universität freiburg

Numerical Optimal Control for Nonsmooth Dynamic Systems

Moritz Diehl[1]

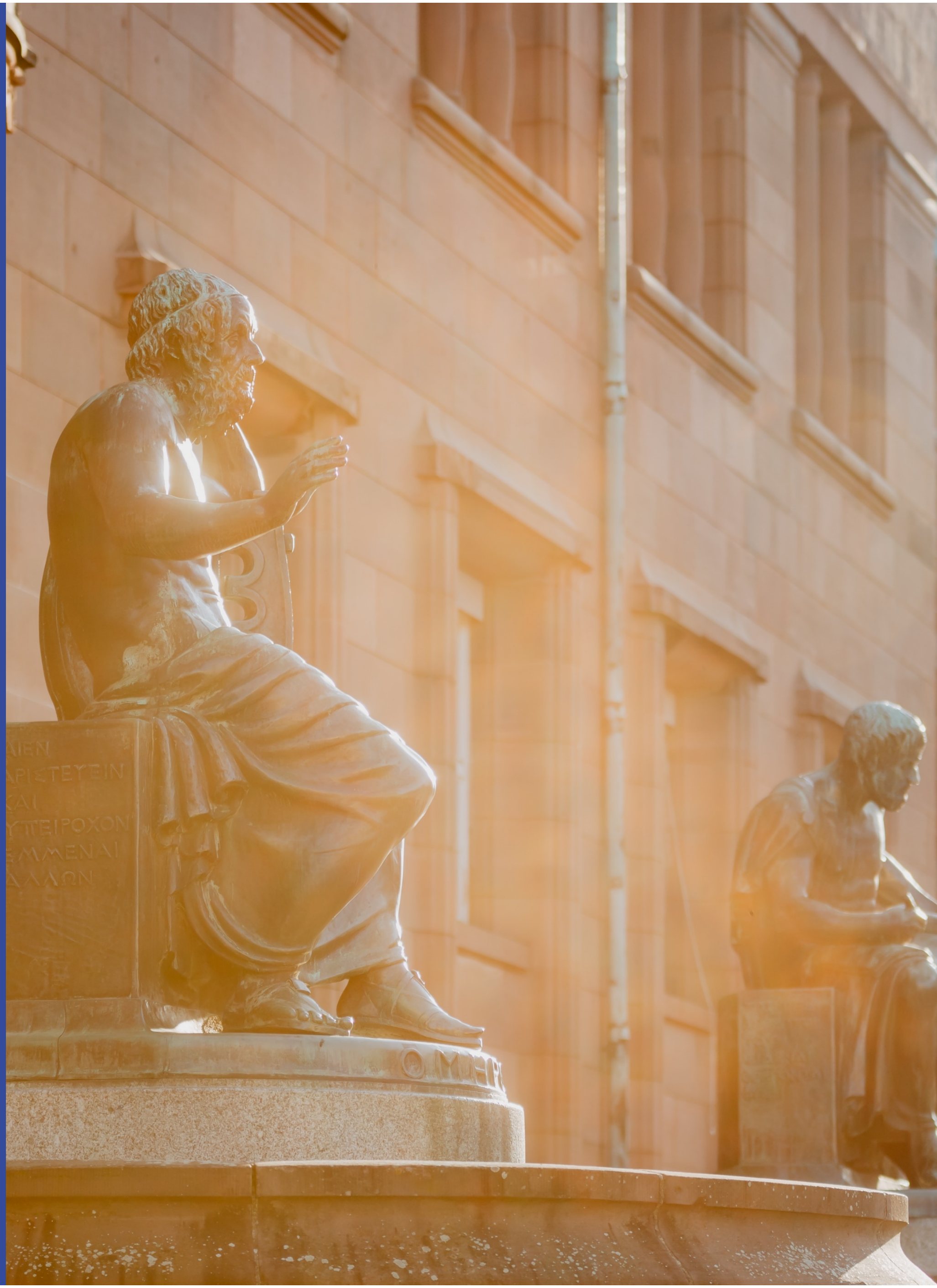
joint work with Armin Nurkanovic[1], Anton Pozharskiy[1], Christian Dietz[1,2], Sebastian Albrecht[2]

[1] Department of Microsystems Engineering and Department of Mathematics, University of Freiburg, Germany

[2] Siemens Foundational Technology, Munich, Germany

13th IFAC Symposium on Nonlinear Control Systems (NOLCOS), Reykjavik, Iceland

July 23-25, 2025



Continuous-Time Optimal Control Problems (OCP)

Continuous-Time OCP with Ordinary Differential Equation (ODE) Constraints

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_{\mathrm{c}}(x(t), u(t))\,\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t), u(t)) \\
& 0 \geq h(x(t), u(t)), \ t \in [0, T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

Continuous-Time Optimal Control Problems (OCP)

Continuous-Time OCP with Ordinary Differential Equation (ODE) Constraints

$$\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_{\mathrm{c}}(x(t), u(t))\,\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t), u(t)) \\
& 0 \geq h(x(t), u(t)), \; t \in [0, T] \\
& 0 \geq r(x(T))
\end{aligned}$$

Can in most applications assume convexity of all "outer" problem functions: L_{c}, E, h, r.

Three Levels of Difficulty in Continuous-Time OCP

Continuous-Time OCP

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_\mathrm{c}(x(t),u(t))\,\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t),u(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

Three levels of difficulty:

Three Levels of Difficulty in Continuous-Time OCP

Continuous-Time OCP

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_c(x(t),u(t))\,\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t),u(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

Three levels of difficulty:

(a) Linear ODE: $f(x,u) = Ax + Bu$

Three Levels of Difficulty in Continuous-Time OCP

Continuous-Time OCP

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_{\mathrm{c}}(x(t),u(t))\,\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t),u(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

Three levels of difficulty:

(a) Linear ODE: $f(x,u) = Ax + Bu$

(b) Nonlinear smooth ODE: $f \in \mathcal{C}^1$

Three Levels of Difficulty in Continuous-Time OCP

Continuous-Time OCP

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_c(x(t),u(t))\,\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t),u(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

Three levels of difficulty:

(a) Linear ODE: $f(x,u) = Ax + Bu$

(b) Nonlinear smooth ODE: $f \in \mathcal{C}^1$

(c) **Nonsmooth Dynamics (NSD):**

- f not differentiable (NSD1),
- f not continuous (NSD2), or even
- f not finite valued, discontinuous state $x(t)$ (NSD3)

Three Levels of Difficulty in Continuous-Time OCP

Continuous-Time OCP

$$\min_{x(\cdot),u(\cdot)} \int_0^T L_c(x(t),u(t))\,\mathrm{d}t + E(x(T))$$

$$\text{s.t.} \quad x(0) = \bar{x}_0$$

$$\dot{x}(t) = f(x(t),u(t))$$

$$0 \geq h(x(t),u(t)),\ t \in [0,T]$$

$$0 \geq r(x(T))$$

Three levels of difficulty:

(a) Linear ODE: $f(x,u) = Ax + Bu$

(b) Nonlinear smooth ODE: $f \in \mathcal{C}^1$

(c) **Nonsmooth Dynamics (NSD):**

- f not differentiable (NSD1),
- f not continuous (NSD2), or even
- f not finite valued, discontinuous state $x(t)$ (NSD3)

First focus on smooth cases (a) and (b).

Three Levels of Difficulty in Continuous-Time OCP

Continuous-Time OCP

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_{\mathrm{c}}(x(t), u(t))\,\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t), u(t)) \\
& 0 \geq h(x(t), u(t)), \; t \in [0, T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

Three levels of difficulty:

(a) Linear ODE: $f(x,u) = Ax + Bu$

(b) Nonlinear smooth ODE: $f \in \mathcal{C}^1$

First focus on smooth cases (a) and (b).

Recall: Runge-Kutta Discretization for Smooth Systems

Ordinary Differential Equation (ODE)

$$\dot{x}(t) = \underbrace{f(x(t), u(t))}_{=:v(t)}$$

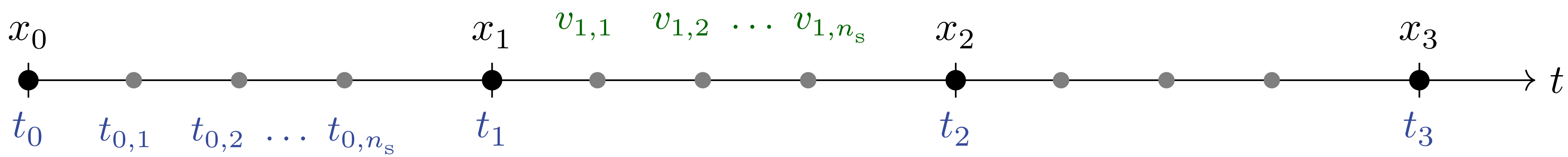
Initial Value Problem (IVP)

$$\begin{aligned} x(0) &= \bar{x}_0 \\ v(t) &= f(x(t), u(t)) \\ \dot{x}(t) &= v(t) \\ & t \in [0, T] \end{aligned}$$

Discretization: N Runge-Kutta steps of each n_s stages

$$\begin{aligned} x_{0,0} &= \bar{x}_0, \qquad\qquad \Delta t = \tfrac{T}{N} \\ v_{k,j} &= f(x_{k,j}, u_k) \\ x_{k,j} &= x_{k,0} + \Delta t \textstyle\sum_{n=1}^{n_s} a_{jn} v_{k,n} \\ x_{k+1,0} &= x_{k,0} + \Delta t \textstyle\sum_{n=1}^{n_s} b_n v_{k,n} \\ & j = 1, \ldots, n_s, \quad k = 0, \ldots, N-1 \end{aligned}$$

For fixed controls and initial value: square system with $n_x + N(2n_s + 1)n_x$ unknowns, implicitly defined via $n_x + N(2n_s + 1)n_x$ equations.
(trivial eliminations in case of explicit RK methods)



Direct Methods Transform OCP into Nonlinear Program (NLP)

Continuous time OCP

$$
\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_\mathrm{c}(x(t),u(t))\,\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t),u(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}
$$

- Direct methods "first discretize, then optimize"

Direct Methods Transform OCP into Nonlinear Program (NLP)

Continuous time OCP

$$\min_{x(\cdot),u(\cdot)} \int_0^T L_\mathrm{c}(x(t),u(t))\,\mathrm{d}t + E(x(T))$$

$$\text{s.t.} \quad x(0) = \bar{x}_0$$

$$\dot{x}(t) = f(x(t),u(t))$$

$$0 \geq h(x(t),u(t)),\ t \in [0,T]$$

$$0 \geq r(x(T))$$

- Direct methods "first discretize, then optimize"

1. Parameterize controls, e.g. $u(t) = u_n, t \in [t_n, t_{n+1}]$.

Direct Methods Transform OCP into Nonlinear Program (NLP)

Continuous time OCP

$$\min_{x(\cdot),u(\cdot)} \int_0^T L_\mathrm{c}(x(t),u(t))\,\mathrm{d}t + E(x(T))$$
$$\text{s.t.} \quad x(0) = \bar{x}_0$$
$$\dot{x}(t) = f(x(t),u(t))$$
$$0 \geq h(x(t),u(t)),\ t \in [0,T]$$
$$0 \geq r(x(T))$$

- Direct methods "first discretize, then optimize"

1. Parameterize controls, e.g. $u(t) = u_n, t \in [t_n, t_{n+1}]$.
2. Discretize cost and dynamics

$$L_\mathrm{d}(x_n, z_k, u_n) \approx \int_{t_n}^{t_{n+1}} L_\mathrm{c}(x(t),u(t))\,\mathrm{d}t$$

Replace $\dot{x} = f(x,u)$ by

$$x_{n+1} = \phi_f(x_n, z_n, u_n)$$
$$0 = \phi_\mathrm{int}(x_n, z_n, u_n)$$

Direct Methods Transform OCP into Nonlinear Program (NLP)

Continuous time OCP

$$\min_{x(\cdot),u(\cdot)} \int_0^T L_c(x(t),u(t))\,\mathrm{d}t + E(x(T))$$

$$\begin{aligned}
\text{s.t.}\quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = f(x(t),u(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}$$

- Direct methods "first discretize, then optimize"

1. Parameterize controls, e.g. $u(t) = u_n, t \in [t_n, t_{n+1}]$.
2. Discretize cost and dynamics

   $$L_\mathrm{d}(x_n, z_k, u_n) \approx \int_{t_n}^{t_{n+1}} L_\mathrm{c}(x(t),u(t))\,\mathrm{d}t$$

   Replace $\dot{x} = f(x,u)$ by

   $$x_{n+1} = \phi_f(x_n, z_n, u_n)$$
   $$0 = \phi_{\mathrm{int}}(x_n, z_n, u_n)$$

3. Also discretize path constraints

   $$0 \geq \phi_h(x_n, z_n, u_n),\ n = 0, \ldots N-1.$$

Direct Methods Transform OCP into Nonlinear Program (NLP)

Continuous time OCP

$$\begin{aligned} \min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_\mathrm{c}(x(t),u(t))\,\mathrm{d}t + E(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = f(x(t),u(t)) \\ & 0 \geq h(x(t),u(t)), \; t \in [0,T] \\ & 0 \geq r(x(T)) \end{aligned}$$

- Direct methods "first discretize, then optimize"

Discrete time OCP (an NLP)

$$\begin{aligned} \min_{\mathbf{x},\mathbf{z},\mathbf{u}} \quad & \textstyle\sum_{k=0}^{N-1} L_\mathrm{d}(x_k,z_k,u_k) + E(x_N) \\ \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & x_{n+1} = \phi_f(x_n,z_n,u_n) \\ & 0 = \phi_\mathrm{int}(x_n,z_n,u_n) \\ & 0 \geq \phi_h(x_n,z_n,u_n), \; n = 0,\dots,N-1 \\ & 0 \geq r(x_N) \end{aligned}$$

Variables $\mathbf{x} = (x_0,\dots,x_N)$, $\mathbf{z} = (z_0,\dots,z_N)$ and $\mathbf{u} = (u_0,\dots,u_{N-1})$.
Here, $\mathbf{z}$ are the intermediate variables of the integrator (e.g. Runge-Kutta)

Simplest Direct Transcription: Single Step Explicit Euler

(not recommended in practice, other Runge-Kutta methods are much more efficient)

Continuous time OCP

$$\min_{x(\cdot),u(\cdot)} \int_0^T L_{\mathrm{c}}(x(t),u(t))\,\mathrm{d}t + E(x(T))$$

$$\begin{aligned} \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = f(x(t),u(t)) \\ & 0 \geq h(x(t),u(t)),\ t \in [0,T] \\ & 0 \geq r(x(T)) \end{aligned}$$

- Direct methods: first discretize, then optimize

Single Step Explicit Euler NLP, with $\Delta t = \frac{T}{N}$

$$\min_{\mathbf{x},\mathbf{u}} \sum_{k=0}^{N-1} L_{\mathrm{c}}(x_k,u_k)\Delta t + E(x_N)$$

$$\begin{aligned} \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & x_{n+1} = x_n + f(x_n,u_n)\Delta t \\ & 0 \geq h(x_n,u_n),\ n = 0,\ldots,N-1 \\ & 0 \geq r(x_N) \end{aligned}$$

Variables $\mathbf{x} = (x_0,\ldots,x_N)$ and $\mathbf{u} = (u_0,\ldots,u_{N-1})$.
(single step explicit Euler has no internal integrator variables $\mathbf{z}$)

Sparse NLP resulting from direct transcription

Discrete time OCP (an NLP)

$$\min_{\mathbf{x},\mathbf{z},\mathbf{u}} \ \sum_{k=0}^{N-1} L_{\mathrm{d}}(x_k, z_n, u_k) + E(x_N)$$

$$\begin{aligned} \text{s.t.} \quad x_0 &= \bar{x}_0 \\ x_{n+1} &= \phi_f(x_n, z_n, u_n) \\ 0 &= \phi_{\text{int}}(x_n, z_n, u_n) \\ 0 &\geq \phi_h(x_n, z_n, u_n), \ \ n = 0, \ldots, N-1 \\ 0 &\geq r(x_N) \end{aligned}$$

Variables $w = (\mathbf{x}, \mathbf{z}, \mathbf{u})$

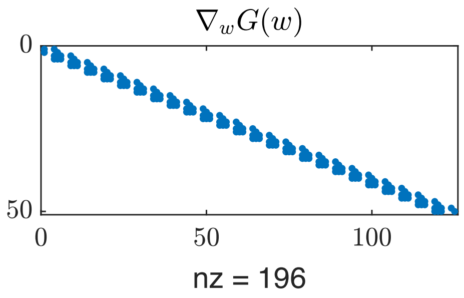
Nonlinear Program (NLP)

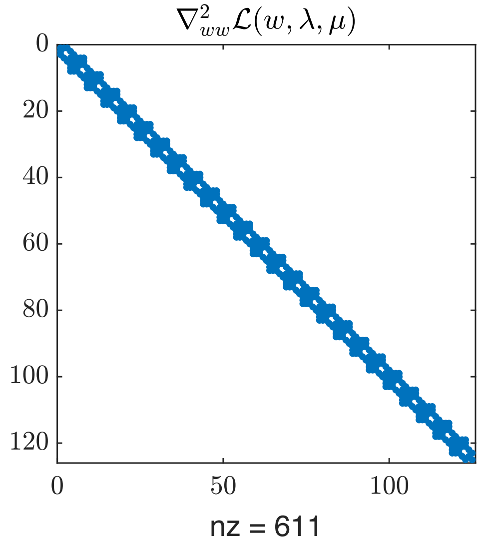
$$\min_{w \in \mathbb{R}^{n_x}} \ F(w)$$

$$\begin{aligned} \text{s.t. } G(w) &= 0 \\ H(w) &\geq 0 \end{aligned}$$

Large and sparse NLP

Sparse NLP resulting from direct transcription





Variables $w = (\mathbf{x}, \mathbf{z}, \mathbf{u})$

Nonlinear Program (NLP)

$$\begin{aligned} \min_{w\in\mathbb{R}^{n_x}} \quad & F(w) \\ \text{s.t.} \quad & G(w) = 0 \\ & H(w) \geq 0 \end{aligned}$$

Large and sparse NLP

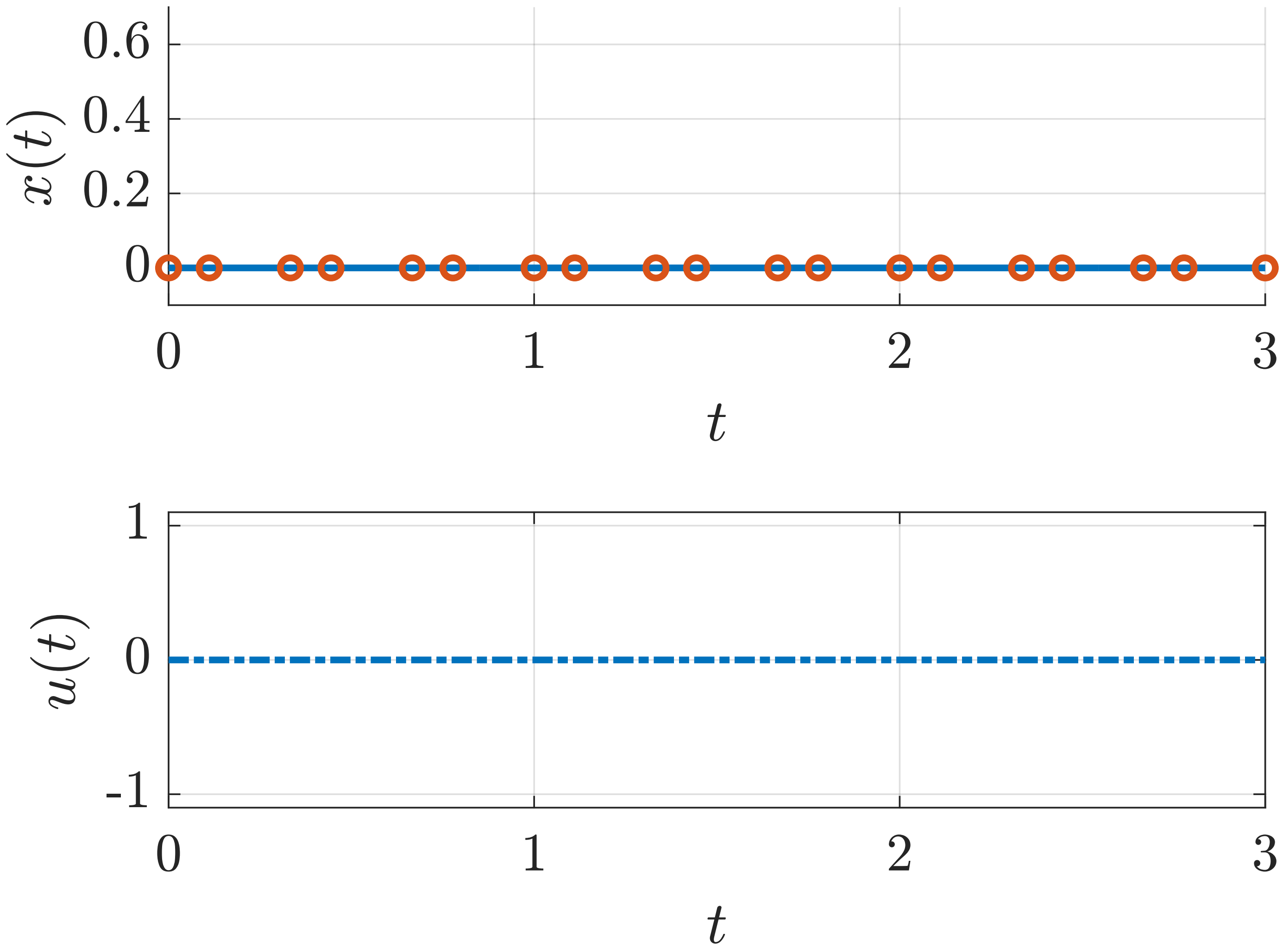
Illustrative example of direct collocation with Newton-type optimization

Illustrative nonlinear optimal control problem (with one state and one control)

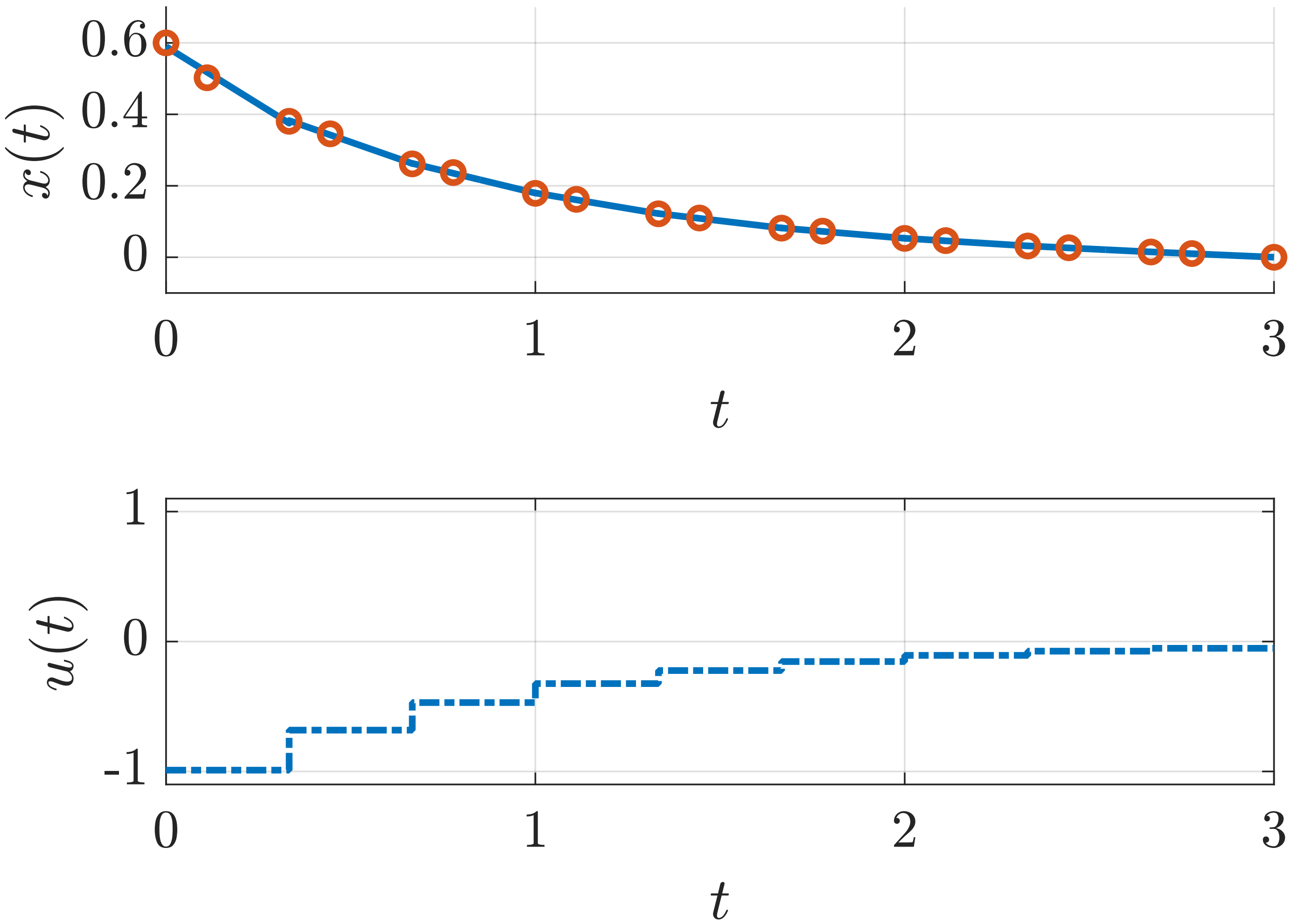
$$\begin{aligned}
&\underset{x(\cdot),u(\cdot)}{\text{minimize}} \quad \int_0^3 x(t)^2 + u(t)^2 \, dt \\
&\text{subject to} \\
&\qquad x(0) = \bar{x}_0 && && \text{(initial value, } \bar{x}_0 = 0.6\text{)} \\
&\qquad \dot{x} = (1+x)x + u, && && \text{(ODE model)} \\
&\qquad -1 \leq u(t) \leq 1, && t \in [0,3] && \text{(bounds)} \\
&\qquad x(3) = 0 && && \text{(terminal constraint)}
\end{aligned}$$

- choose $N = 9$ equal intervals and Radau-IIA collocation with $n_s = 2$ stages
- obtain nonlinear program with $n_x + (2n_s + 1)Nn_x + Nn_u$ variables
- initialize with zeros everywhere, solve with CasADi and Ipopt (interior point)

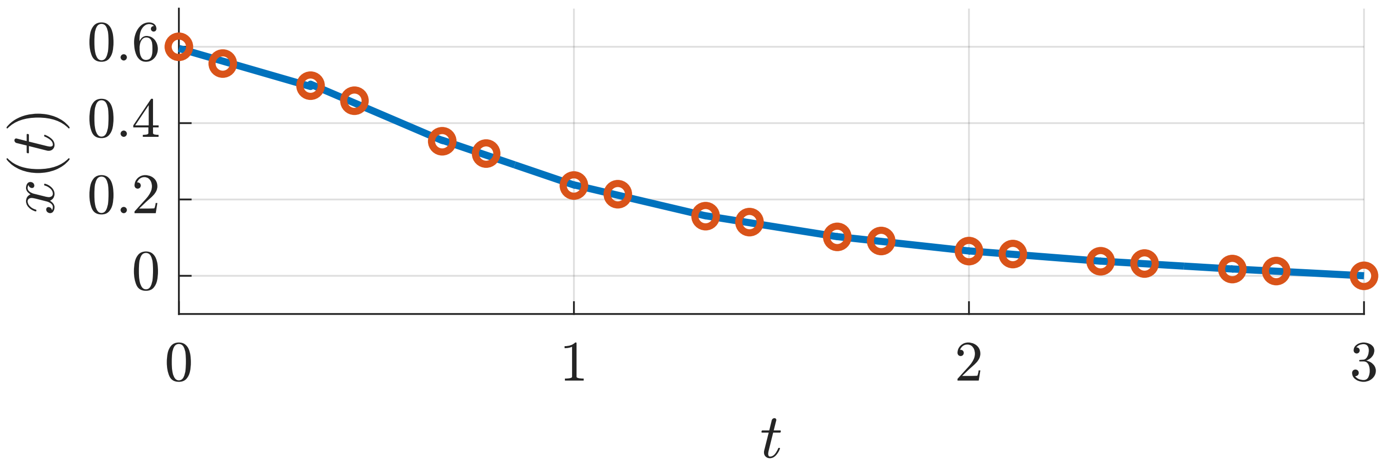
Illustrative example: Initialization

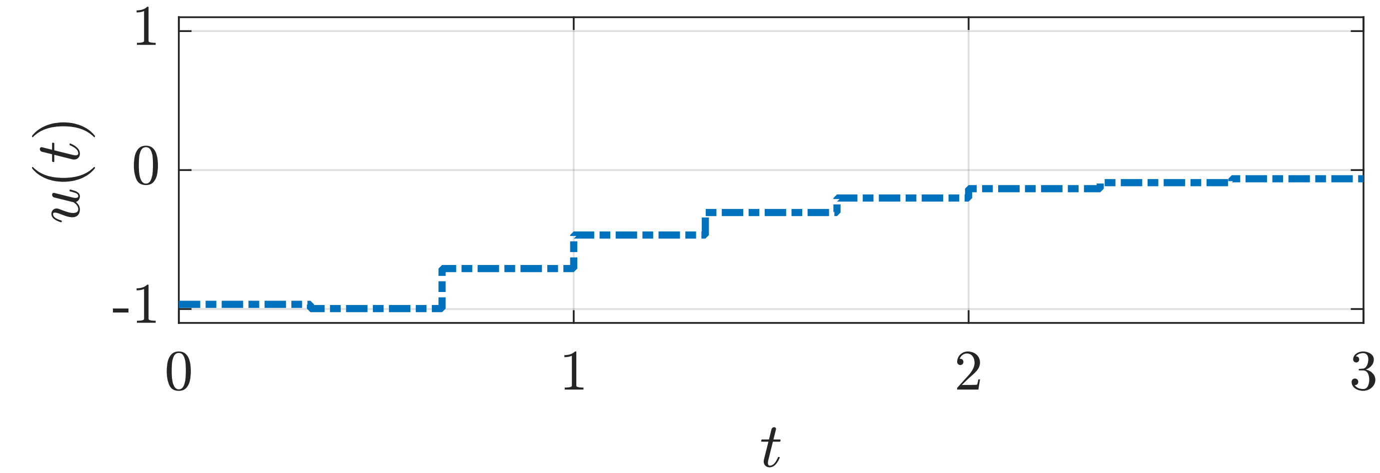


Illustrative example: First Iterate

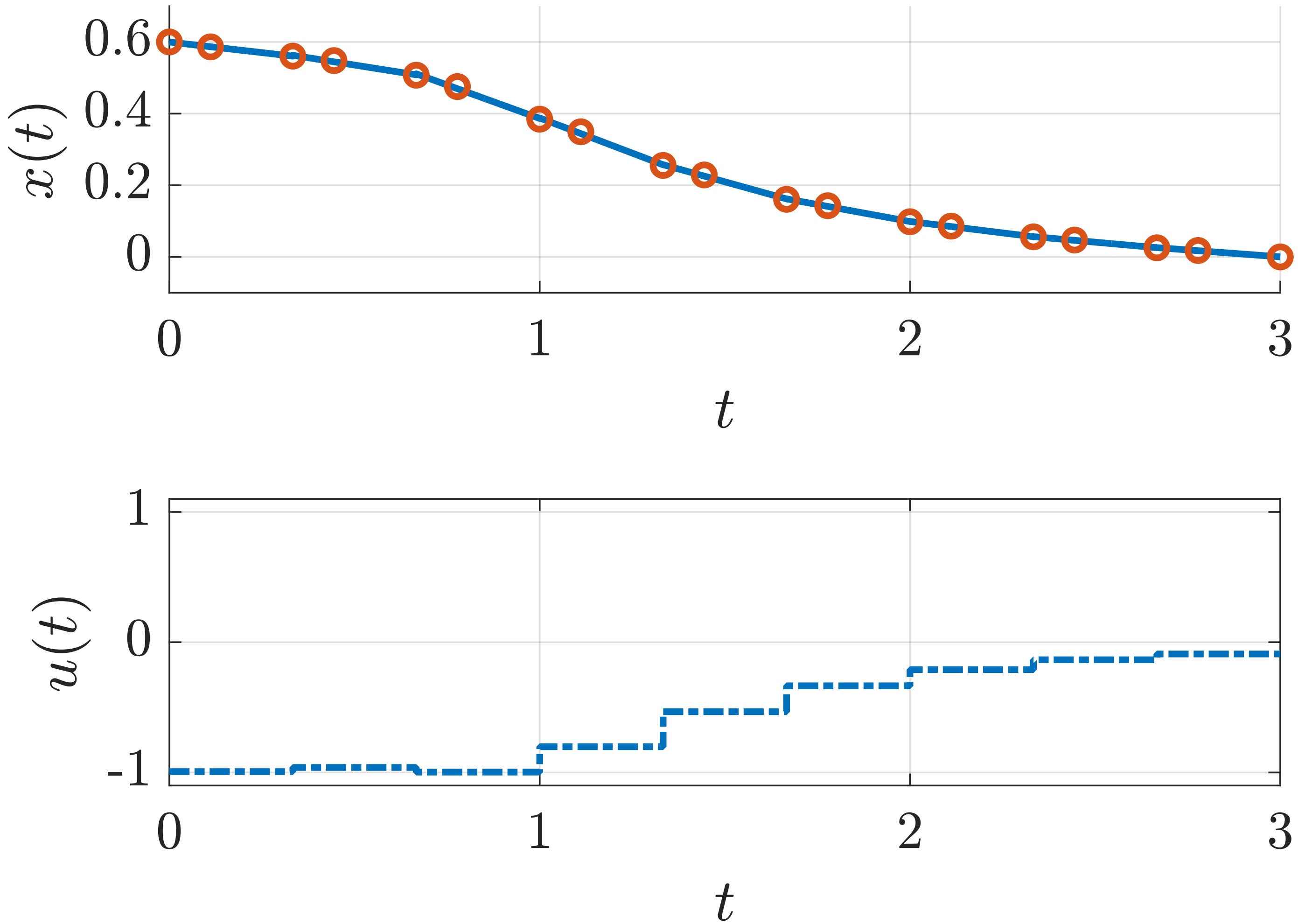


Illustrative example: Second Iterate

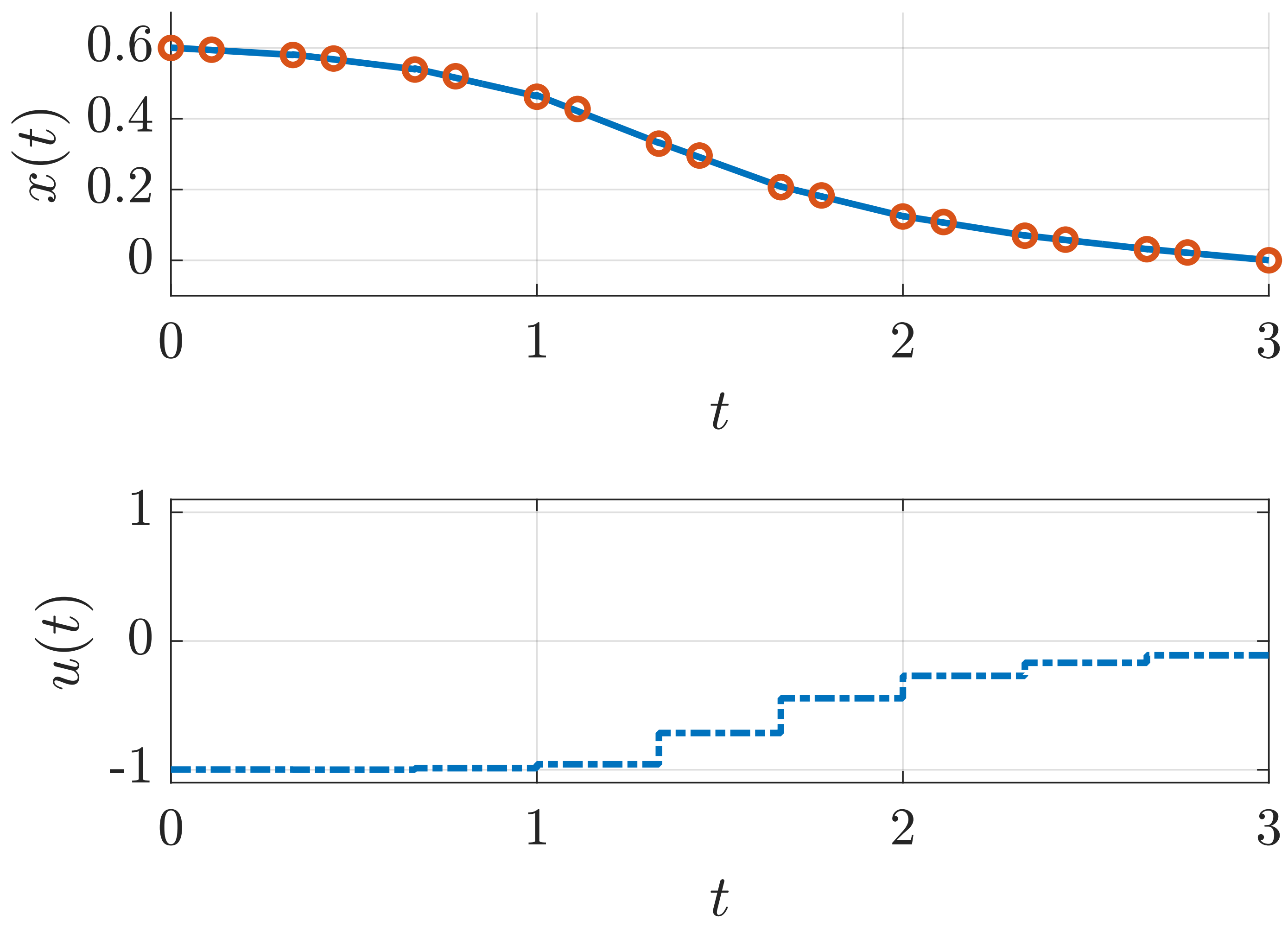




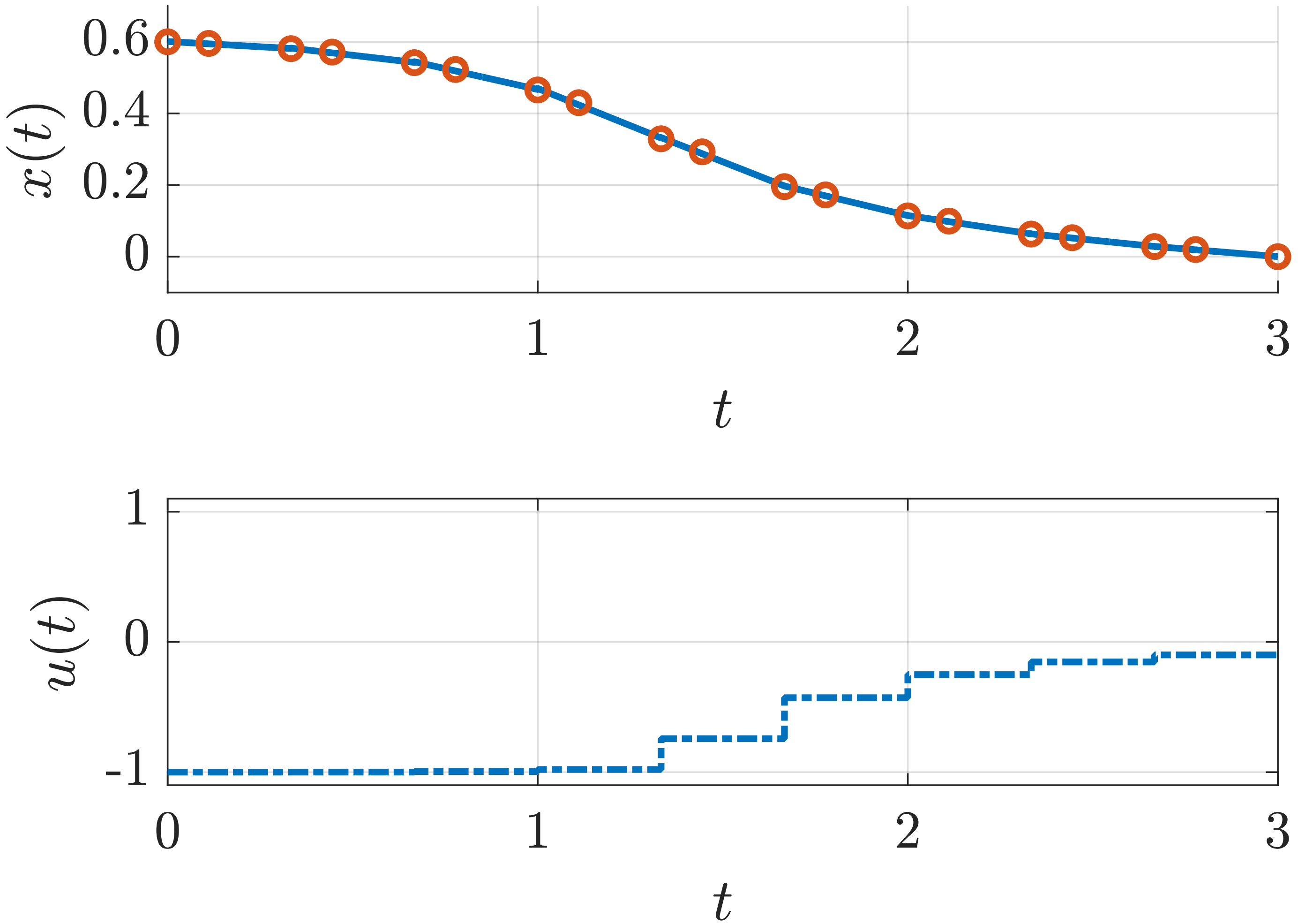
Illustrative example: Third Iterate



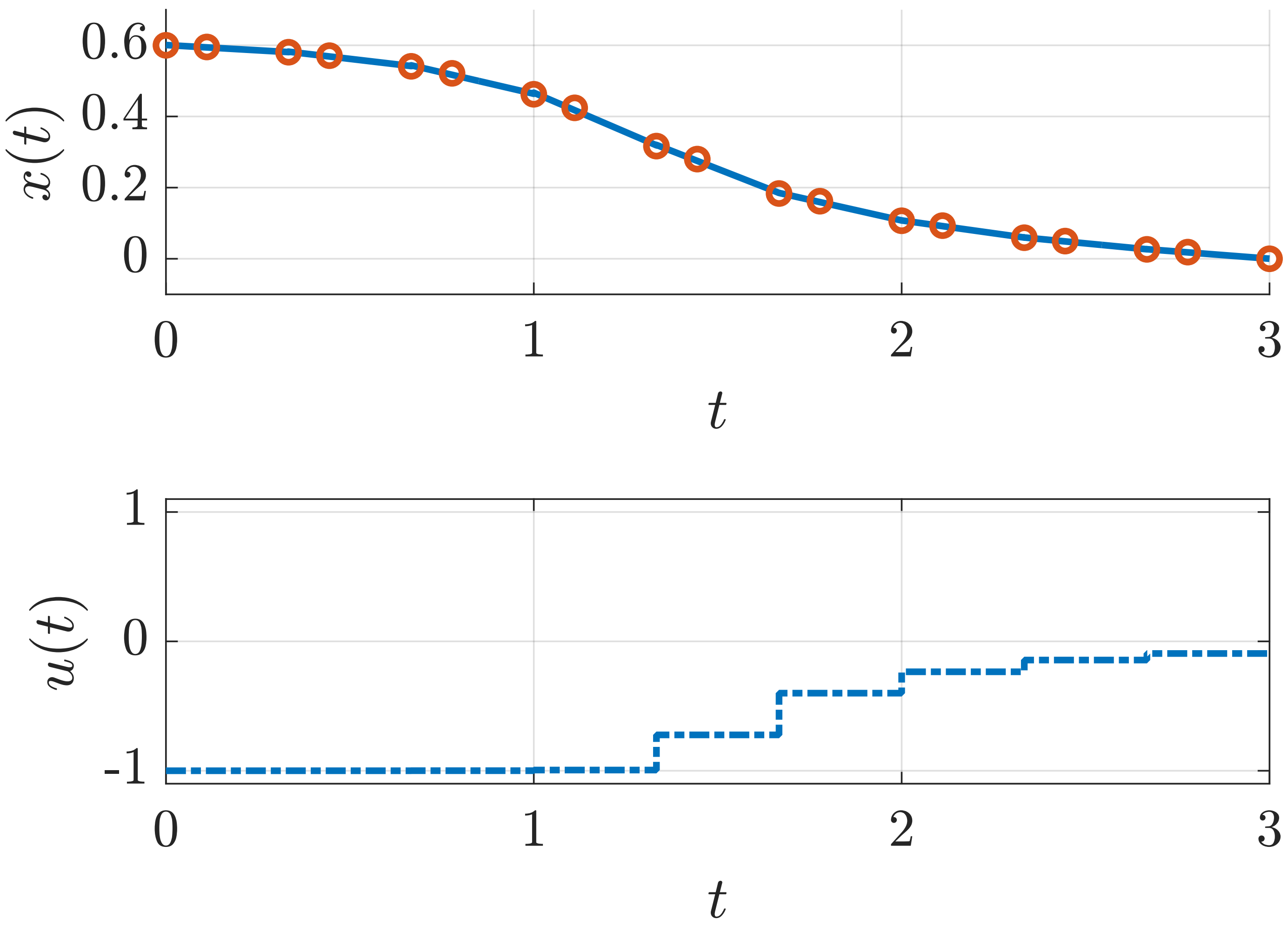
Illustrative example: Fourth Iterate



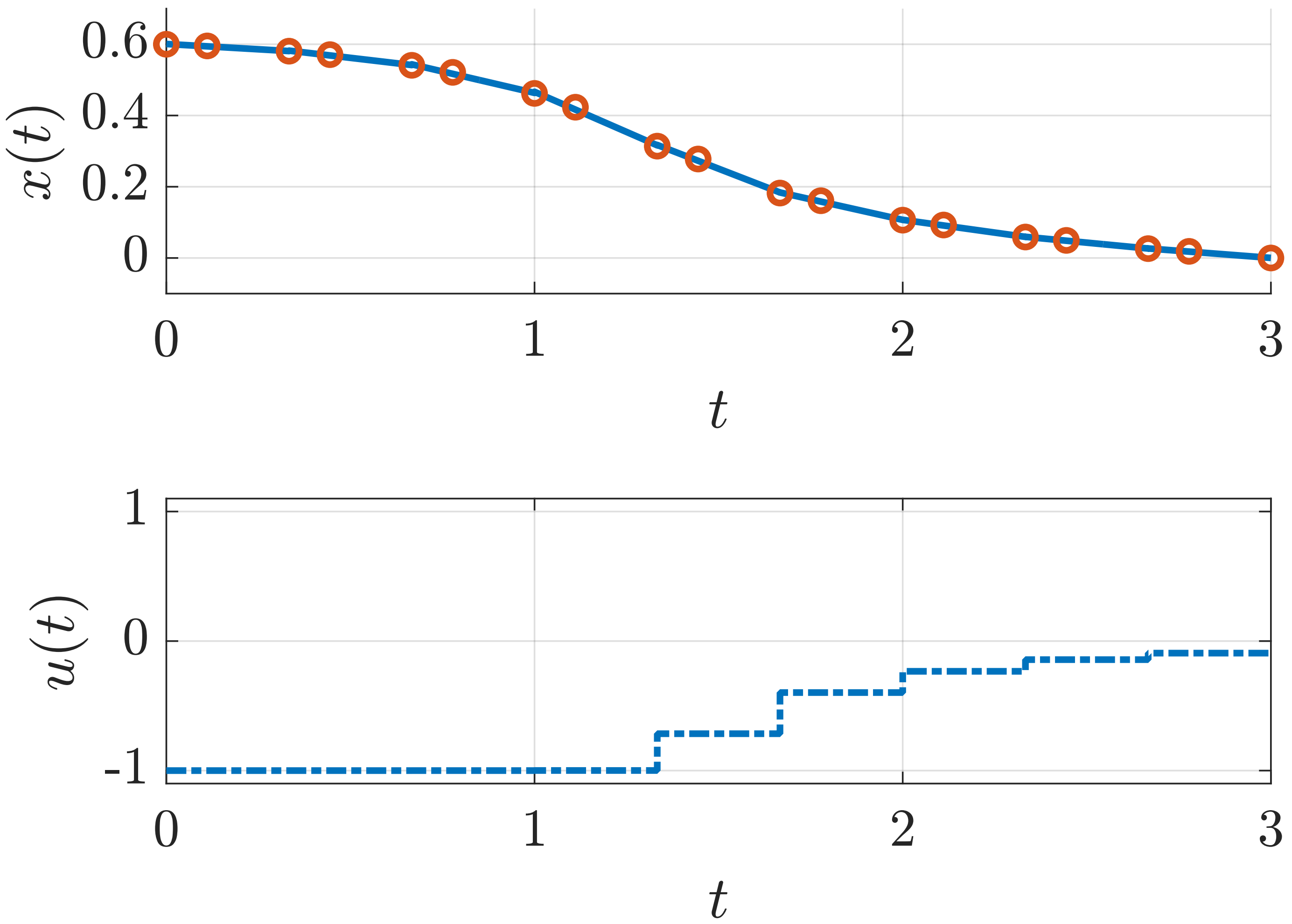
Illustrative example: Fifth Iterate



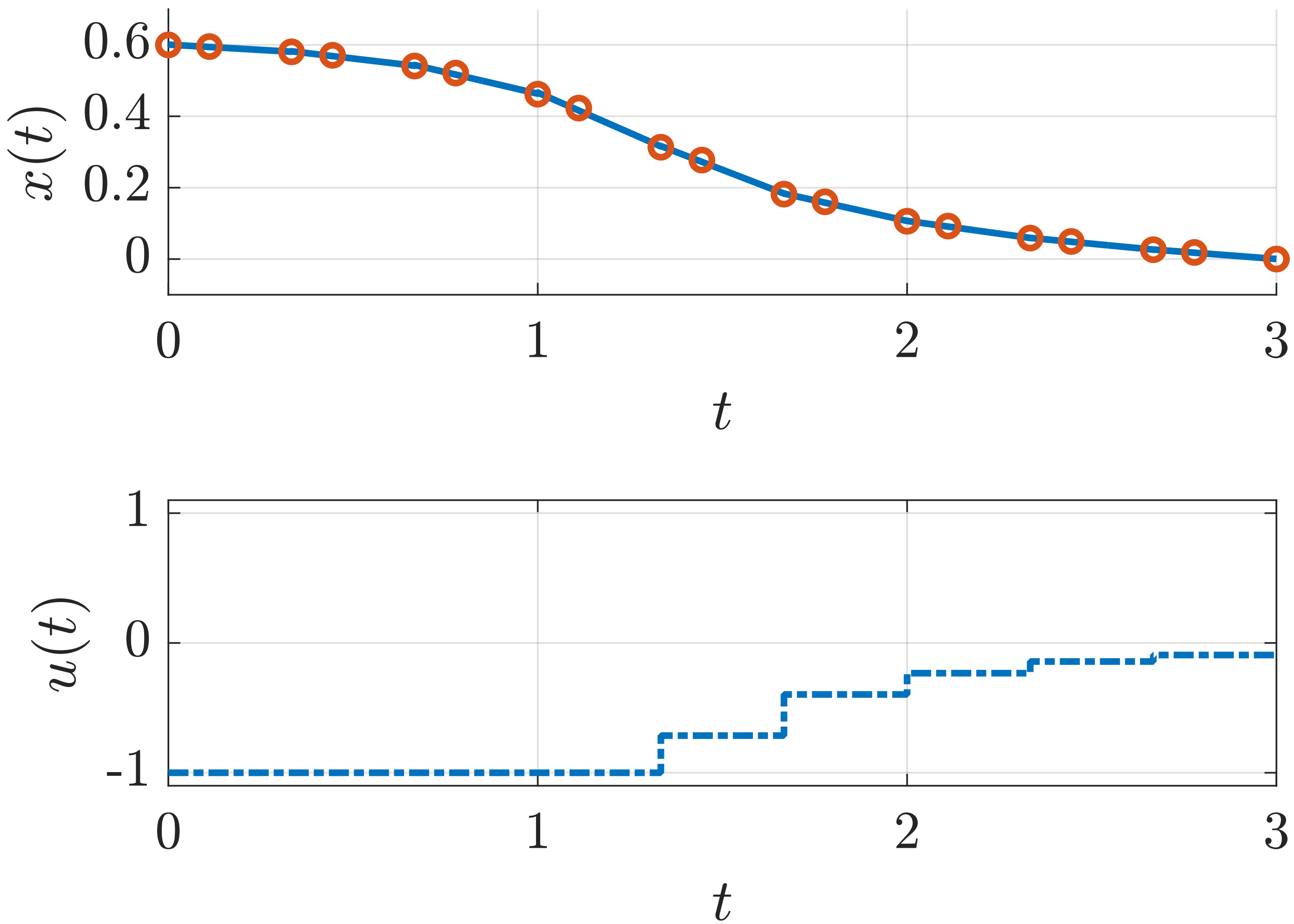
Illustrative example: Sixth Iterate



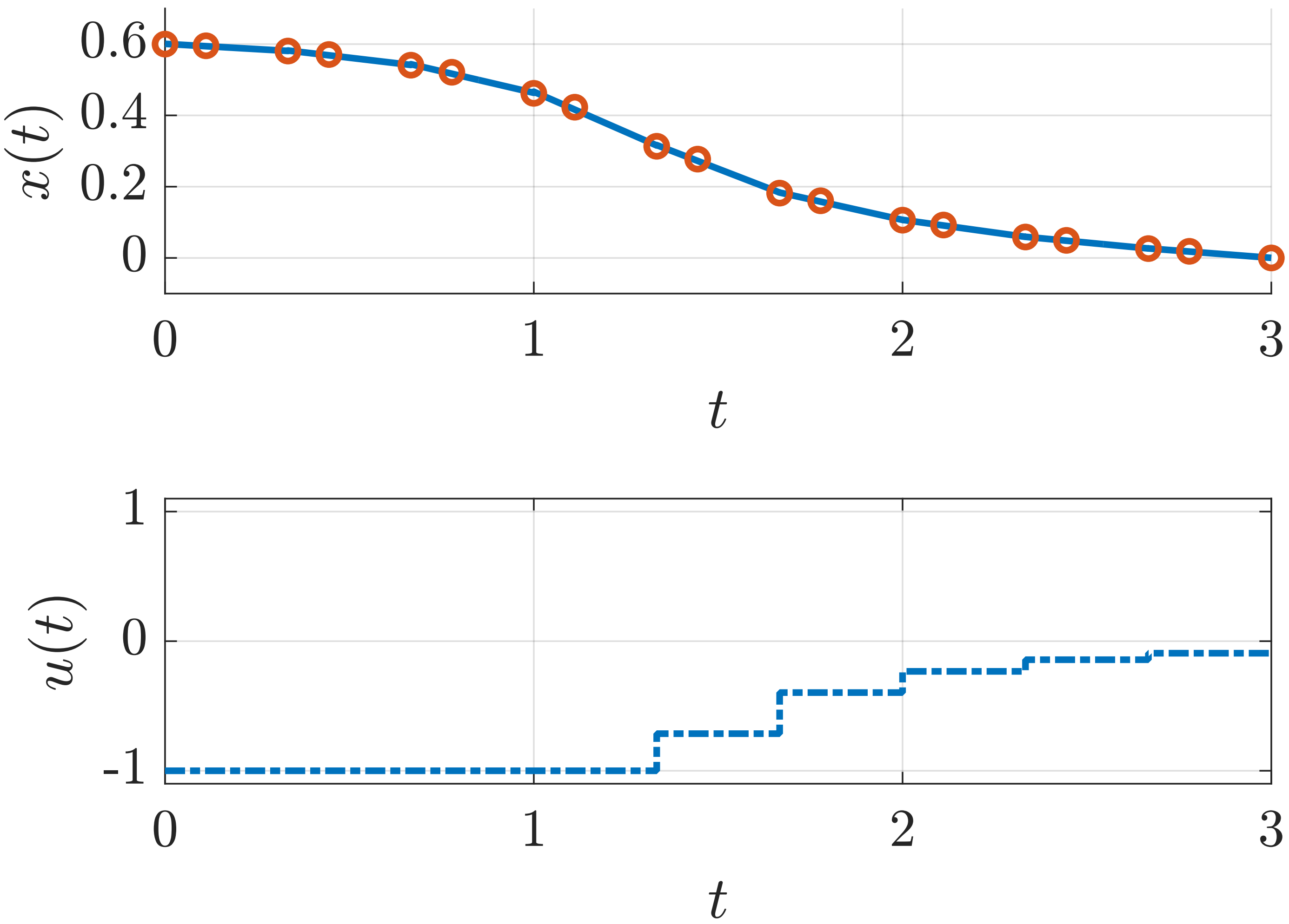
Illustrative example: Seventh Iterate



Illustrative example: Eighth Iterate



Illustrative example: Solution after Nine Newton-type Iterations



More Complex Example: Power Optimal Trajectories in Airborne Wind Energy (AWE)

formulated and solved daily by practitioners using open-source python package “AWEBox” [De Schutter et al. 2023]

For simple plane attached to a tether:

- 20 differential states (3+3 trans, 9+3 rotation, 1+1 tether)
- 1 algebraic state (tether force)
- 8 invariants (6 rotation, 2 due to tether constraint)
- 3 control inputs (aileron, elevator, tether length)

Translational:

$$\begin{bmatrix} m & 0 & 0 & x \\ 0 & m & 0 & y \\ 0 & 0 & m & z \\ x & y & z & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \lambda \end{bmatrix} = \begin{bmatrix} F_x + m\left(\dot{\delta}^2 r_A + \dot{\delta}^2 x + 2\dot{\delta}\dot{y} + \ddot{\delta} y\right) \\ F_y + m\left(y\dot{\delta}^2 - 2\dot{x}\dot{\delta} - \ddot{\delta}(rA + x)\right) \\ F_z - gm \\ -\dot{x}^2 - \dot{y}^2 - \dot{z}^2 \end{bmatrix}$$

Rotational:

$$\dot{R} = R\omega_\times - R^T \begin{bmatrix} 0 \\ 0 \\ \dot{\delta} \end{bmatrix}, \qquad J\dot{\omega} = T - \omega \times J\omega, \qquad R = \begin{bmatrix} \vec{E}_x & \vec{E}_y & \vec{E}_z \end{bmatrix}$$

Aero. coefficients:

$$\vec{v} = \begin{bmatrix} \dot{x} - \dot{\delta} y \\ \dot{y} + \dot{\delta}(r_{\mathrm{A}} + x) \\ \dot{z} \end{bmatrix} - \vec{w}(x, y, z, \delta, t), \qquad \alpha = -\frac{\vec{E}_z^T \vec{v}}{\vec{E}_x^T \vec{v}}, \qquad \beta = \frac{\vec{E}_y^T \vec{v}}{\vec{E}_x^T \vec{v}}$$

Aero. forces/torques:

$$\vec{F}_{\mathrm{A}} = \frac{1}{2}\rho A \|\vec{v}\| (C_{\mathrm{L}} \vec{v} \times \vec{E}_y - C_{\mathrm{D}} \vec{v}), \quad \vec{T}_{\mathrm{A}} = \frac{1}{2}\rho A \|\vec{v}\|^2 \begin{bmatrix} C_{\mathrm{R}} \\ C_{\mathrm{P}} \\ C_{\mathrm{Y}} \end{bmatrix}$$

Newton-Type Optimization Iterations for Power Optimal Flight

(video by Greg Horn, using CasADi and Ipopt as optimization engine)