate this problem as a smooth NLP, as follows.

In the final smooth NLP, the constraints encode n_F Wolfe dual problems.

Favourable Case 4: Lifted Wolfe Dual NLP Formulation

Convex uncertainty set, maximization function F concave in w

An exact and numerically well behaved NLP formulation is given by the "Lifted Wolfe Dual", cf. [Diehl, Houska, Stein, Steuermann. A lifting method for generalized semi-infinite programs based on lower level Wolfe duality, Comput. Optim. Appl., 54, pp. 189–210, 2013]

Lifted Wolfe NLP formulation (with optional constraints)

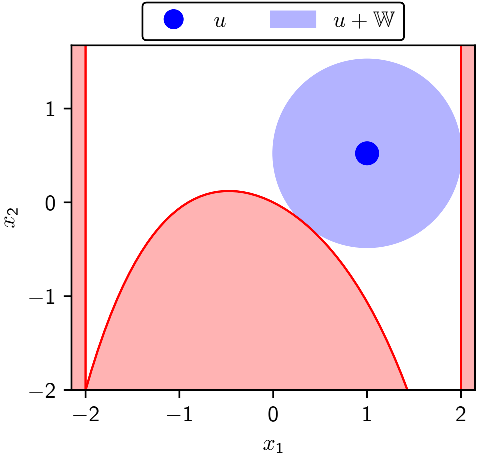
$$\begin{aligned} \underset{u,W,\Lambda}{\text{minimize}} \quad & F_0(u) \\ \text{subject to} \quad & \mathcal{L}_i(u,w_i,\lambda_i) \leq 0 \quad \text{for} \quad i=1,\ldots,n_F, \\ & \nabla_w \mathcal{L}_i(u,w_i,\lambda_i) = 0, \\ & \lambda_i \geq 0, \\ & (H(w_i) \leq 0) \quad \text{(optional constraints)} \end{aligned}$$

NLP with $n_u + n_F(n_w + n_H)$ variables, $n_F(n_w+1)$ equalities, $n_F(1+n_H)$ inequalities.

Optional constraints $H(w_i) \leq 0$ allow one to treat $F_i(u,w)$ which are only concave inside $\mathbb{W}$.

Case 4: Visualization of Inner Problem Convexity

F concave in w, with $c_1 = -1, c_2 = -1$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$



Favourable Case 5: Euclidean ball uncertainty, F quadratic in w

Maximization function F quadratic in w and $\mathbb{W}$ is an ℓ_2 ball

Assume uncertainty is a Euclidean norm ball

$$\mathbb{W} = \{w \in \mathbb{R}^{n_w} \mid \underbrace{(1/2)(\|w\|_2^2 - 1)}_{=:H(w)} \leq 0\}$$

and inner functions F are quadratic in w, i.e., equal their second order Taylor series in w

$$\boxed{F_i(u,w) = F_i(u,0) + \nabla_w F_i(u,0)^\top w + (1/2) w^\top \nabla_w^2 F_i(u,0) w}$$

If largest eigenvalue $\lambda_i^{\max}(u)$ of Hessian $\nabla_w^2 F_i(u,0) \in \mathbb{R}^{n_w \times n_w}$ is negative or zero, the inner functions are concave, so we would have Case 4. Otherwise, the quadratic function F_i is **not concave** in w.

Define "non-concavity constant" $L_i(u) := \max\{0, \lambda_i^{\max}(u)\}$.

Observation: the inner problem Lagrangian is quadratic in w, and concave for any $\lambda \geq L_i(u)$.

$$\begin{aligned} \mathcal{L}_i(u,w,\lambda) &= F(u,w) - \lambda H(w) \\ &= F_i(u,0) + (1/2)\lambda + \nabla_w F_i(u,0)^\top w + (1/2) w^\top (\nabla_w^2 F_i(u,0) - \lambda I) w \end{aligned}$$

Favourable Case 5: Exact NLP Formulation

Maximization function F quadratic in w and $\mathbb{W}$ is an ℓ_2 ball

Idea: replace $F_i(u,w)$ by concave $\tilde{F}_i(u,w) := \mathcal{L}_i(u,w,L_i(u))$, then treat as Case 4.

Lemma: $\boxed{\max_{w\in\mathbb{W}} \tilde{F}_i(u,w) = \max_{w\in\mathbb{W}} F(u,w)}$.

Proof is involved, based on S-Lemma [Yakubovich 1971], cf. literature on "trust region subproblem" e.g. [Beck, Vaisbourd. Globally Solving the Trust Region Subproblem Using Simple First-Order Methods, SIAM J. Optim., vol. 28, no. 3, pp. 1951–1967, 2018]

Exact Wolfe dual NLP formulation for F quadratic in w on ℓ_2-ball

$$\begin{aligned}
\underset{u,W,\Lambda}{\text{minimize}} \quad & F_0(u) \\
\text{subject to} \quad & \mathcal{L}_i(u,w_i,\lambda_i) \leq 0 \quad \text{for} \quad i=1,\ldots,n_F, \\
& \nabla_w \mathcal{L}_i(u,w_i,\lambda_i) = 0, \\
& \lambda_i \geq L_i(u)
\end{aligned}$$

with $\mathcal{L}_i(u,w,\lambda) = F(u,w) - (\lambda/2)(\|w\|_2^2 - 1)$ and $L_i(u) := \max\{0, \lambda^{\max}(\nabla_w^2 F_i(u,0))\}$

Note: last constraint equivalent to $\lambda_i \geq 0$ and matrix inequality $\lambda_i I \succeq \nabla_w^2 F_i(u,0)$

Case 5 application - min-max of jointly convex quadratic function

Regard a jointly convex quadratic function

$$F(u,w) = \frac{1}{2}\begin{bmatrix}1\\u\\w\end{bmatrix}^\top \begin{bmatrix}a & b^\top & c^\top\\ b & A & C^\top \\ c & C & B\end{bmatrix}\begin{bmatrix}1\\u\\w\end{bmatrix}$$

with non-concavity constant $L = \lambda^{\max}(B)$, and the min-max problem on ℓ_2-ball

$$\boxed{\min_{u\in\mathbb{R}^{n_u}} \max_{\|w\|_2\leq 1} F(u,w)}$$

What is its exact Wolfe dual formulation?

Case 5 application - min-max of jointly convex quadratic function

Regard a jointly convex quadratic function

$$F(u,w) = \frac{1}{2} \begin{bmatrix} 1 \\ u \\ w \end{bmatrix}^\top \begin{bmatrix} a & b^\top & c^\top \\ b & A & C^\top \\ c & C & B \end{bmatrix} \begin{bmatrix} 1 \\ u \\ w \end{bmatrix}$$

with non-concavity constant $L = \lambda^{\max}(B)$, and the min-max problem on ℓ_2-ball

$$\boxed{\min_{u \in \mathbb{R}^{n_u}} \max_{\|w\|_2 \leq 1} F(u,w)}$$

What is its exact Wolfe dual formulation?

Exact Wolfe dual NLP formulation for jointly convex quadratic F on ℓ_2-ball

$$\begin{aligned} \underset{u,\,w,\,\lambda}{\text{minimize}} \quad & F(u,w) - (\lambda/2)(\|w\|_2^2 - 1) \\ \text{subject to} \quad & \nabla_w F(u,w) - \lambda w = 0, \\ & \lambda \geq L \end{aligned}$$

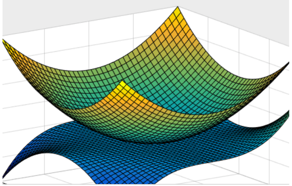
Overview

- Robust optimization problem as semi-infinite optimization problem and guiding example
- Five favourable cases (that can exactly be formulated as finite NLPs)
  1. Finite uncertainty
  2. Polytopic uncertainty, maximization function convex in uncertainty
  3. Affine, norm bounded uncertainty
  4. Concave maximization on convex uncertainty set
  5. Quadratic maximization on ℓ_2-ball
- **Approximate NLP formulations (conservative in special cases)**
  - Linearization (conservative in concave case)
  - Lagrangian relaxation (exact in concave and quadratic case)

From now on: smooth F with bounded non-concavity

Nonlinear robust optimization problem:

$$\begin{array}{ll} \underset{u \in \mathbb{U}}{\text{minimize}} & F_0(u) \\ \text{subject to} & \underset{w \in \mathbb{W}}{\max} F_i(u,w) \le 0, \quad i = 1, \ldots, n_F \end{array}$$



ASSUMPTION 1 - Smoothness on Uncertainty Set

All functions $F_i : \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}$ are twice continuously differentiable on the domain $\mathbb{U} \times \mathbb{W}$.

ASSUMPTION 2 - Bounded Non-Concavity

The convex hull $\bar{\mathbb{W}}$ of $\mathbb{W}$ contains the origin, and there exist smooth non-negative functions $L_i : \mathbb{U} \to \mathbb{R}$ so that for all $u \in \mathbb{U}$ and $w \in \bar{\mathbb{W}}$ holds

$$\boxed{w^\top \nabla_w^2 F_i(u,w) w \le L_i(u) \|w\|^2}$$

For concave F_i, the Hessian's eigenvalues are nonpositive, so we have $L_i(u) = 0$.

First Approach: Linearization (for norm ball uncertainty)

Regard norm ball uncertainty $\mathbb{W} = \{w \in \mathbb{R}^{n_w} \mid \|w\| \leq 1\}$ for arbitrary norm $\|\cdot\|$.

Using Taylor's theorem, for each $w \in \mathbb{W}$ there exists a $t \in [0,1]$ such that

$$F_i(u,w) = F_i(u,0) + \nabla_w F_i(u,0)^\top w + \frac{1}{2} \underbrace{w^\top \nabla_w^2 F_i(u,tw) w}_{\leq L_i(u)}.$$

This yields an upper bound, using the dual norm (as before in the affine case 3)

$$\underbrace{\max_{w \in \mathbb{W}} F_i(u,w)}_{=:\varphi_i(u)} \leq \underbrace{F_i(u,0) + \|\nabla_w F_i(u,0)\|_* + \frac{1}{2} L_i(u)}_{=:\tilde{\varphi}_i^{\mathrm{lin}}(u)}$$

Thus, we have obtained a conservative approximation.

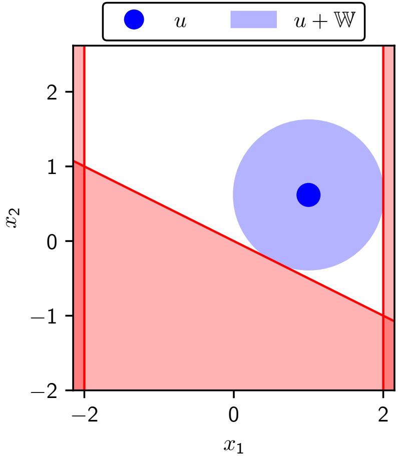
Approximation by Linearization (Conservative)

$$\begin{aligned} &\underset{u \in \mathbb{R}^{n_u}}{\text{minimize}} \quad F_0(u) \\ &\text{subject to } F_i(u,0) + \|\nabla_w F_i(u,0)\|_* + \frac{1}{2} L_i(u) \leq 0, \; i = 1, \dots, n_F. \end{aligned}$$

- in case of ℓ_2-norm, this is a nonlinear Second Order Cone Program (SOCP)
- solve with Newton-type methods e.g. Sequential Convex Programming (SCP)
- need high-order derivatives and sophisticated differentiation tools e.g. CasADi
- for dynamic systems, there exist different ways to obtain $\nabla_w F_i(u,0)$
  - forward sensitivities [Nagy & Braatz, JPC, 2004]
  - adjoint sensitivities [D., Bock, Kostina, Math. Prog., 2006]
  - Lyapunov matrix propagation [Houska & D., CDC, 2009], cf. RDO lecture
- if no Hessian bound is known, one can just set $L_i(u) = 0$, but looses feasibility guarantee

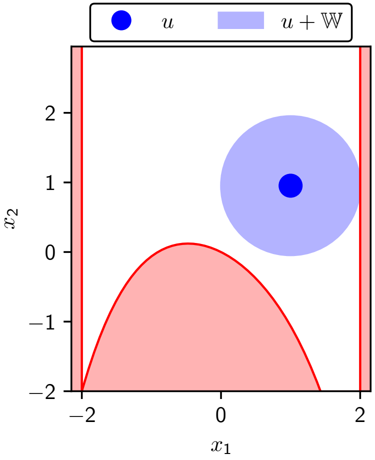
Linearization-based approximation - exact for affine F

F linear in w, with $c_1 = 0, c_2 = 0$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L(u) = 0$



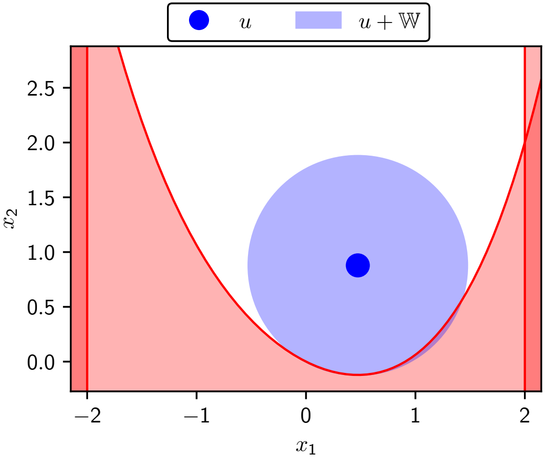
Linearization-based approximation - conservative, when F concave

F concave in w, with $c_1 = -1, c_2 = -1$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L(u) = 0$



Linearization-based approximation - underestimated non-concavity

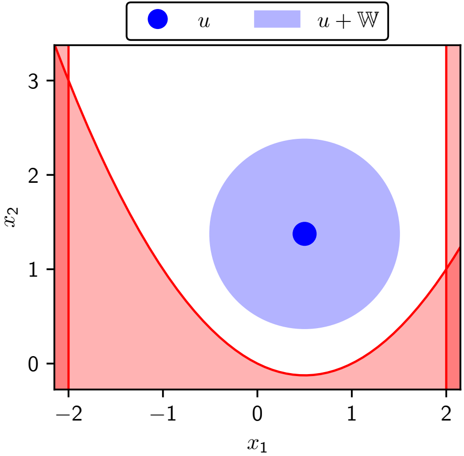
F convex in w, with $c_1 = 1, c_2 = 1$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L(u) = 0$



The approximation with $L_i(u) = 0$ is too optimistic because F is not concave.

Linearization-based approximation - correctly estimated non-concavity

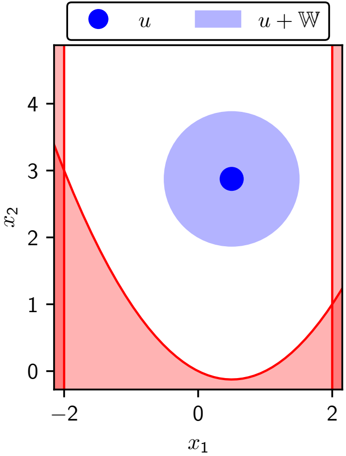
F convex in w, with $c_1 = 1, c_2 = 0$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L_i(u) = 1$



The approximation with $L_i(u) = 1$ is conservative, as predicted by the theory.

Linearization-based approximation - overestimated non-concavity

F convex in w, with $c_1 = 1, c_2 = 0$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L_i(u) = 4$



The approximation with (wrong) $L_i(u) = 4$ is (very) conservative, as predicted by the theory.

Linearization is conservative for convex obstacle constraints

Constraints for avoiding circular obstacles with center c and radius R have the form

$$F_i(u,w) = R^2 - \|P(u,w) - c\|_2^2 \le 0$$

If position function $P(u,w)$ is affine in w, function F_i is concave, so linearization and solution with $L_i(u) = 0$ delivers a conservative approximation (this is true for all convex obstacles).

We prefer the unsquared distance - which preserves concavity - because it leads to less conservatism [Carlos, Sartor, Zanelli, Diehl, Oriolo. Least Conservative Linearized Constraint Formulation for Real-Rime Motion Generation. IFAC WC, 2020]:

$$F_i(u,w) = R - \|P(u,w) - c\|_2 \le 0$$

To avoid nonsmoothness at $P(u,w) = c$, one can approximate it it with a small $\epsilon > 0$ as

$$\boxed{F_i(u,w) = R - \sqrt{\|P(u,w) - c\|_2^2 + \epsilon^2} \le 0}$$

Second Approach: Lagrangian Relaxation (here for ℓ_2-norm balls only)

Regard again the lower level (inner) maximization problem on ℓ_2-ball $\mathbb{W} = \{w \mid \|w\|_2 \leq 1\}$:

$$\boxed{\varphi_i(u) = \max_{w \in \mathbb{R}^{n_w}} F_i(u, w) \quad \text{s.t.} \quad \frac{1}{2}(w^\top w - 1) \leq 0}$$

Its Lagrangian is: $\mathcal{L}_i(u, w, \lambda) = F_i(u, w) - \frac{\lambda}{2}(w^\top w - 1)$.
Define a (modified) Lagrange dual function:

$$Q_i(u, \lambda) := \max_{w \in \mathbb{W}} \left(F_i(u, w) - \frac{\lambda}{2}(w^\top w - 1) \right)$$

Weak duality and relaxation gives an upper bound:

$$\varphi_i(u) \quad \leq \quad \min_{\lambda \geq 0} Q_i(u, \lambda)$$

Second Approach: Lagrangian Relaxation (here for ℓ_2-norm balls only)

Regard again the lower level (inner) maximization problem on ℓ_2-ball $\mathbb{W} = \{w \mid \|w\|_2 \leq 1\}$:

$$\boxed{\varphi_i(u) = \max_{w \in \mathbb{R}^{n_w}} F_i(u,w) \quad \text{s.t.} \quad \frac{1}{2}(w^\top w - 1) \leq 0}$$

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Define a (modified) Lagrange dual function:

$$Q_i(u,\lambda) := \max_{w \in \mathbb{W}} \left(F_i(u,w) - \frac{\lambda}{2}(w^\top w - 1) \right)$$

Weak duality and relaxation gives an upper bound:

$$\varphi_i(u) \quad \leq \quad \min_{\lambda \geq 0} Q_i(u,\lambda) \quad \leq \quad \boxed{\min_{\lambda \geq L_i(u)} Q_i(u,\lambda) =: \tilde{\varphi}_i^{\text{lagr}}(u)}$$

Second Approach: Lagrangian Relaxation (here for ℓ_2-norm balls only)

Regard again the lower level (inner) maximization problem on ℓ_2-ball $\mathbb{W} = \{w \mid \|w\|_2 \leq 1\}$:

$$\boxed{\varphi_i(u) = \max_{w \in \mathbb{R}^{n_w}} F_i(u,w) \quad \text{s.t.} \quad \frac{1}{2}(w^\top w - 1) \leq 0}$$

Its Lagrangian is: $\mathcal{L}_i(u,w,\lambda) = F_i(u,w) - \frac{\lambda}{2}(w^\top w - 1)$.
Define a (modified) Lagrange dual function:

$$Q_i(u,\lambda) := \max_{w \in \mathbb{W}} \left(F_i(u,w) - \frac{\lambda}{2}(w^\top w - 1) \right)$$

Weak duality and relaxation gives an upper bound:

$$\varphi_i(u) \quad \leq \quad \min_{\lambda \geq 0} Q_i(u,\lambda) \quad \leq \quad \boxed{\min_{\lambda \geq L_i(u)} Q_i(u,\lambda) =: \tilde{\varphi}_i^{\text{lagr}}(u)}$$

Requiring both $w \in \mathbb{W}$ and $\lambda \geq L_i(u)$ ensures that that the Hessian is always negative semi-definite (given that $L_i(u) \geq \max_{w \in \mathbb{W}} \lambda^{\max}(\nabla_w^2 F_i(u,w))$). Thus, we can use Wolfe duality, i.e., characterize maximizers by stationarity of the Lagrange gradient. [Houska & D., Math. Prog. Ser. A, 2013]

Approximate NLP formulation resulting from Lagrangian relaxation

Lagrangian relaxation based NLP

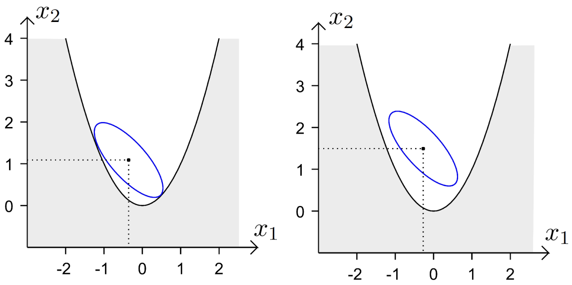
$$\begin{aligned} \underset{u,\lambda_1,w_1,\ldots,\lambda_{n_F},w_{n_F}}{\text{minimize}} \quad & F_0(u) \\ \text{subject to} \quad & F_i(u,w_i) - (\lambda_i/2)(w_i^\top w_i - 1) \leq 0, \\ & \nabla_w F_i(u,w_)) - \lambda_i w_i = 0, \\ & \lambda_j \geq L_i(u), \quad \|w_i\|_2^2 \leq 1, \quad i = 1,\ldots,n_F. \end{aligned}$$

- need $n_F(n_w+1)$ additional optimization variables (as in Favourable Cases 4 and 5)
- can use any NLP solver, or Sequential Convex Bilevel Programming (SCBP) [Houska & D., Math. Prog. Ser. A, 2013] (no third order derivatives needed for quadratic convergence)
- exact in two cases:

  (a) concave F_i with $L_i = 0$ and

  (b) nonconcave quadratic F_i with $L_i(u) = \lambda^{\max}\left(\nabla_w^2 F_i(0,0)\right)$ [Yakubovich, 1971]

How much conservatism is introduced?



THEOREM [Houska & D., Math. Prog. Ser. A, 2013] based on [Yakubovich, Vestnik Leningrad Univ., 1971]

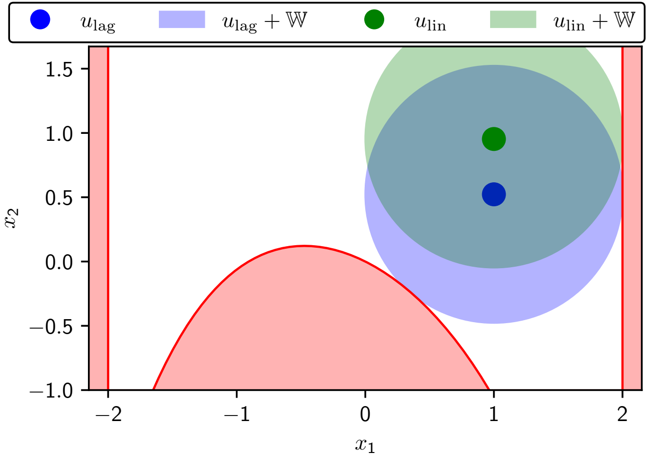
Given a valid Hessian bound $L_i(u)$, Lagrangian relaxation is always tighter than linearization:

$$\boxed{\varphi_i(u) \quad \leq \quad \tilde{\varphi}_i^{\text{lagr}}(u) \quad \leq \quad \tilde{\varphi}_i^{\text{lin}}(u)}$$

and exact if $F_i(u, w)$ is concave or quadratic in w and $L_i(u)$ is tight.

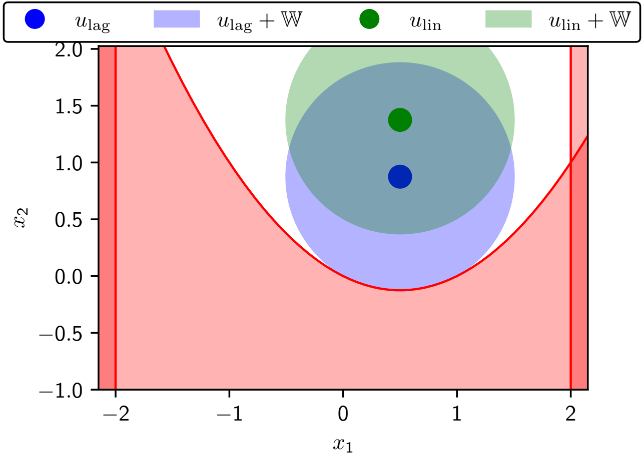
Lagrangian relaxation - exact, when F concave

F concave in w, with $c_1 = -1, c_2 = -1$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L(u) = 0$



Lagrangian relaxation - correctly estimated non-concavity

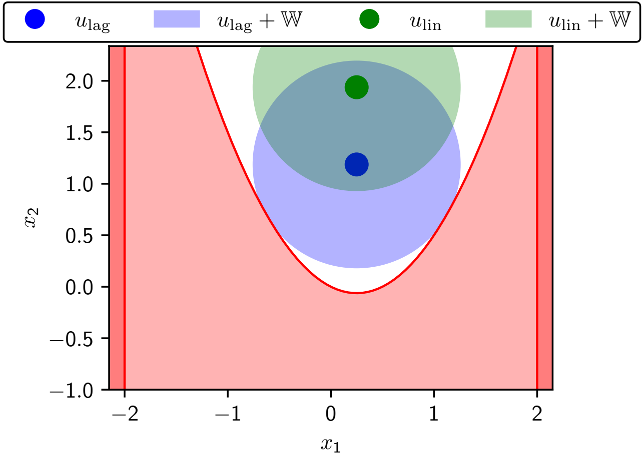
F convex quadratic in w, with $c_1 = 1, c_2 = 0$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L_i(u) = 1$



The approximation for quadratic F with correct $L_i(u) = 1$ is exact, as predicted by the theory.

Lagrangian relaxation - correctly estimated strong non-concavity

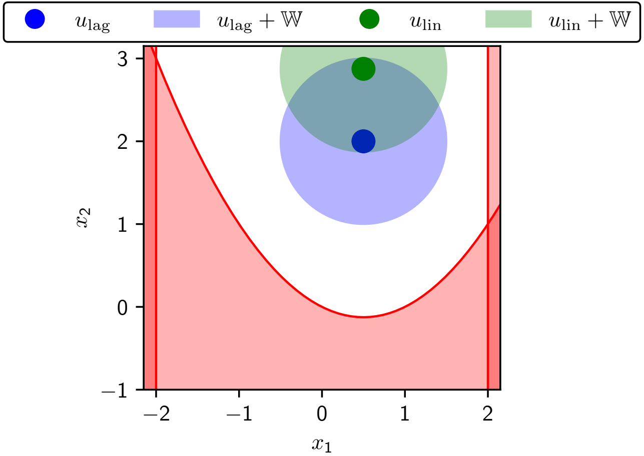
F convex quadratic in w, with $c_1 = 2, c_2 = 0$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L_i(u) = 2$



The approximation for quadratic F with correct $L_i(u) = 2$ is exact, as predicted by the theory

Lagrangian relaxation - overestimated non-concavity

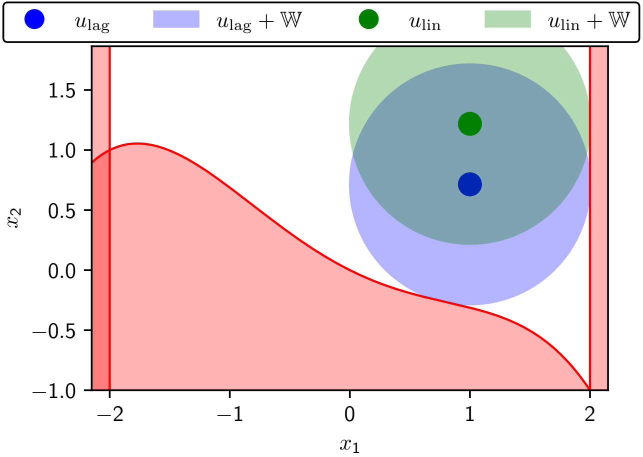
F convex quadratic in w, with $c_1 = 1, c_2 = 0$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L_i(u) = 4$



The approximation with too large bound $L_i(u) = 4$ is conservative, as predicted by the theory.

Lagrangian relaxation - inner problem not convex

F nonlinear in w, with $c_1 = 0.5, c_2 = -1$ and ℓ_2-norm uncertainty $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_2 \leq 1\}$, $L_i(u) = 1$



Lagrangian relaxation can be exact despite lower level nonconvexity and wrong estimate of non-concavity bound ($L_i(u) = 1$ while $c_1 = 0.5$).

Conclusions

- Robust optimization is in general a semi-infinite program (SIP)
- Five favourable cases can exactly be formulated as finite nonlinear programs (NLP)
  1. Finite uncertainty
  2. Polytopic uncertainty, maximization function convex in uncertainty
  3. Affine, norm bounded uncertainty (doubly affine case even leads to convex NLP)
  4. Concave maximization on convex uncertainty set
  5. Quadratic maximization on ℓ_2-ball
- Linearization-based approximation
  - based on dual norms
  - needs higher order derivatives that should be computed efficiently
  - exact in affine case
  - conservative in concave case (e.g. convex obstacle collision constraints)
  - basis for many robust MPC approaches
- Lagrangian relaxation: expensive, but tighter, and exact in concave and quadratic case

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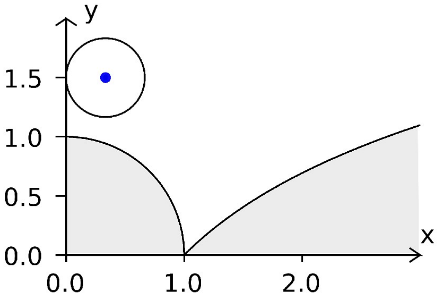
Appendix - Iterates of Sequential Convex Bilevel Programming (SCBP)

Les us look at the iterates of an efficient and quadratically convergent algorithm for Lagrangian relaxation problems called

Sequential Convex Bilevel Programming (SCBP)

introduced in [Houska & D., Math. Prog. Ser. A, 2013]

Tutorial Example (2 uncertainties, 3 constraints) solved by SCBP



$$\min_{x,y} \quad \left(x - \tfrac{1}{2}\right)^2 + y^2$$

subject to

$$\begin{cases} 0 \geq -x + w \\ 0 \geq 1 - (x+w)^2 - (y+v)^2 \\ 0 \geq \log(x+w) - (y+v) \end{cases}$$

Uncertainty Set:

Ball with radius $r = \frac{1}{3}$, $B(v,w) := v^2 + w^2 - r^2 \leq 0$

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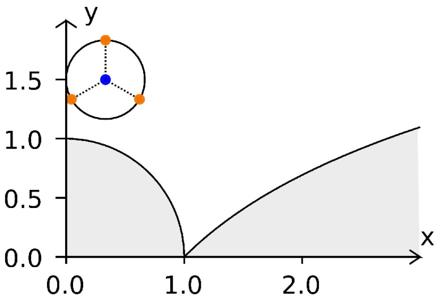

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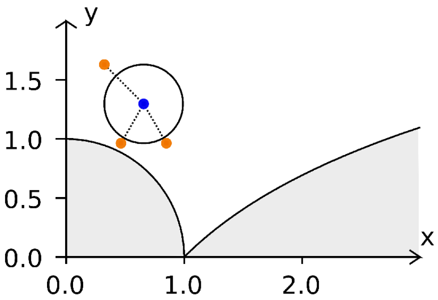

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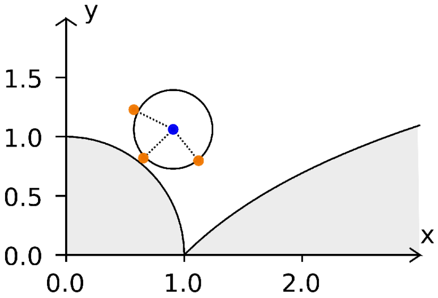

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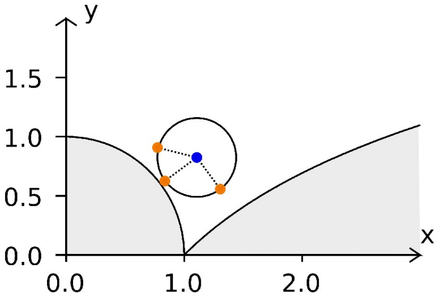

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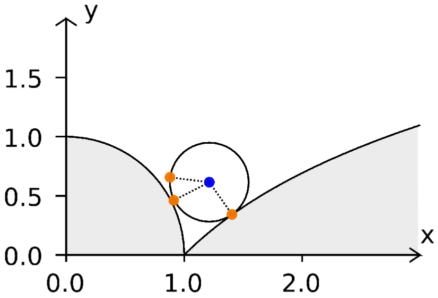

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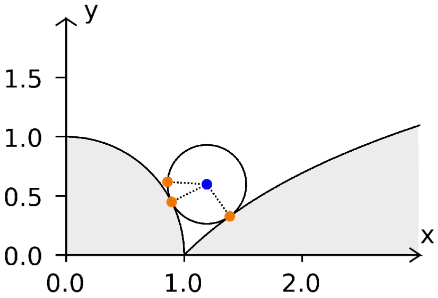

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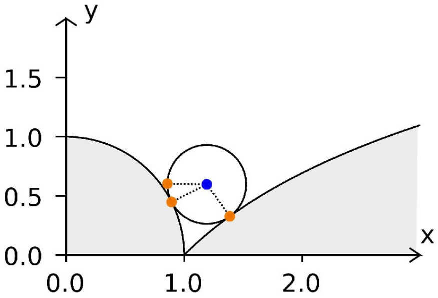

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Iteration	1	2	3	4	5	6	7	8
$-\log_{10}$(KKT-TOL)	0.3	0.5	0.7	1.0	1.5	3.4	7.0	12.1

Can achieve high accuracy