

# Lecture 6: Finite Elements with Switch Detection for Filippov Systems

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Systems Control and Optimization Laboratory (syscop)  
Summer School on Direct Methods for Optimal Control of Nonsmooth Systems  
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**universität freiburg**

# Outline of the lecture



- 1 Time stepping and smoothing in nonsmooth optimal control
- 2 Finite Elements with Switch Detection (FESD)
- 3 FESD-Discretization of Optimal Control Problems
- 4 Conclusions and summary

# How to discretize optimal control problems subject to Filippov systems?



In direct optimal control, we first discretize, and then solve a finite-dimensional nonlinear program.

Original optimal control problem  
in continuous time

$$\begin{aligned} \min_{x(\cdot), u(\cdot)} \quad & \int_0^T L(x, u) dt + E(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) \in F_{\mathbb{F}}(x(t), u(t)) \\ & 0 \geq h(x(t), u(t)), \quad t \in [0, T] \\ & 0 \geq r(x(T)) \end{aligned}$$

Assume smooth (convex)  $L, E, h, r$   
Nonsmooth dynamics make problem  
nonconvex.

# How to discretize optimal control problems subject to Filippov systems?



In direct optimal control, we first discretize, and then solve a finite-dimensional nonlinear program.

Optimal control problem  
with Stewart's formulation

$$\begin{aligned} \min_{\substack{x(\cdot), u(\cdot), \\ \theta(\cdot), \lambda(\cdot), \mu(\cdot)}} \quad & \int_0^T L(x, u) dt + E(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = F(x(t), u(t)) \theta(t) \\ & 0 = G_{\text{LP}}(x(t), \theta(t), \lambda(t), \mu(t)) \\ & 0 \geq h(x(t), u(t)), \quad t \in [0, T] \\ & 0 \geq r(x(T)) \end{aligned}$$

Assume smooth (convex)  $L, E, h, r$

Nonsmooth dynamics make problem  
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# How to discretize optimal control problems subject to Filippov systems?



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Assume smooth (convex)  $L, E, h, r$   
Nonsmooth dynamics make problem  
nonconvex.

Goal: discretized optimal control problem  
(an NLP)

$$\begin{aligned} \min_{s, z, u} \sum_{k=0}^{N-1} \Phi_L(s_k, z_k, u_k) + E(s_N) \\ \text{s.t. } s_0 = \bar{x}_0 \\ s_{k+1} = \Phi_f(s_k, z_k, u_k) \\ 0 = \Phi_{\text{int}}(s_k, z_k, u_k) \\ 0 \geq h(s_k, u_k), k = 0, \dots, N-1 \\ 0 \geq r(s_N) \end{aligned}$$

Variables  $s = (s_0, \dots)$ ,  $z = (z_0, \dots)$  and  
 $u = (u_0, \dots, u_{N-1})$   
Nonsmooth  $\Phi_{\text{int}}$

What happens if we use time stepping methods  
in direct optimal control?

# Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]



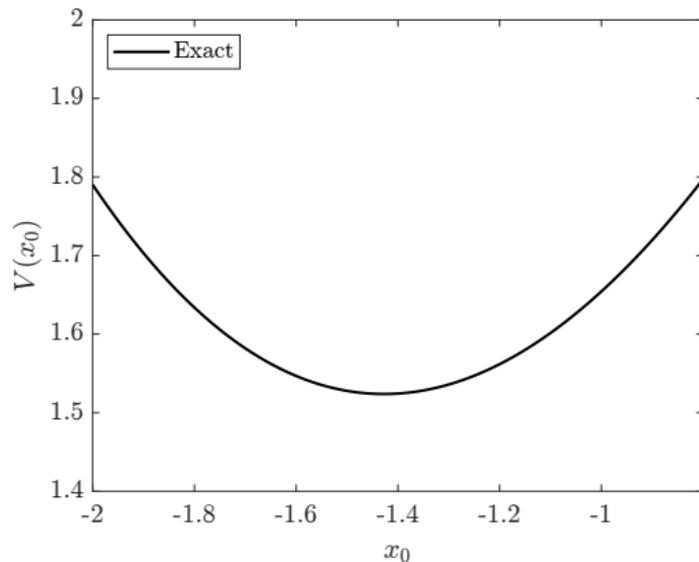
## Continuous-time OCP

$$\begin{aligned} \min_{x(\cdot) \in C^0([0,2])} & \int_0^2 x(t)^2 dt + (x(2) - 5/3)^2 \\ \text{s.t.} & \dot{x}(t) = 2 - \text{sign}(x(t)), \quad t \in [0, 2] \end{aligned}$$

Free initial value  $x(0)$  is the effective degree of freedom.

## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V(x_0)$$



- Denote by  $V(x_0)$  the nonsmooth objective value for the unique feasible trajectory starting at  $x(0) = x_0$ .

# Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]



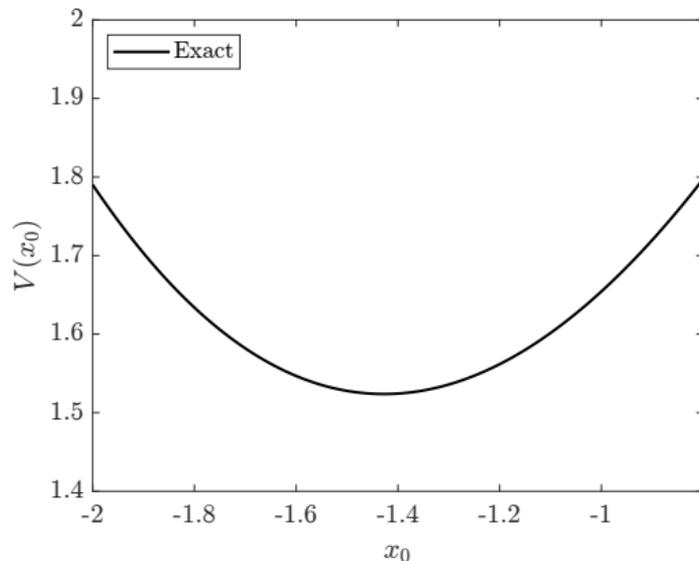
## Continuous-time OCP

$$\begin{aligned} \min_{x(\cdot), \lambda(\cdot), s(\cdot)} \quad & \int_0^2 x(t)^2 dt + (x(2) - 5/3)^2 \\ \text{s.t.} \quad & \dot{x}(t) = 2 - s(t) \\ & 0 \leq \lambda(t) - x(t) \perp 1 + s(t) \geq 0 \\ & 0 \leq \lambda(t) \perp 1 - s(t) \geq 0, \quad t \in [0, 2] \end{aligned}$$

Free initial value  $x(0)$  is the effective degree of freedom.

## Equivalent reduced problem

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# Direct optimal control with a time stepping IRK discretization

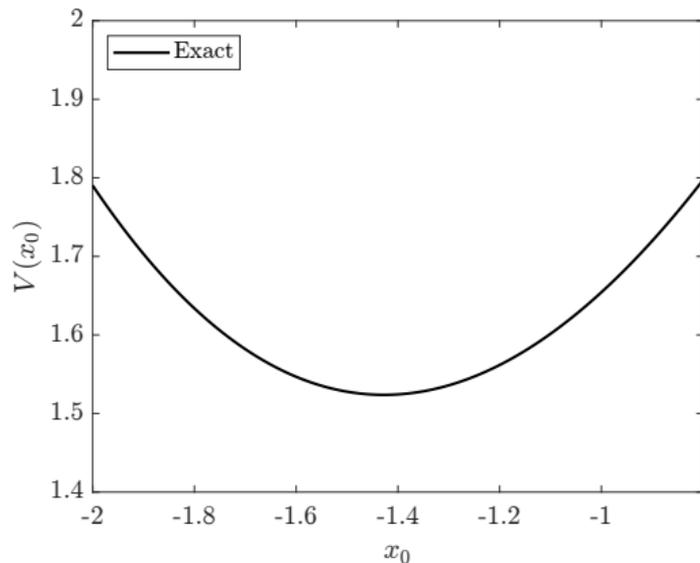
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## Continuous-time OCP

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- ▶ discretize the DCS with **fixed step size** IRK methods
- ▶ e.g., midpoint rule, Gauss-Legendre IRK with  $n_s = 1$ , accuracy  $O(h^2)$



Locally quadratic objective.

# Direct optimal control with a time stepping IRK discretization

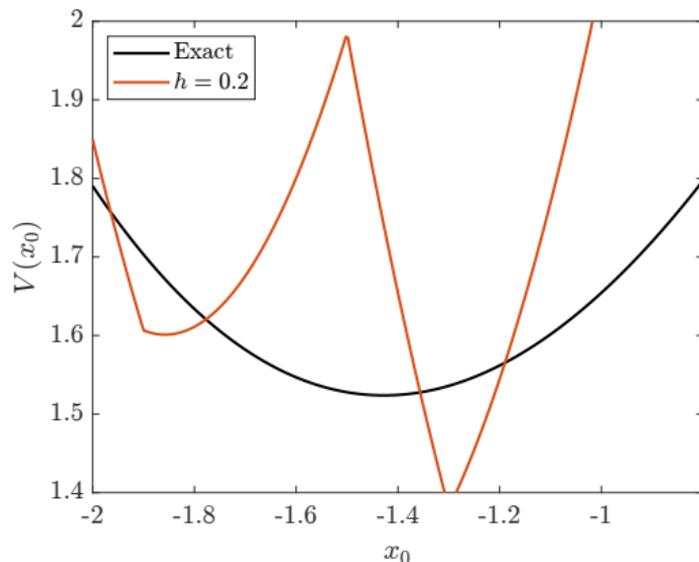
Tutorial example inspired by [Stewart & Anitescu, 2010]



## Discrete-time OCP

$$\begin{aligned} \min_{\mathbf{x}, \mathbf{z}} \quad & \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2 \\ \text{s.t.} \quad & x_{n+1} = \phi_f(x_n, z_n) \\ & 0 = \phi_{\text{int}}(x_n, z_n), \quad n = 0, \dots, N-1 \end{aligned}$$

- ▶ discretize the DCS with **fixed step size** IRK methods
- ▶ e.g., midpoint rule, Gauss-Legendre IRK with  $n_s = 1$ , accuracy  $O(h^2)$
- ▶ step size  $h = 0.2$ , i.e.,  $N = 10$  **integration steps**



Many artificial local minima and wrong derivatives.

# Direct optimal control with a time stepping IRK discretization

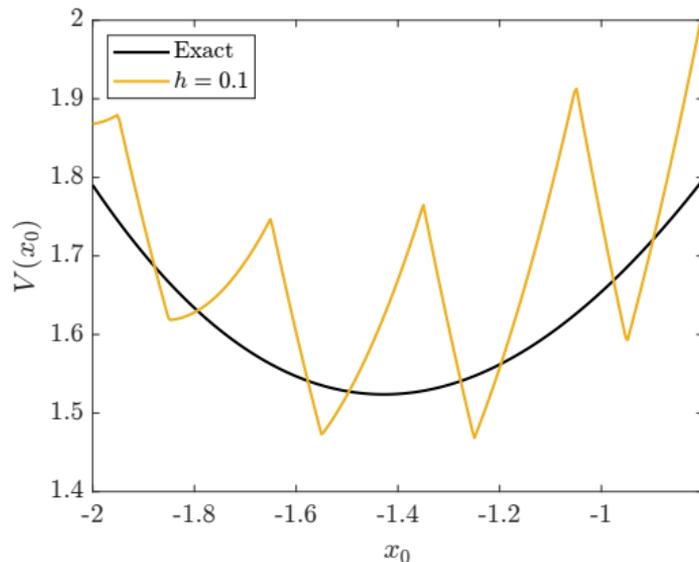
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- ▶ discretize the DCS with **fixed step size** IRK methods
- ▶ e.g., midpoint rule, Gauss-Legendre IRK with  $n_s = 1$ , accuracy  $O(h^2)$
- ▶ step size  $h = 0.1$ , i.e.,  $N = 20$  **integration steps**



Many artificial local minima and wrong derivatives.

# Direct optimal control with a time stepping IRK discretization

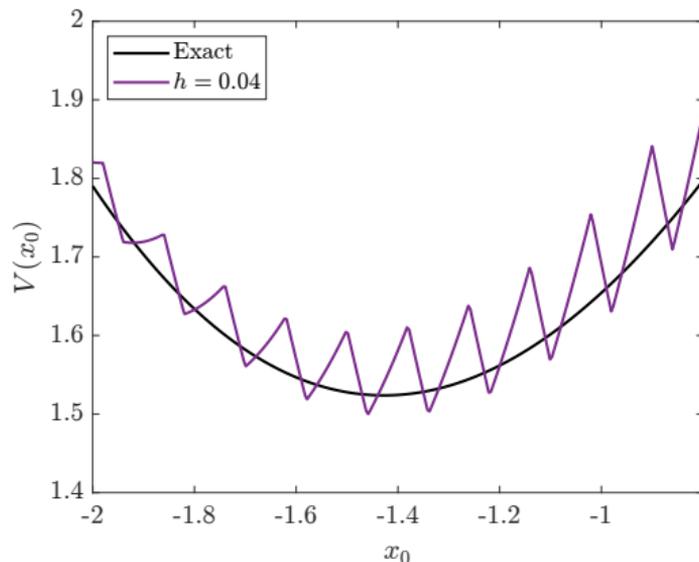
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## Discrete-time OCP

$$\begin{aligned} \min_{\mathbf{x}, \mathbf{z}} \quad & \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2 \\ \text{s.t.} \quad & x_{n+1} = \phi_f(x_n, z_n) \\ & 0 = \phi_{\text{int}}(x_n, z_n), \quad n = 0, \dots, N-1 \end{aligned}$$

- ▶ discretize the DCS with **fixed step size** IRK methods
- ▶ e.g., midpoint rule, Gauss-Legendre IRK with  $n_s = 1$ , accuracy  $O(h^2)$
- ▶ step size  $h = 0.04$ , i.e.,  $N = 50$  **integration steps**



Many artificial local minima and wrong derivatives.

# Direct optimal control with a time stepping IRK discretization

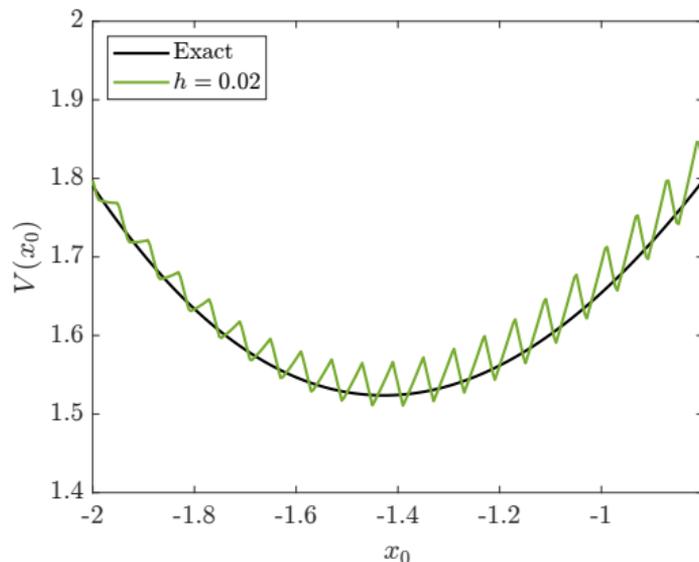
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## Discrete-time OCP

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- ▶ discretize the DCS with **fixed step size** IRK methods
- ▶ e.g., midpoint rule, Gauss-Legendre IRK with  $n_s = 1$ , accuracy  $O(h^2)$
- ▶ step size  $h = 0.02$ , i.e.,  $N = 100$  **integration steps**



Many artificial local minima and wrong derivatives.

# Direct optimal control with a time stepping IRK discretization

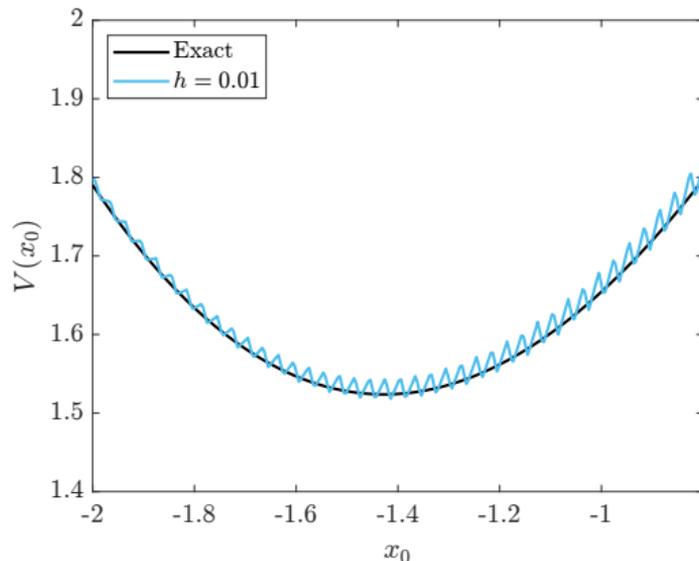
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- ▶ discretize the DCS with **fixed step size** IRK methods
- ▶ e.g., midpoint rule, Gauss-Legendre IRK with  $n_s = 1$ , accuracy  $O(h^2)$
- ▶ step size  $h = 0.01$ , i.e.,  $N = 200$  **integration steps**

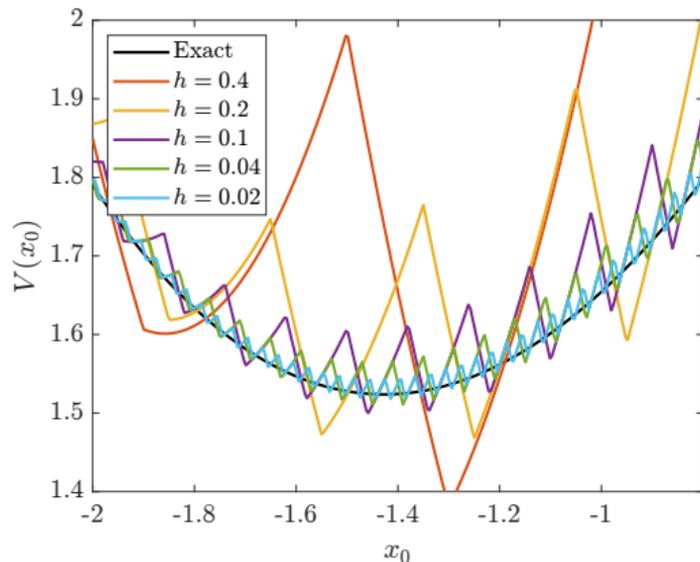


Many artificial local minima and wrong derivatives.

## Discrete-time OCP

$$\begin{aligned} \min_{\mathbf{x}, \mathbf{z}} \quad & \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2 \\ \text{s.t.} \quad & x_{n+1} = \phi_f(x_n, z_n) \\ & 0 = \phi_{\text{int}}(x_n, z_n), \quad n = 0, \dots, N-1 \end{aligned}$$

- ▶ discretize the DCS with **fixed step size** IRK methods
- ▶ e.g., midpoint rule, Gauss-Legendre IRK with  $n_s = 1$ , accuracy  $O(h^2)$
- ▶ decreasing the step size might worsen the situation



Many artificial local minima and wrong derivatives.

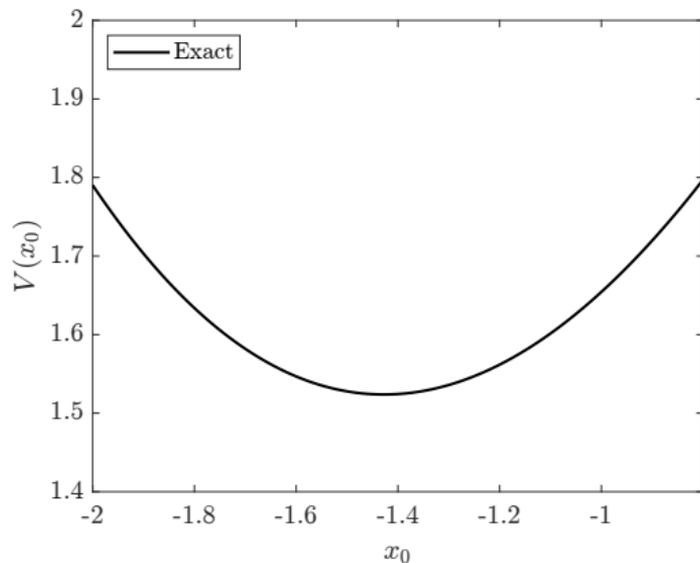
What happens if we use smoothed models in direct optimal control?



## Continuous-time OCP

$$\begin{aligned} \min_{x(\cdot) \in \mathcal{C}^0([0,2])} & \int_0^2 x(t)^2 dt + (x(2) - 5/3)^2 \\ \text{s.t.} & \dot{x}(t) = 2 - \text{sign}(x(t)), \quad t \in [0, 2] \end{aligned}$$

- ▶ midpoint rule, with  $h = 0.05$ ;  $N = 40$





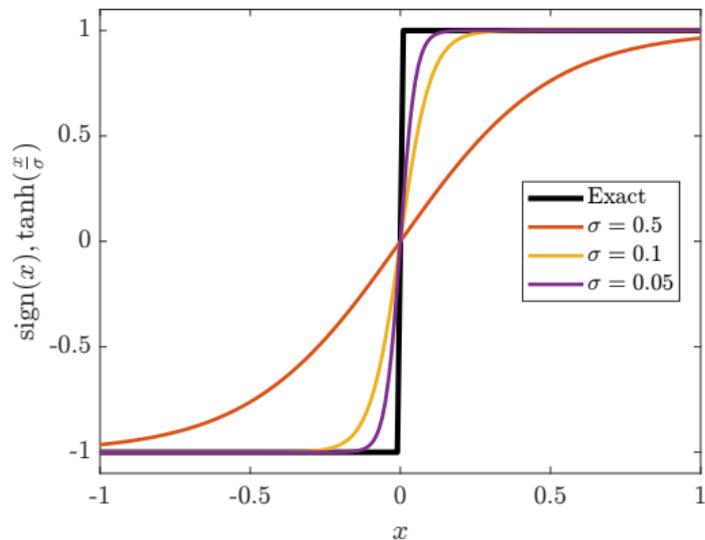
## Smoothed continuous-time OCP

$$\begin{aligned} \min_{x(\cdot) \in \mathcal{C}^\infty([0,2])} & \int_0^2 x(t)^2 dt + (x(2) - 5/3)^2 \\ \text{s.t.} & \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0, 2] \end{aligned}$$

## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- ▶ midpoint rule, with  $h = 0.05$ ;  $N = 40$
- ▶ solve smoothed OCP for different  $\sigma$





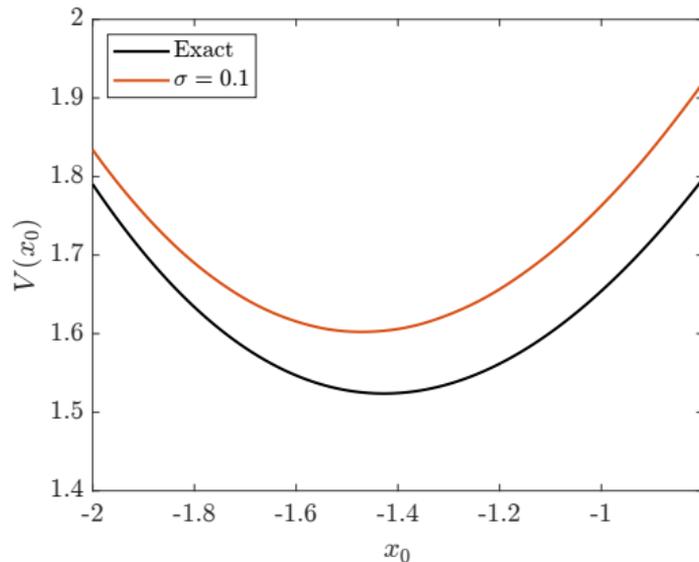
## Smoothed continuous-time OCP

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## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- ▶ midpoint rule, with  $h = 0.05$ ;  $N = 40$
- ▶ solve smoothed OCP with  $\sigma = 0.1$





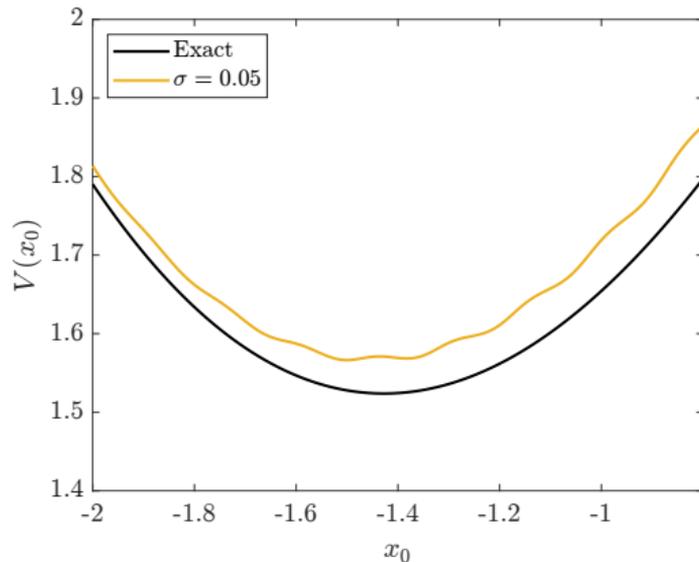
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## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- ▶ midpoint rule, with  $h = 0.05$ ;  $N = 40$
- ▶ solve smoothed OCP with  $\sigma = 0.05$





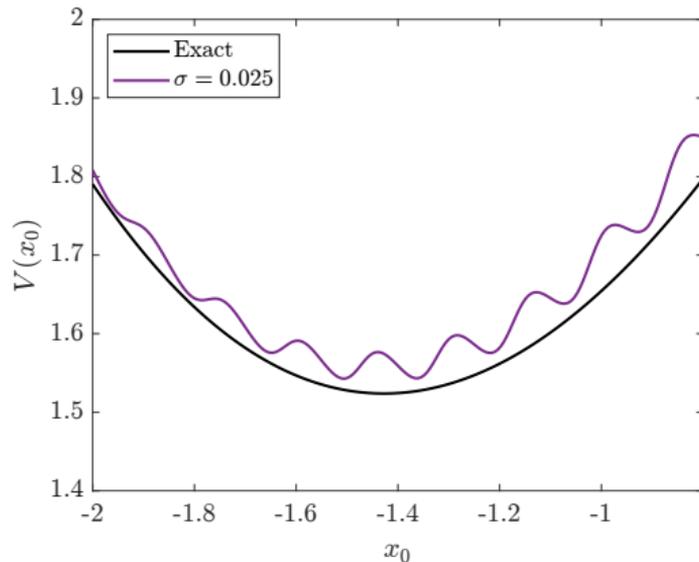
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## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- ▶ midpoint rule, with  $h = 0.05$ ;  $N = 40$
- ▶ solve smoothed OCP with  $\sigma = 0.025$





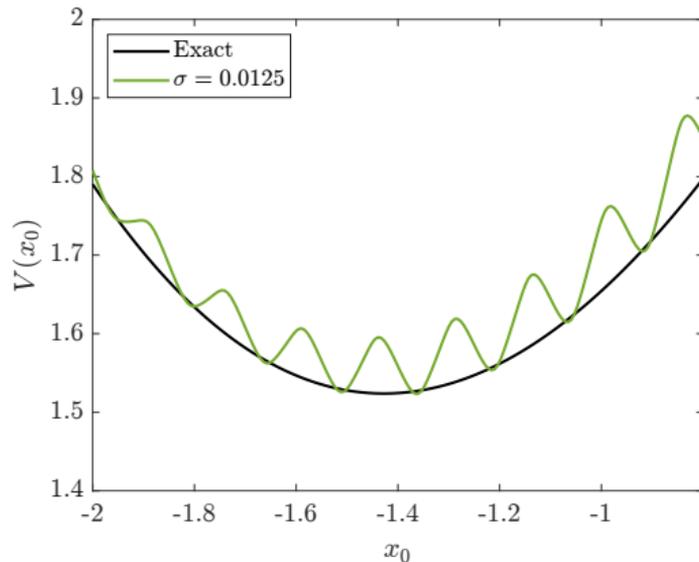
## Smoothed continuous-time OCP

$$\begin{aligned} \min_{x(\cdot) \in \mathcal{C}^\infty([0,2])} & \int_0^2 x(t)^2 dt + (x(2) - 5/3)^2 \\ \text{s.t.} \quad \dot{x}(t) &= 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0, 2] \end{aligned}$$

## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- ▶ midpoint rule, with  $h = 0.05$ ;  $N = 40$
- ▶ solve smoothed OCP with  $\sigma = 0.0125$





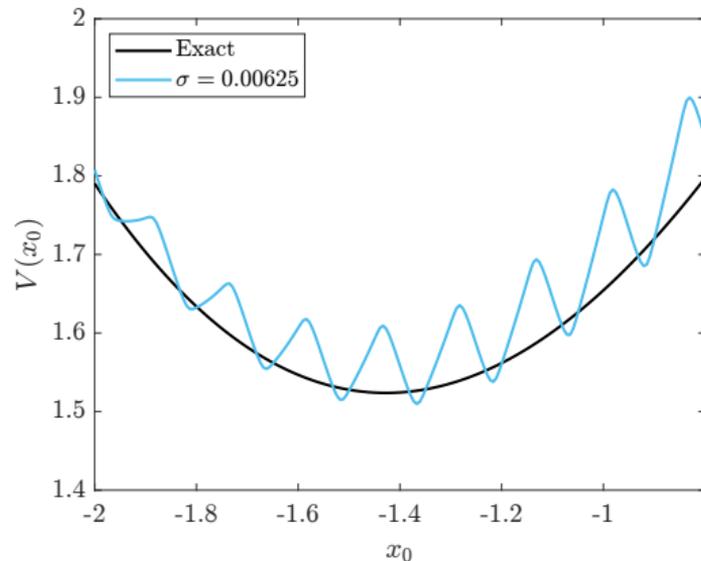
## Smoothed continuous-time OCP

$$\begin{aligned} \min_{x(\cdot) \in \mathcal{C}^\infty([0,2])} & \int_0^2 x(t)^2 dt + (x(2) - 5/3)^2 \\ \text{s.t.} & \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0, 2] \end{aligned}$$

## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- ▶ midpoint rule, with  $h = 0.05$ ;  $N = 40$
- ▶ solve smoothed OCP with  $\sigma = 0.00625$





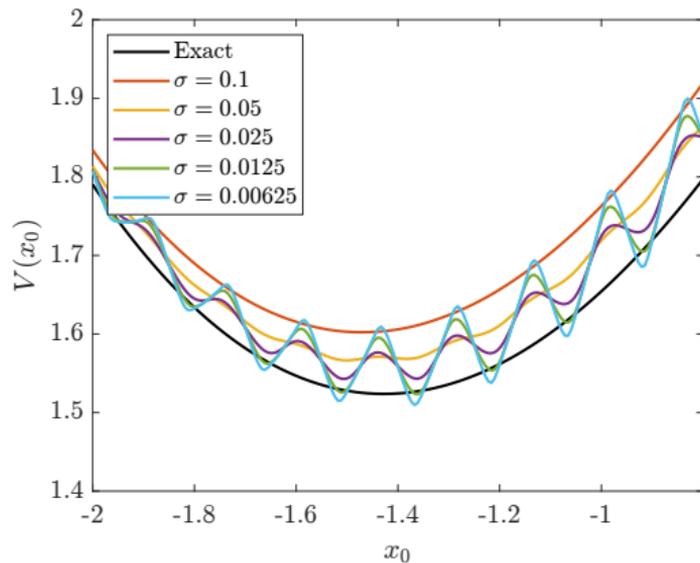
## Smoothed continuous-time OCP

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## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- ▶ midpoint rule, with  $h = 0.05$ ;  $N = 40$



If  $h \gg \sigma$ , then the smooth approximation behaves **the same as the nonsmooth problem!**



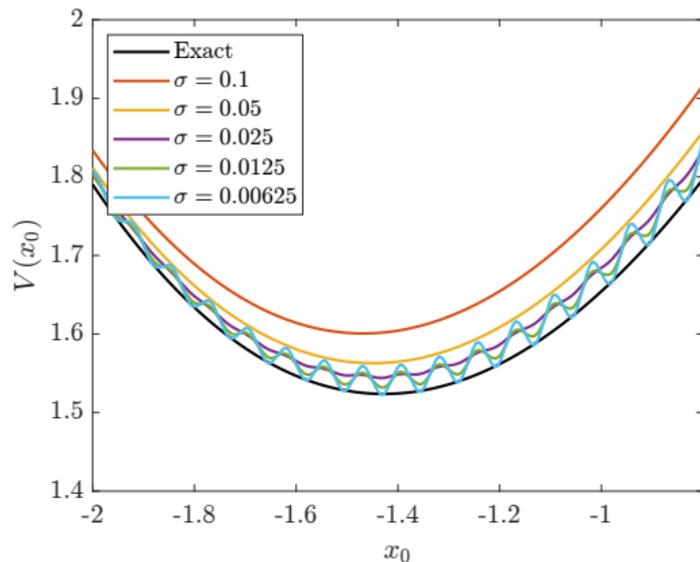
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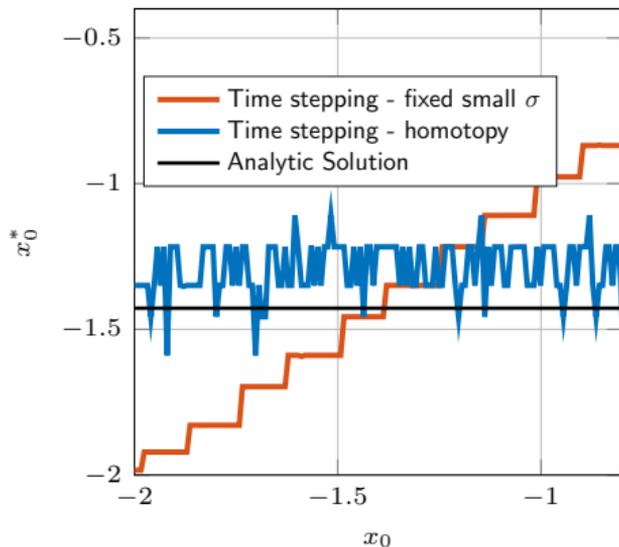
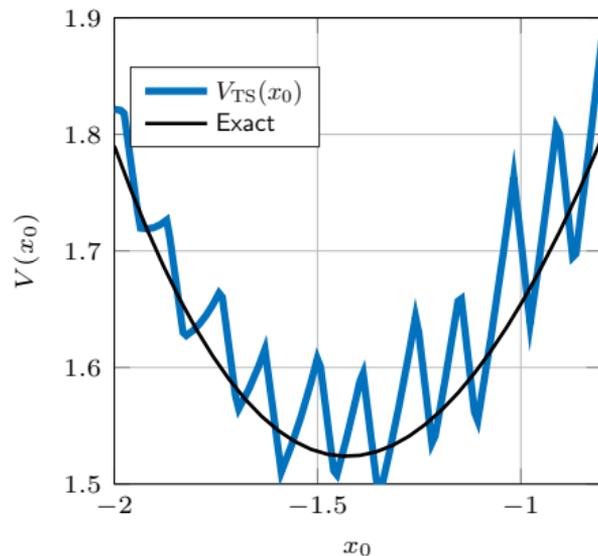
## Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

► midpoint rule, with  $h = 0.025$ ;  $N = 80$



If  $h \gg \sigma$ , then the smooth approximation behaves **the same as the nonsmooth problem!**



- ▶ spurious local minima, optimizer gets trapped close to initialization
- ▶ sensitivity only correct if step sizes are smaller than smoothing parameter [Stewart & Anitescu, 2010]: homotopy improves convergence
- ▶ even for the best local minimizer, only  $O(h)$  accuracy can be expected



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# Main ideas of FESD

Based on [Baumrucker & Biegler, 2009], [N. et. al, 2022, 2022a, 2023]



## FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)

$$\dot{x} \in F_F(x, u)$$



$$\begin{aligned}\dot{x} &= F(x, u)\theta \\ 0 &= G_{\text{DCS}}(x, z, \theta)\end{aligned}$$

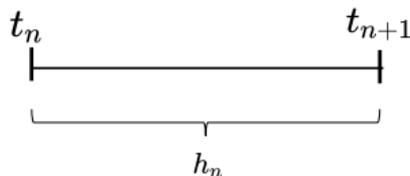
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## FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements



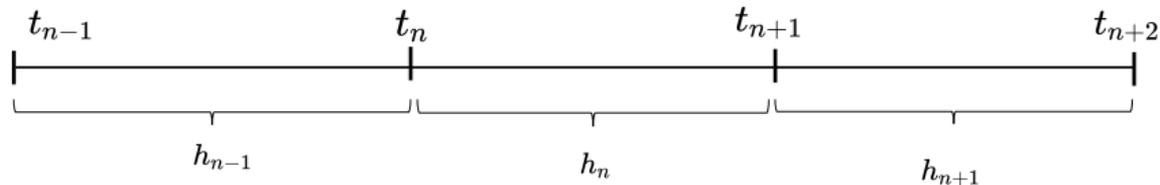
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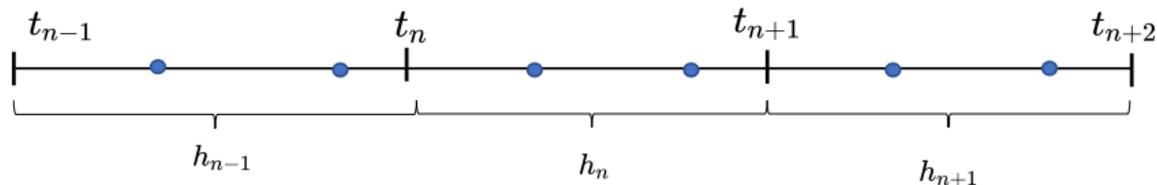
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## FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2 and 3)



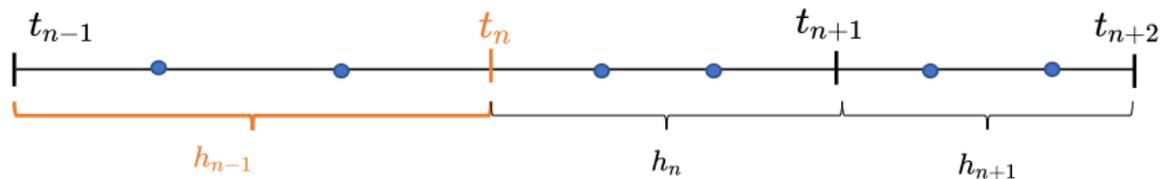
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## FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2 and 3)
4. Let step sizes  $h_n$  be degrees of freedom (under-determined system)



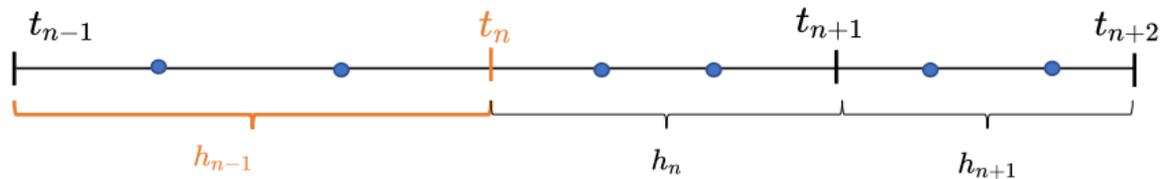
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## FESD overview

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2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2 and 3)
4. Let step sizes  $h_n$  be degrees of freedom
5. Cross complementarity conditions - adapt  $h_n$  for switch detection



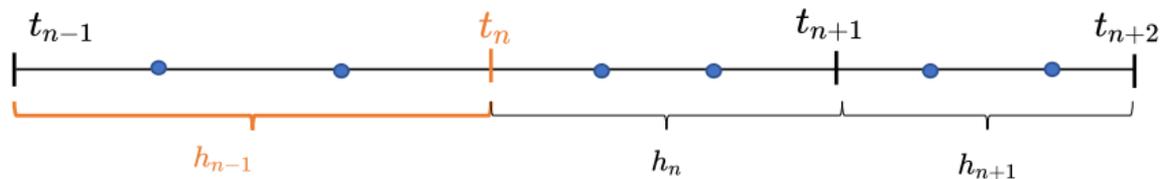
# Main ideas of FESD

Based on [Baumrucker & Biegler, 2009], [N. et. al, 2022, 2022a, 2023]



## FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2 and 3)
4. Let step sizes  $h_n$  be degrees of freedom
5. Cross complementarity conditions - adapt  $h_n$  for switch detection
6. Step equilibration - remove degrees of freedom if no switch



# Recap on Filippov Convexification



Switched ODE not well-defined on region boundaries  $\partial R_i$ . Idea by A.F. Filippov (1923-2006): replace ODE by differential inclusion, using convex combination of neighboring vector fields.

## Filippov Differential Inclusion

$$\dot{x} \in F_F(x, u) := \left\{ \begin{array}{l} \sum_{i=1}^{n_f} f_i(x, u) \theta_i \quad \left| \quad \sum_{i=1}^{n_f} \theta_i = 1, \right. \\ \theta_i \geq 0, \quad i = 1, \dots, n_f, \\ \left. \theta_i = 0, \quad \text{if } x \notin \overline{R_i} \right\}$$



Aleksei F. Filippov  
(1923-2006)  
image source: wikipedia

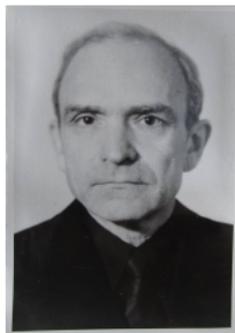
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Aleksei F. Filippov  
(1923-2006)  
image source: wikipedia

- ▶ for interior points  $x \in R_i$  nothing changes:  $F_F(x, u) = \{f_i(x, u)\}$
- ▶ Provides meaningful generalization on region boundaries.  
E.g. on  $\overline{R_1} \cap \overline{R_2}$  both  $\theta_1$  and  $\theta_2$  can be nonzero



## LP representation

$$\dot{x} = F(x, u) \theta$$

$$\begin{aligned} \text{with } \theta \in \operatorname{argmin}_{\tilde{\theta} \in \mathbb{R}^{n_f}} \quad & g(x)^\top \tilde{\theta} \\ \text{s.t.} \quad & 0 \leq \tilde{\theta} \\ & 1 = e^\top \tilde{\theta} \end{aligned}$$

where

$$F(x, u) := [f_1(x, u), \dots, f_{n_f}(x, u)] \in \mathbb{R}^{n_x \times n_f}$$

$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$

$$e := [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}$$

# From Filippov to dynamic complementarity systems

Using the KKT conditions of the parametric LP



## LP representation

$$\dot{x} = F(x, u) \theta$$

$$\text{with } \theta \in \underset{\tilde{\theta} \in \mathbb{R}^{n_f}}{\text{argmin}} \quad g(x)^\top \tilde{\theta}$$

$$\text{s.t. } 0 \leq \tilde{\theta}$$

$$1 = e^\top \tilde{\theta}$$

where

$$F(x, u) := [f_1(x, u), \dots, f_{n_f}(x, u)] \in \mathbb{R}^{n_x \times n_f}$$

$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$

$$e := [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}$$

Express equivalently by optimality conditions:

## Dynamic Complementarity System (DCS)

$$\dot{x} = F(x, u) \theta \quad (1a)$$

$$0 = g(x) - \lambda - e\mu \quad (1b)$$

$$0 \leq \theta \perp \lambda \geq 0 \quad (1c)$$

$$1 = e^\top \theta \quad (1d)$$

## Compact notation

$$\dot{x} = F(x, u) \theta$$

$$0 = G_{\text{LP}}(x, \theta, \lambda, \mu),$$

- ▶  $\mu \in \mathbb{R}$  and  $\lambda \in \mathbb{R}^{n_f}$  are Lagrange multipliers
- ▶ (1c)  $\Leftrightarrow \min\{\theta, \lambda\} = 0 \in \mathbb{R}^{n_f}$
- ▶ Together, (1b), (1c), (1d) determine the  $(2n_f + 1)$  variables  $(\theta, \lambda, \mu)$  uniquely



## Continuous time DCS

$$x(0) = \bar{x}_0,$$

$$\dot{x}(t) = v(t)$$

$$v(t) = F(x(t), u(t)) \theta(t)$$

$$0 = g(x(t)) - \lambda(t) - e\mu(t)$$

$$0 \leq \theta(t) \perp \lambda(t) \geq 0$$

$$1 = e^\top \theta(t), \quad t \in [0, T]$$

# Conventional discretization by Implicit Runge Kutta (IRK) method

## Continuous time DCS

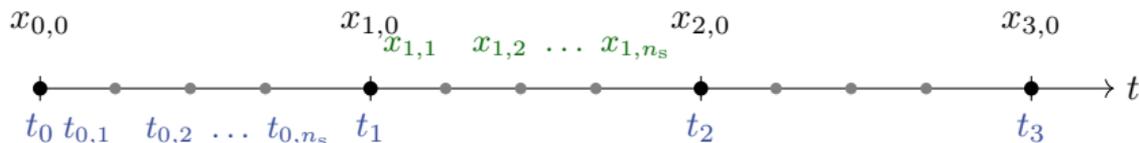
$$\begin{aligned}
 x(0) &= \bar{x}_0, \\
 \dot{x}(t) &= v(t) \\
 v(t) &= F(x(t), u(t)) \theta(t) \\
 0 &= g(x(t)) - \lambda(t) - e\mu(t) \\
 0 &\leq \theta(t) \perp \lambda(t) \geq 0 \\
 1 &= e^\top \theta(t), \quad t \in [0, T]
 \end{aligned}$$

## Discrete time IRK-DCS equation

$$\begin{aligned}
 x_{0,0} &= \bar{x}_0, \quad x_{n+1,0} = x_{n,0} + h \sum_{i=1}^{n_s} b_i v_{n,i} \\
 x_{n,i} &= x_{n,0} + h \sum_{j=1}^{n_s} a_{i,j} v_{n,j} \\
 v_{n,i} &= F(x_{n,i}, u_{n,i}) \theta_{n,i} \\
 0 &= g(x_{n,i}) - \lambda_{n,i} - e\mu_{n,i} \\
 0 &\leq \theta_{n,i} \perp \lambda_{n,i} \geq 0 \\
 1 &= e^\top \theta_{n,i}, \quad i = 1, \dots, n_s, \quad n = 0, \dots, N-1
 \end{aligned}$$

Notation:  $x_{n,i} \in \mathbb{R}^{n_x}$ ,  $\theta_{n,i} \in \mathbb{R}^m$  etc. RK stage values with:

- ▶  $n \in \{0, 1, \dots, N\}$  - index of integration step; step length  $h := T/N$
- ▶  $i, j \in \{0, 1, \dots, n_s\}$  - index of intermediate IRK stage / collocation point
- ▶  $a_{i,j}$  and  $b_i$  - Butcher tableau entries of Implicit Runge Kutta method



# Conventional time stepping - illustrative example

Regard example with  $x \in \mathbb{R}^2$  and constants  $a, k, c > 0$ :

$$\dot{x} = \begin{cases} f_1(x), & x_1 > 0, \\ f_2(x), & x_1 < 0. \end{cases}$$

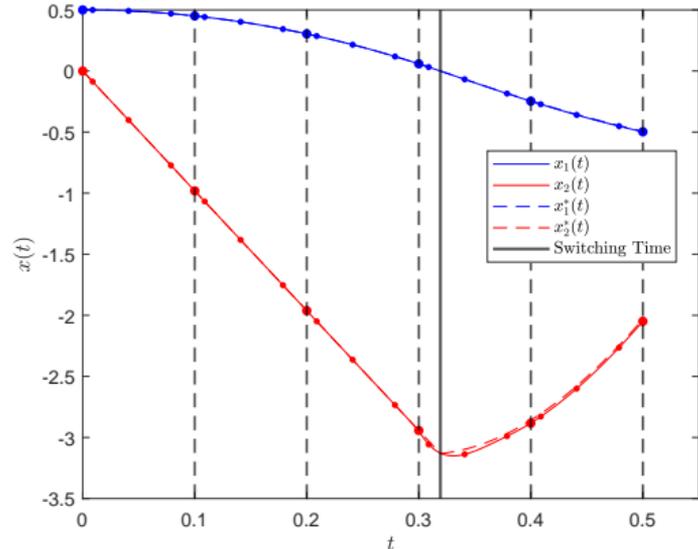
$$f_1(x) = \begin{pmatrix} x_2 \\ -a \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} x_2 \\ -kx_1 - cx_2 \end{pmatrix}$$

$$g_1(x) = -x_1,$$

$$g_2(x) = x_1,$$

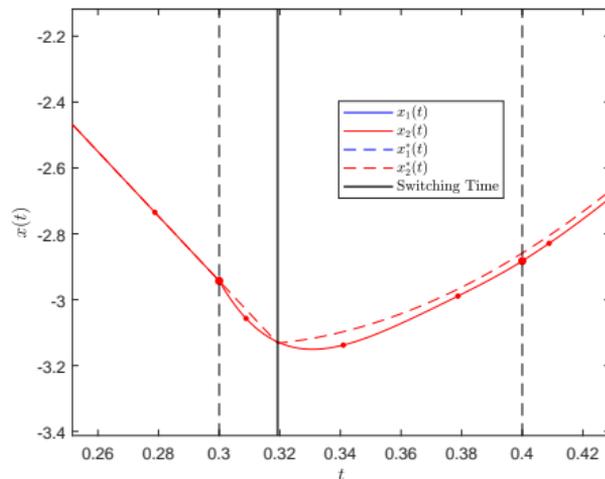
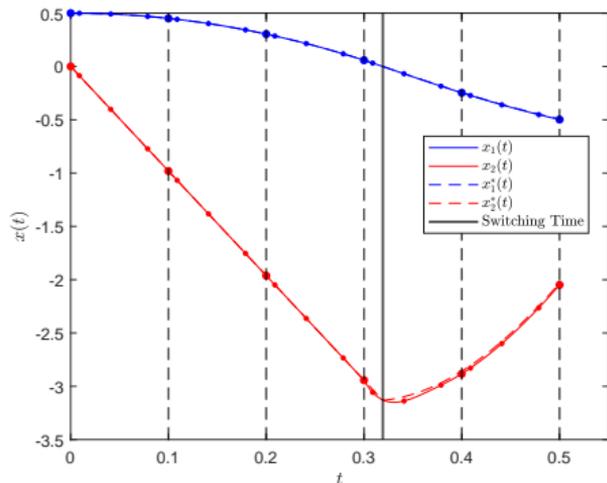
$$\bar{x}_0 = [0.5, 0]^\top.$$

Solve with IRK Radau IIA method of **order 7**  
 $s = 4, N = 5, T = 0.5, h = 0.1$



# Conventional time stepping - illustrative example

Zoom in



High integration accuracy of 7th order IRK method is lost in fourth time step.  
Reason: we try to approximate a nonsmooth function by a (smooth) polynomial.

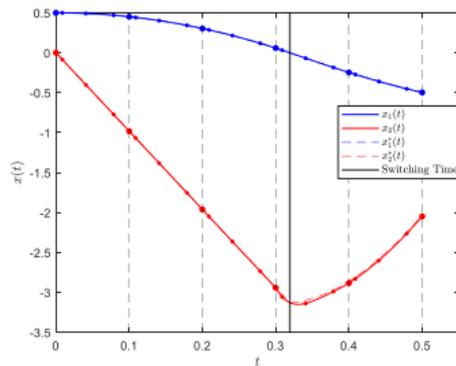
**Question: could we ensure that switches happen only at element boundaries?**

→ **Finite Elements with Switch Detection (FESD)**

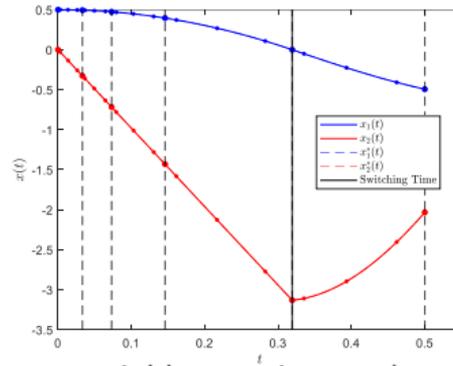
# Finite Elements with Switch Detection (FESD)

FESD is a novel DCS discretization method based on three ideas:

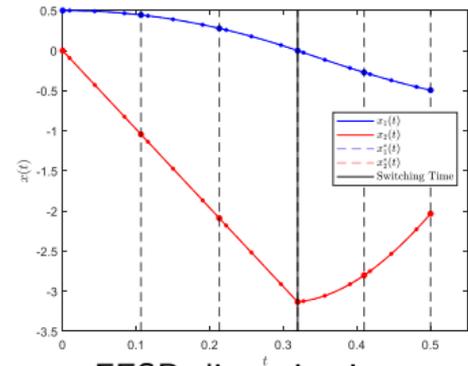
- ▶ make stepsizes  $h_n$  free, ensure  $\sum_{n=0}^{N-1} h_n = T$  [cf. Baumrucker & Biegler, 2009]
- ▶ allow switches only at element boundaries, enforce via *cross-complementarities*
- ▶ remove spurious degrees of freedom via *step equilibration*



conventional  
discretization



variable stepsizes and  
cross-complementarities



FESD discretization  
with step equilibration



## Time-stepping discretization

$$\begin{aligned}
 x_{0,0} &= \bar{x}_0, & h &= T/N \\
 x_{n+1,0} &= x_{n,0} + h \sum_{i=1}^{n_s} b_i v_{n,i} \\
 x_{n,i} &= x_{n,0} + h \sum_{j=1}^{n_s} a_{i,j} v_{n,j} \\
 v_{n,i} &= F(x_{n,i}, u_{n,i}) \theta_{n,i} \\
 0 &= g(x_{n,i}) - \lambda_{n,i} - e \mu_{n,i} \\
 0 &\leq \theta_{n,i} \perp \lambda_{n,i} \geq 0 \\
 1 &= e^\top \theta_{n,i}
 \end{aligned}$$

for  $i = 1, \dots, n_s$   
and  $n = 0, \dots, N-1$

## FESD discretization without step equilibration

$$\begin{aligned}
 x_{0,0} &= \bar{x}_0, & \sum_{n=0}^{N-1} h_n &= T \\
 x_{n+1,0} &= x_{n,0} + h_n \sum_{i=1}^{n_s} b_i v_{n,i} \\
 x_{n,i} &= x_{n,0} + h_n \sum_{j=1}^{n_s} a_{i,j} v_{n,j} \\
 v_{n,i} &= F(x_{n,i}, u_{n,i}) \theta_{n,i} \\
 0 &= g(x_{n,i'}) - \lambda_{n,i'} - e \mu_{n,i'} \\
 0 &\leq \theta_{n,i} \perp \lambda_{n,i'} \geq 0 \quad (\text{cross-complementarities}) \\
 1 &= e^\top \theta_{n,i}
 \end{aligned}$$

for  $i = 1, \dots, n_s$  and  $n = 0, \dots, N-1$   
and  $i' = 0, 1, \dots, n_s$

- ▶  $N$  extra variables ( $h_0, \dots, h_{N-1}$ ) restricted by one extra equality
- ▶ Additional multipliers  $\lambda_{n,0}, \mu_{n,0}$  are uniquely determined

# Conventional DCS and FESD discretization with step equilibration

## Time-stepping discretization

$$x_{0,0} = \bar{x}_0, \quad h = T/N$$

$$x_{n+1,0} = x_{n,0} + h \sum_{i=1}^{n_s} b_i v_{n,i}$$

$$x_{n,i} = x_{n,0} + h \sum_{j=1}^{n_s} a_{i,j} v_{n,j}$$

$$v_{n,i} = F(x_{n,i}, u_{n,i}) \theta_{n,i}$$

$$0 = g(x_{n,i}) - \lambda_{n,i} - e \mu_{n,i}$$

$$0 \leq \theta_{n,i} \perp \lambda_{n,i} \geq 0$$

$$1 = e^\top \theta_{n,i}$$

for  $i = 1, \dots, n_s$

and  $n = 0, \dots, N-1$

## FESD discretization with step equilibration

$$x_{0,0} = \bar{x}_0, \quad \sum_{n=0}^{N-1} h_n = T$$

$$x_{n+1,0} = x_{n,0} + h_n \sum_{i=1}^{n_s} b_i v_{n,i}$$

$$x_{n,i} = x_{n,0} + h_n \sum_{j=1}^{n_s} a_{i,j} v_{n,j}$$

$$v_{n,i} = F(x_{n,i}, u_{n,i}) \theta_{n,i}$$

$$0 = g(x_{n,i'}) - \lambda_{n,ii'} - e \mu_{n,i'}$$

$$0 \leq \theta_{n,i} \perp \lambda_{n,ii'} \geq 0 \quad (\text{cross-complementarities})$$

$$1 = e^\top \theta_{n,i}$$

$$0 = \nu(\theta_{n'}, \theta_{n'+1}, \lambda_{n'}, \lambda_{n'+1}) \cdot (h_{n'} - h_{n'+1})$$

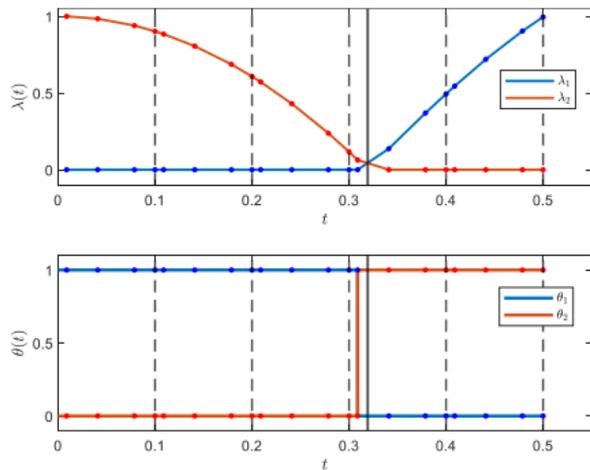
for  $i = 1, \dots, n_s$  and  $n = 0, \dots, N-1$

and  $i' = 0, 1, \dots, n_s$  and  $n' = 0, \dots, N-2$

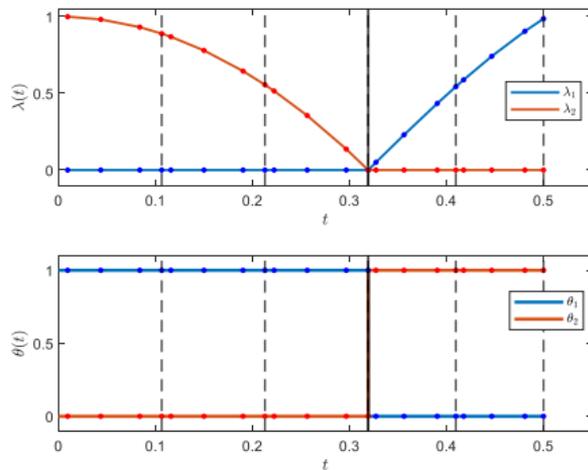
- ▶  $N$  extra variables  $(h_0, \dots, h_{N-1})$  restricted by one extra equality
- ▶ Additional multipliers  $\lambda_{n,0}, \mu_{n,0}$  are uniquely determined
- ▶ Indicator function  $\nu(\theta_{n'}, \theta_{n'+1}, \lambda_{k'}, \lambda_{k'+1})$  only zero if a switch occurs

# Multipliers in conventional and FESD discretization

Time stepping discretization:



FESD discretization:



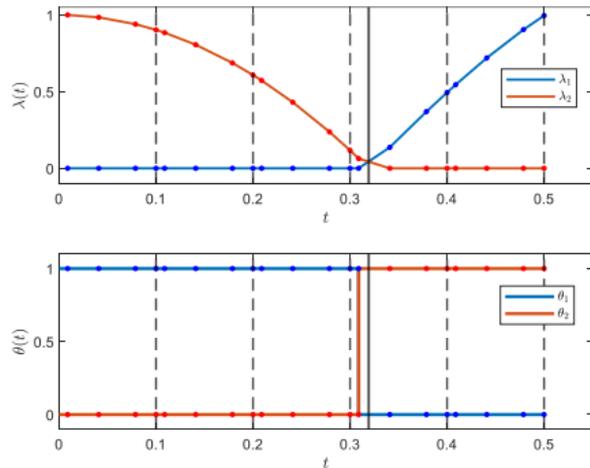
## Lemma (Cross complementarity)

*If any  $\theta_{n,j,i}$  with  $j = 1, \dots, n_s$  is positive, then all  $\lambda_{n,j',i}$  with  $j' = 0, \dots, n_s$  must be zero. Conversely, if any  $\lambda_{n,j',i}$  is positive, then all  $\theta_{n,j,i}$  are zero.*

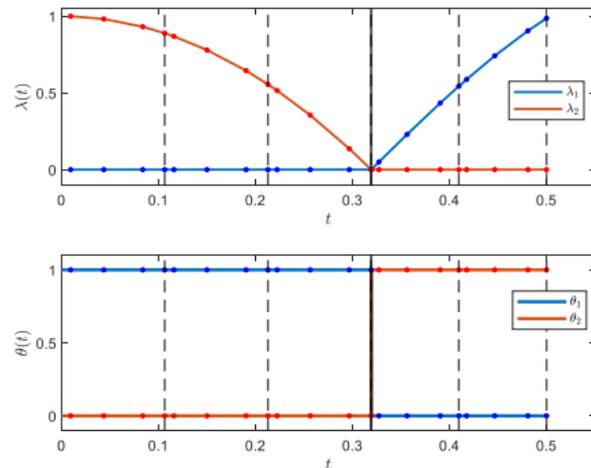
Notation  $\lambda_{n,j,i}$  -  $n$  - finite element,  $j$  - RK stage,  $i$  - component of vector

# Multipliers in conventional and FESD discretization

Time stepping discretization:



FESD discretization:



FESD's cross-complementarities exploit the fact that the multiplier  $\lambda_i(t)$  is continuous in time. On boundary,  $\lambda_i(t_n)$  **must be zero** if  $\theta_i(t) > 0$  for any  $t \in [t_{n-1}, t_{n+1}]$  on the adjacent intervals. This implicitly imposes the constraint  $g_i(x_n) - g_j(x_n) = 0$ .

$\implies h_n$  **adapts for exact switch detection**



- ▶ if no switches happen, cross complementarity implied by standard complementarity
- ▶ spurious degrees of freedom in  $h_n$ : more degrees of freedom than equations

# Step equilibration

- ▶ if no switches happen, cross complementarity implied by standard complementarity
- ▶ spurious degrees of freedom in  $h_n$ : more degrees of freedom than equations
- ▶ exploit complementarity of  $\theta_n, \lambda_n$  to encode switching logic
- ▶ define (very complicated) switch indicator function  $\nu$  (cf. PhD Nurkanović):

$$\nu(\theta_n, \theta_{n+1}, \lambda_n, \lambda_{n+1}) := \begin{cases} \text{positive,} & \text{if no switch at } t_{n+1} \\ 0, & \text{if switch at } t_{n+1} \end{cases}$$

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- ▶ step equilibration:

$$0 = \nu(\theta_n, \theta_{n+1}, \lambda_n, \lambda_{n+1}) \cdot (h_n - h_{n+1}), \quad n = 0, \dots, N-2$$

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- ▶ step equilibration:

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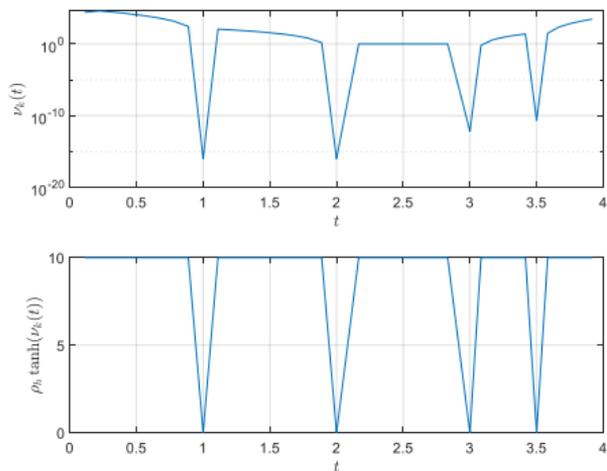
- ▶ Summary:
  - ▶ If switch happens, then  $h_n$  is determined by cross complementarity.
  - ▶ If no switch happens, then  $h_n$  is determined by step equilibration.

# Numerical solution without equilibration

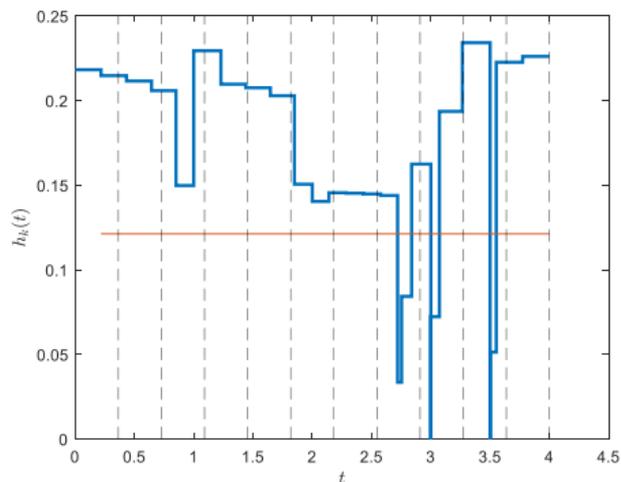
Example with four switches



Indicator function over time:



Step size over time:



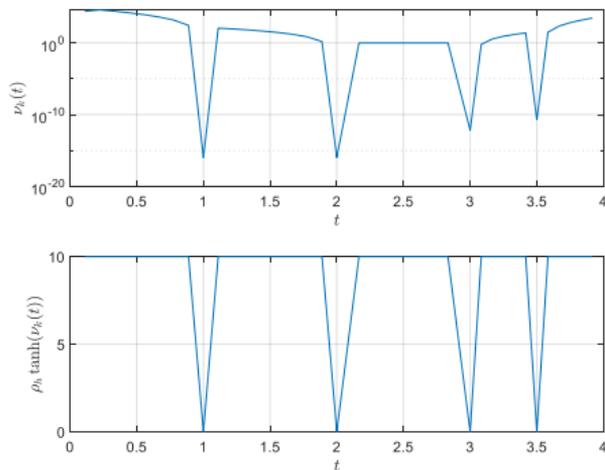
Optimizer varies step size randomly, potentially playing with integration errors.

# Numerical solution with equilibration

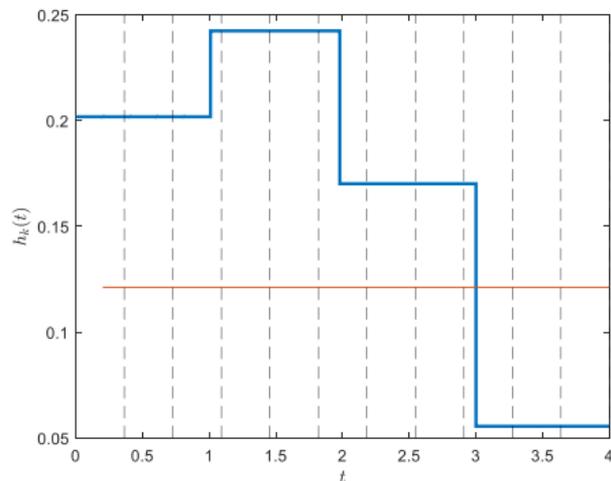
Example with four switches



Indicator function over time:



Step size over time:



Equidistant grid on each "switching stage". Jumps exactly at switching times.



1. An FESD problem needs to solve a nonlinear complementarity problem (NCP) to advance the integration. The solutions of these NCP are locally unique.
  - ▶ For a given point determine which constraint cross comp. and step eq. are binding, and which implicitly satisfied.
  - ▶ Obtain square system and apply implicit function theorem.
2. Convergence of the FESD method to a Filippov solution of the underlying system with accuracy  $O(h^p)$  is proven. Here,  $p$  is the order of the underlying smooth IRK method.
  - ▶ Solution approximation and true solution predict same active set.
  - ▶ Switching time accuracy also  $O(h^p)$ .



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  - ▶ Solution approximation and true solution predict same active set.
  - ▶ Switching time accuracy also  $O(h^p)$ .
3. Convergence of numerical sensitivities to the true value with  $O(h^p)$  is given.
  - ▶ Cross. comp. implicitly enforce switching condition and lead to correct sensitivities.
  - ▶ The Stewart & Anitescu problem is solved.

## Tutorial example

$$\dot{x} = \begin{cases} A_1 x, & \|x\|_2^2 < 1, \\ A_2 x, & \|x\|_2^2 > 1, \end{cases}$$

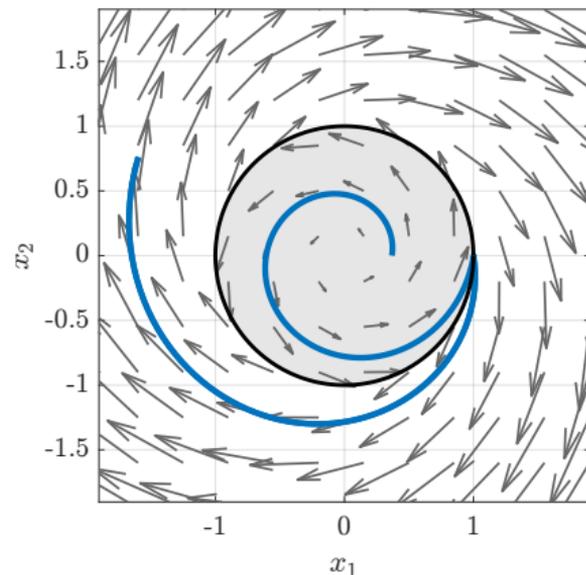
$$\text{with } A_1 = \begin{bmatrix} 1 & 2\pi \\ -2\pi & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -2\pi \\ 2\pi & 1 \end{bmatrix}$$

$$x(0) = (e^{-1}, 0) \text{ for } t \in [0, \frac{\pi}{2}].$$

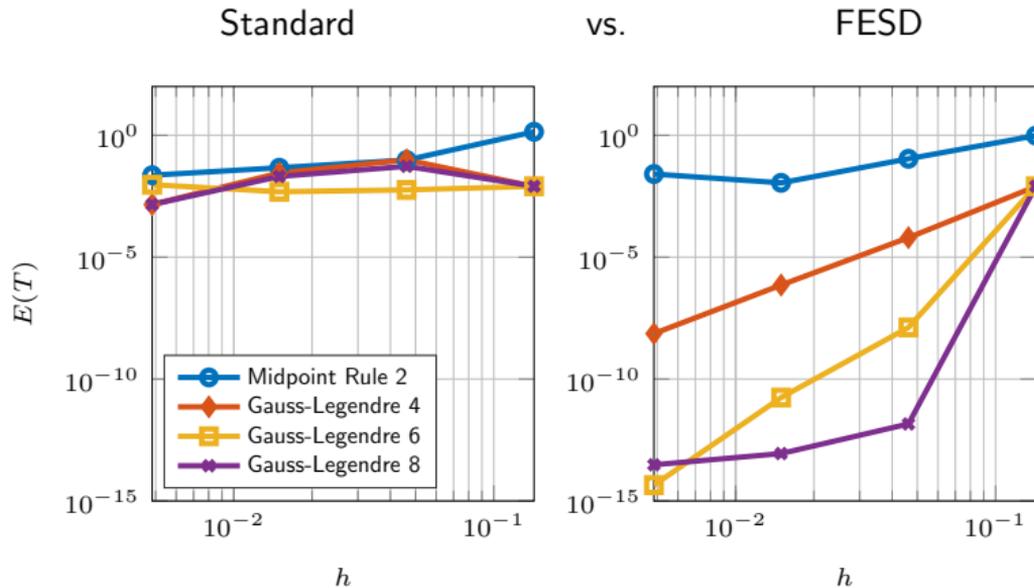
Compute global integration error  $E(T)$  using different strategies.

Compute solution approximation:

1. With fixed step size IRK methods (time-stepping).
2. FESD with same underlying IRK methods.

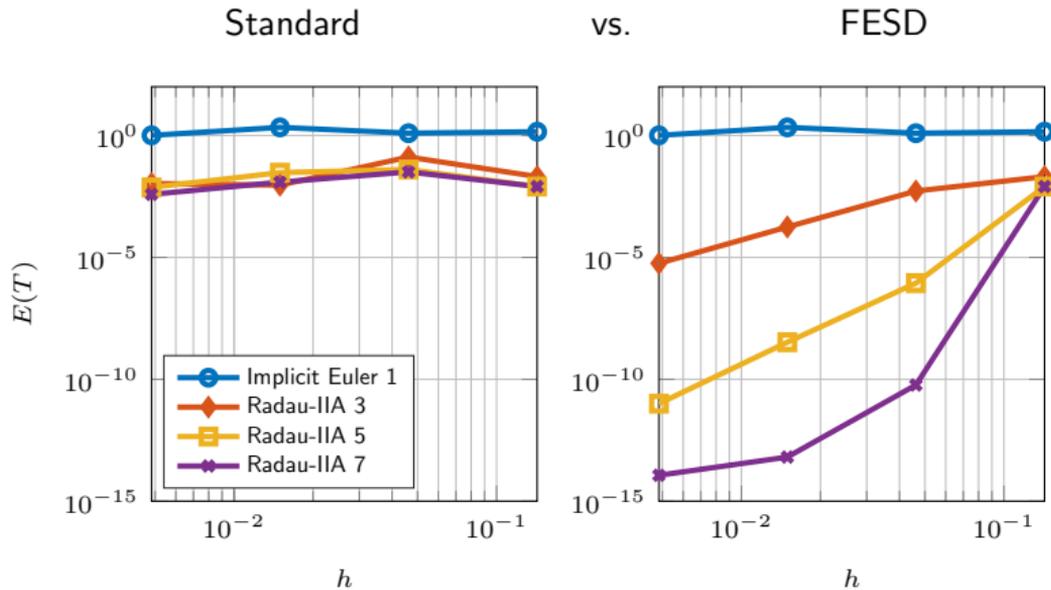


# FESD recovers high integration order for switched systems



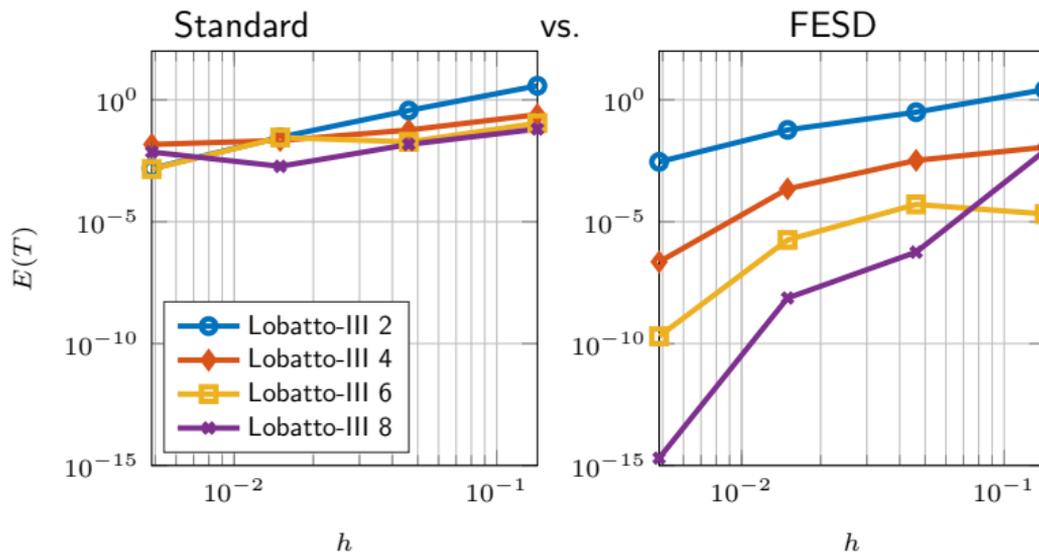
Integration error  $E(T)$  at time  $T = \pi/2$  vs. step-size  $h$ , for different IRK methods.  
**FESD discretization recovers high integration order**

# FESD recovers high integration order for switched systems



Integration error  $E(T)$  at time  $T = \pi/2$  vs. step-size  $h$ , for different IRK methods.  
**FESD discretization recovers high integration order**

# FESD recovers high integration order for switched systems



Integration error  $E(T)$  at time  $T = \pi/2$  vs. step-size  $h$ , for different IRK methods.  
**FESD discretization recovers high integration order**



- 1 Time stepping and smoothing in nonsmooth optimal control
- 2 Finite Elements with Switch Detection (FESD)
- 3 FESD-Discretization of Optimal Control Problems
- 4 Conclusions and summary

# Discretizing optimal control problems with FESD

## Discretized optimal control problem

$$\min_{s, z, u} \sum_{k=0}^{N-1} \Phi_L(s_k, z_k, u_k) + E(s_N)$$

s.t.  $s_0 = \bar{x}_0$

$$s_{k+1} = \Phi_f(s_k, z_k, u_k)$$

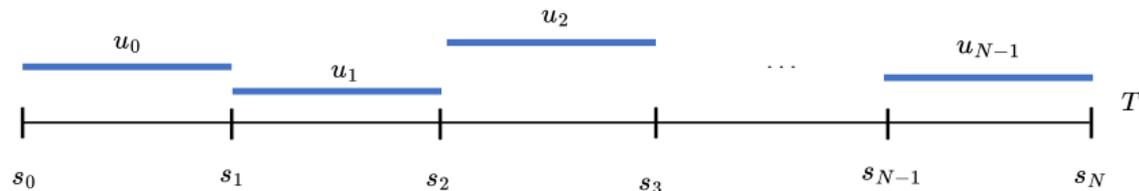
$$0 = \Phi_{\text{int}}(s_k, z_k, u_k)$$

$$0 \geq h(s_k, u_k), \quad k = 0, \dots, N-1$$

$$0 \geq r(s_N)$$

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- ▶ FESD with  $N_{\text{FE}}$  finite elements applied on every control interval

Control horizon  $[0, T]$  with  $N$  control stages



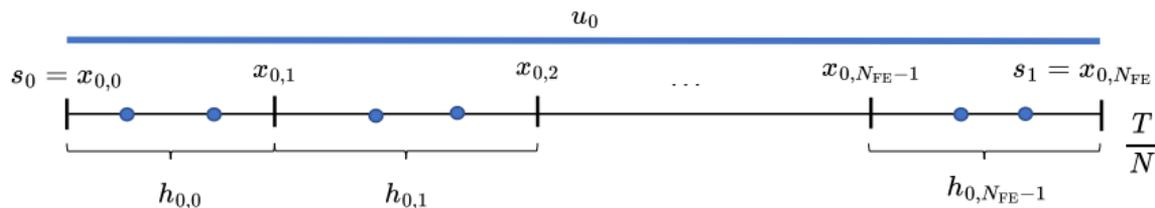
# Discretizing optimal control problems with FESD

## Discretized optimal control problem

$$\begin{aligned}
 & \min_{s,z,u} \sum_{k=0}^{N-1} \Phi_L(s_k, z_k, u_k) + E(s_N) \\
 & \text{s.t. } s_0 = \bar{x}_0 \\
 & \quad s_{k+1} = \Phi_f(s_k, z_k, u_k) \\
 & \quad 0 = \Phi_{\text{int}}(s_k, z_k, u_k) \\
 & \quad 0 \geq h(s_k, u_k), \quad k = 0, \dots, N-1 \\
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- ▶  $\Phi_{\text{int}}$  summarizes all internal FESD equations: RK, cross complementarity, step equilibration,...

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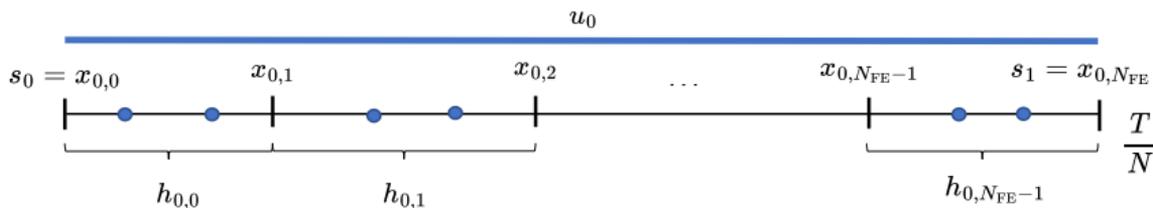
# Discretizing optimal control problems with FESD

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- ▶  $\Phi_{\text{int}}$  summarizes all internal FESD equations: RK, cross complementarity, step equilibration, ...
- ▶  $z = (z_0, \dots, z_{N-1})$  - all interval variables: internal states, stage values of states and multipliers, step sizes, ...

Control horizon  $[0, T]$  with  $N$  control stages



# FESD-discretized optimal control problems are MPCC

## Discretized optimal control problem

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 & 0 = \Phi_{\text{int}}(s_k, z_k, u_k) \\
 & 0 \geq h(s_k, u_k), \quad k = 0, \dots, N-1 \\
 & 0 \geq r(s_N)
 \end{aligned}$$

Collect  $w = (s, z, u) \in \mathbb{R}^{n_w}$   
 Mathematical programs with  
 complementarity constraints (MPCC) are  
 more difficult than standard NLPs

## NLP with Complementarity Constraints

$$\begin{aligned}
 \min_{w \in \mathbb{R}^{n_w}} \quad & F(w) \\
 \text{s.t.} \quad & 0 = G(w) \\
 & 0 \geq H(w) \\
 & 0 \leq G_1(w) \perp G_2(w) \geq 0
 \end{aligned}$$

Standard and cross complementarity  
 constraints summarized in

$$0 \leq G_1(w) \perp G_2(w) \geq 0$$



Newton-type methods generate a sequence  $w_0, w_1, w_2, \dots$  by linearizing and solving convex subproblems.

## Summarized NLP

$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & F(w) \\ \text{s.t.} \quad & 0 = G(w) \\ & 0 \geq H(w) \end{aligned}$$

Still assume smooth convex  $F, H$ .  
Nonlinear  $G$  makes problem nonconvex.



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## NLP with complementarity constraints

$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & F(w) \\ \text{s.t.} \quad & 0 = G(w) \\ & 0 \geq H(w) \\ & 0 \leq G_1(w) \perp G_2(w) \geq 0 \end{aligned}$$

There is significant **nonconvex and nonsmooth** structure in the NLP.

# Mathematical Programs with Complementarity Constraints (MPCC)

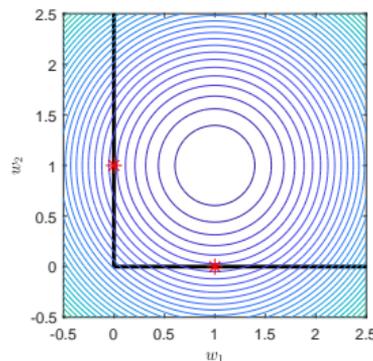
NLP with additional constraints of complementarity type:

$$x \perp y \Leftrightarrow x^\top y = 0$$

## MPCC as an NLP

$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & F(w) \\ \text{s.t.} \quad & 0 = G(w) \\ & 0 \geq H(w) \\ & 0 \leq G_1(w) \\ & 0 \leq G_2(w) \\ & 0 \geq G_1(w)^\top G_2(w) \end{aligned}$$

Convex  $J, H$  and smooth  $F$ .  
Smooth  $G_1, G_2$ .



Due to complementarity constraints, MPCC are nonsmooth and nonconvex.

Toy MPCC example:

$$\begin{aligned} \min_{w \in \mathbb{R}^2} \quad & (w_1 - 1)^2 + (w_2 - 1)^2 \\ \text{s.t.} \quad & 0 \leq w_1 \perp w_2 \geq 0 \end{aligned}$$

Two local minimizers.  
One local maximizer  
(without constraint qualification).

**MPCCs treated in detail in three lectures by C. Kirches.**

# Revisiting the OCP example - now with FESD

Tutorial example inspired by [Stewart & Anitescu, 2010]



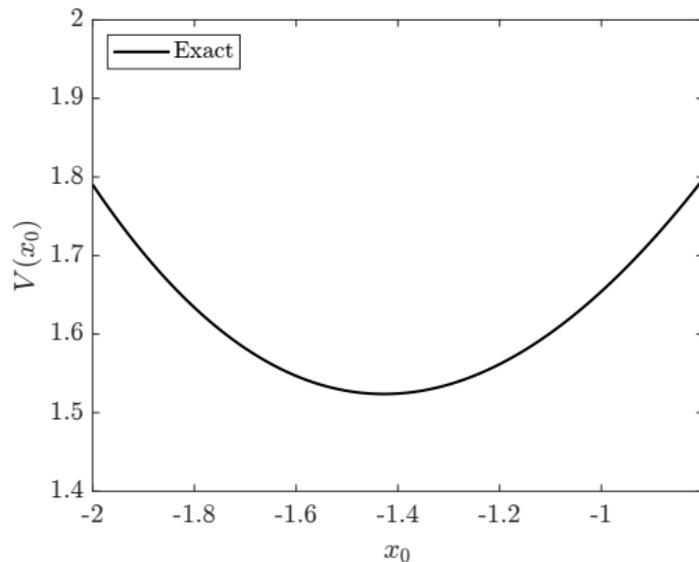
## Continuous-time OCP

$$\begin{aligned} \min_{x(\cdot) \in \mathcal{C}^0([0,2])} & \int_0^2 x(t)^2 dt + (x(2) - 5/3)^2 \\ \text{s.t.} & \dot{x}(t) = 2 - \text{sign}(x(t)), \quad t \in [0, 2] \end{aligned}$$

Free initial value  $x(0)$  is the effective degree of freedom.

## Equivalent reduced problem

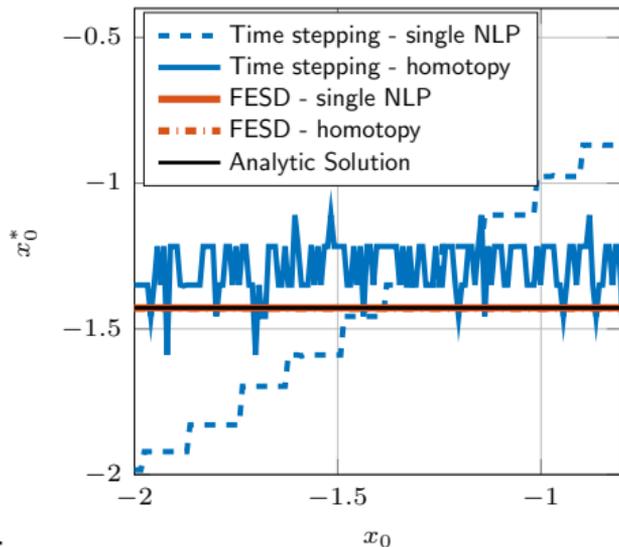
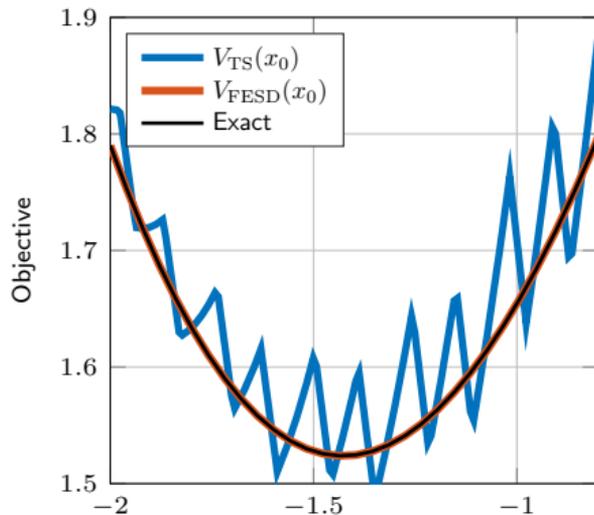
$$\min_{x_0 \in \mathbb{R}} V(x_0)$$



- Denote by  $V(x_0)$  the nonsmooth objective value for the unique feasible trajectory starting at  $x(0) = x_0$ .

# Revisiting the OCP example - now with FESD

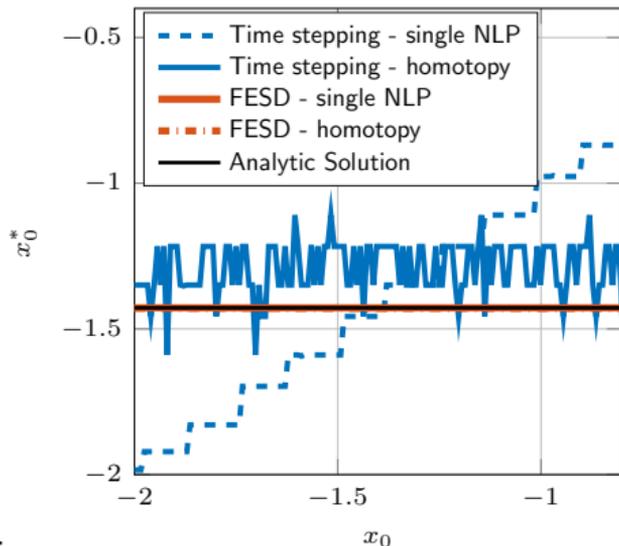
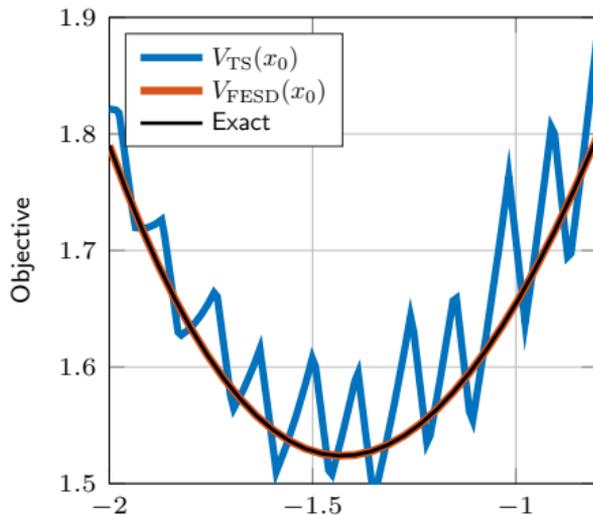
Tutorial example inspired by [Stewart & Anitescu, 2010]



- ▶ no spurious local minima, correct sensitivities
- ▶ convergence to the "true" local minimum, both with homotopy and without it
- ▶ accuracy of order  $O(h^p)$ , in contrast to standard approach with only  $O(h)$

# Revisiting the OCP example - now with FESD

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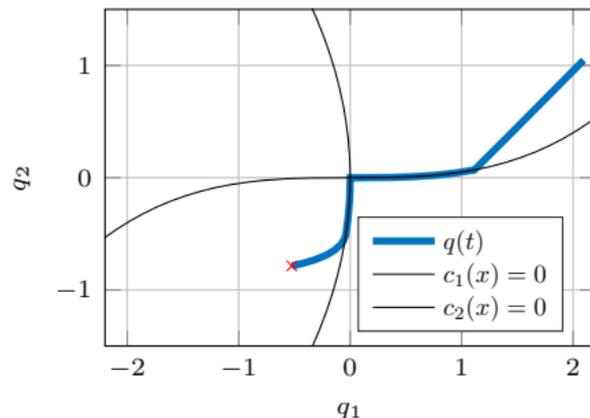
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- ▶ convergence to the "true" local minimum, both with homotopy and without it
- ▶ accuracy of order  $O(h^p)$ , in contrast to standard approach with only  $O(h)$
- ▶ FESD solves the accuracy and convergence issues

## OCP with sliding modes

$$\begin{aligned} \min_{x(\cdot), u(\cdot)} \quad & \int_0^4 u(t)^\top u(t) + v(t)^\top v(t) dt \\ \text{s.t.} \quad & x(0) = \left( \frac{2\pi}{3}, \frac{\pi}{3}, 0, 0 \right) \\ & \dot{x}(t) = \begin{bmatrix} -\text{sign}(c(x(t))) + v(t) \\ u(t) \end{bmatrix} \\ & -2e \leq v(t) \leq 2e \\ & -10e \leq u(t) \leq 10e \quad t \in [0, 4], \\ & q(T) = \left( -\frac{\pi}{6}, -\frac{\pi}{4} \right) \end{aligned}$$

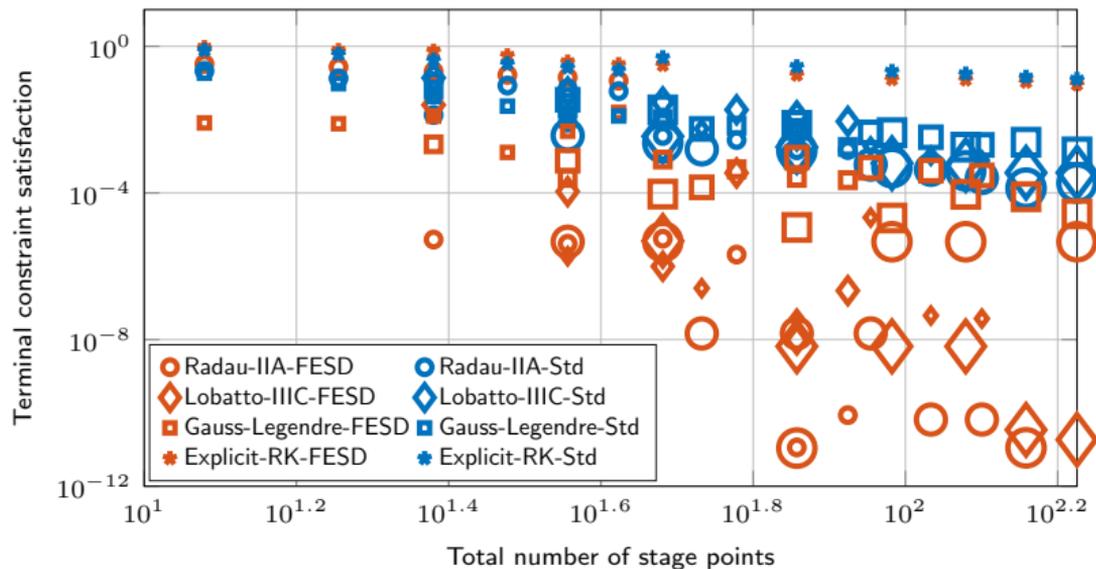
States  $q, v \in \mathbb{R}^2$  and control  $u \in \mathbb{R}^2$ ,  
 $x = (q, v)$

$$\text{Switching functions } c(x) = \begin{bmatrix} q_1 + 0.15q_2^2 \\ 0.05q_1^3 + q_2 \end{bmatrix}$$



# FESD vs standard IRK - number of function evaluations

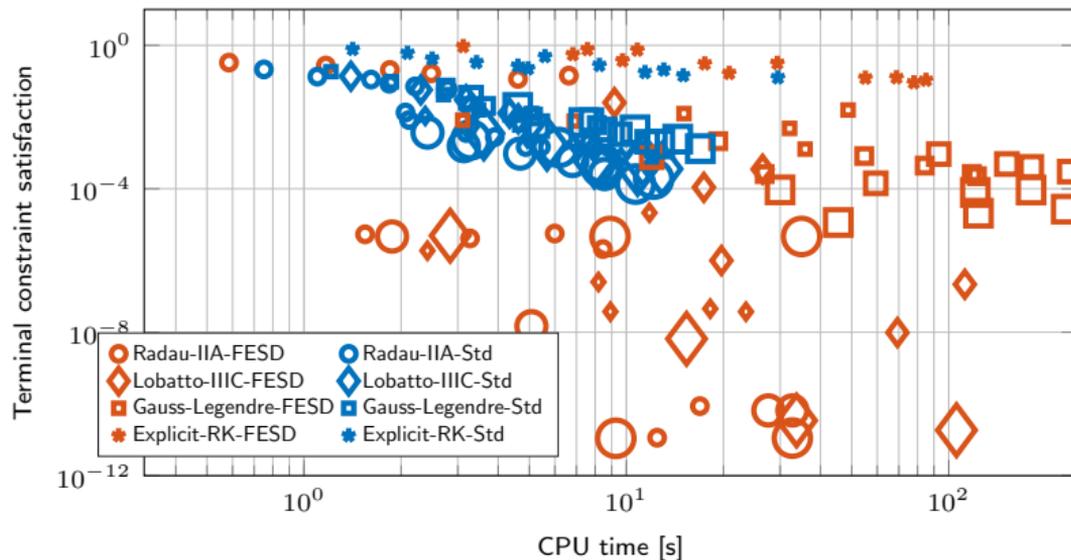
Benchmark on an optimal control problem with nonlinear sliding modes



Terminal constraint satisfaction vs. number of stage points

# FESD vs standard IRK - CPU Time

Benchmark on an optimal control problem with nonlinear sliding modes



Terminal constraint satisfaction vs. CPU time  
FESD one million times more accurate than Std. for CPU time of  $\approx 2$  s



- ▶ Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for nonsmooth systems of level NSD2
- ▶ Following similar lines, FESD can be derived for the Heaviside step reformulation



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- ▶ Key ideas: make step sizes degrees of freedom and introduce implicit relations that locate the switches
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- ▶ Key ideas: make step sizes degrees of freedom and introduce implicit relations that locate the switches
- ▶ Switch detection not only essential for high accuracy, **but also for correct sensitivities** (no spurious solutions)
- ▶ FESD solves many of the issues that standard methods have: integration accuracy, convergence of sensitivities
- ▶ Main practical difficulty: solving Mathematical Programs with Complementarity Constraints (MPCC)



- ▶ Brian T. Baumrucker and Lorenz T. Biegler. MPEC strategies for optimization of a class of hybrid dynamic systems. *Journal of Process Control*, 19(8):1248–1256, 2009.
- ▶ David E Stewart and Mihai Anitescu. Optimal control of systems with discontinuous differential equations. *Numerische Mathematik*, 114(4):653–695, 2010.
- ▶ Armin Nurkanović, Mario Sperl, Sebastian Albrecht, and Moritz Diehl. Finite Elements with Switch Detection for Direct Optimal Control of Nonsmooth Systems. Submitted to *Numerische Mathematik* 2022.
- ▶ Armin Nurkanović, Sebastian Albrecht, and Moritz Diehl. Limits of MPCC Formulations in Direct Optimal Control with Nonsmooth Differential Equations. In *2020 European Control Conference (ECC)*, pages 2015–2020, 2020.
- ▶ Armin Nurkanović and Moritz Diehl. NOSNOC: A software package for numerical optimal control of nonsmooth systems. *IEEE Control Systems Letters*, 2022.
- ▶ Armin Nurkanović, Anton Pozharskiy, Jonathan Frey, and Moritz Diehl. Finite elements with switch detection for numerical optimal control of nonsmooth dynamical systems with set-valued step functions. *arXiv preprint arXiv:2307.03482*, 2023.
- ▶ Armin Nurkanović, Jonathan Frey, Anton Pozharskiy, and Moritz Diehl. Finite elements with switch detection for direct optimal control of nonsmooth systems with set-valued step functions. In *Conference on Decision on Control*, 2023.



Suppose that  $x(t)$  crosses from  $R_1$  to  $R_2$  and recall that  $\mu = \min_j g_j(x)$

**Continuous time:**

- ▶ Before switch:  $\theta_1(t) > 0, \lambda_1(t) = 0$ , and  $\theta_2(t) = 0, \lambda_2 \geq 0$
- ▶ After switch:  $\theta_1(t) = 0, \lambda_1(t) \geq 0$ , and  $\theta_2(t) > 0, \lambda_2 = 0$



Suppose that  $x(t)$  crosses from  $R_1$  to  $R_2$  and recall that  $\mu = \min_j g_j(x)$

**Discrete time** (switch between the  $n$ -th and  $n + 1$ -st finite element):

- ▶ Before switch:  $\theta_{n,j,1}(t) > 0$ ,  $\lambda_{n,j,1}(t) = 0$ , and  $\theta_{n,j,2}(t) = 0$ ,  $\lambda_{n,j,2} \geq 0$
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# Switch detection - example

Suppose that  $x(t)$  crosses from  $R_1$  to  $R_2$  and recall that  $\mu = \min_j g_j(x)$

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From Lemma 1 it follows that  $\lambda_{n,n_s,1} = \lambda_{n,n_s,2} = 0$

## Switch detection conditions

$$g_1(x_{n+1}) = \lambda_{n,n_s,1} - \mu_{n,n_s}$$



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## Switch detection conditions

$$0 = g_1(x_{n+1}) - g_2(x_{n+1}) = \psi_{12}(x_{n+1})$$

Implies constraint such that  $h_n$  must adapt for exact switch detection!