Hochschule Karlsruhe
Technik und Wirtschaft
UNIVERSITY OF APPLIED SCIENCES

UNI
FREIBURG

Newton-Type Algorithms for Nonlinear Constrained Optimization

Angelika Altmann-Dieses

Faculty of Management Science and Engineering

Karlsruhe University of Applied Sciences

Moritz Diehl

Department of Microsystems Engineering (IMTEK) and Department of Mathematics

University of Freiburg

(some slide material was provided by W. Bangerth and K. Mombaur)

Overview Constrained Optimization

- Necessary Optimality Conditions (KKT-conditions)
- Newton-type methods for equality constrained optimisation
- How to treat inequalites? (example: Interior Point Methods)

Nonlinear Programming

- General problem formulation:

$$\boxed{\min_{x} \quad f(x) \quad \text{s.t.} \quad \left\{ \begin{array}{lcl} g(x) & = & 0 \\ h(x) & \geq & 0 \end{array} \right.}$$

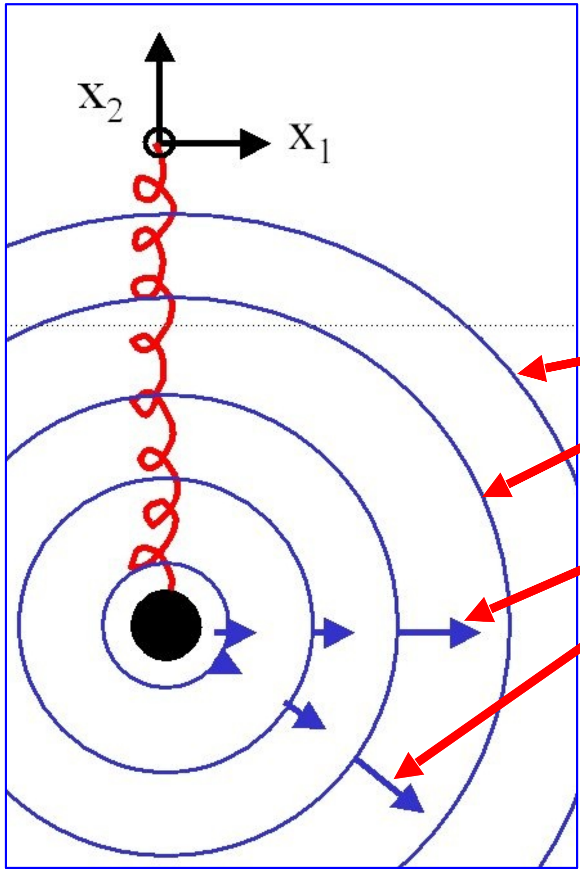
f objective function / cost function

g equality constraints

h inequality constraints

f,g,h shall be smooth (twice differentiable) functions

Recall: ball on a spring without constraints



$$\boxed{\min_{x \in R^2} x_1^2 + x_2^2 + m x_2}$$

contour lines of *f(x)*

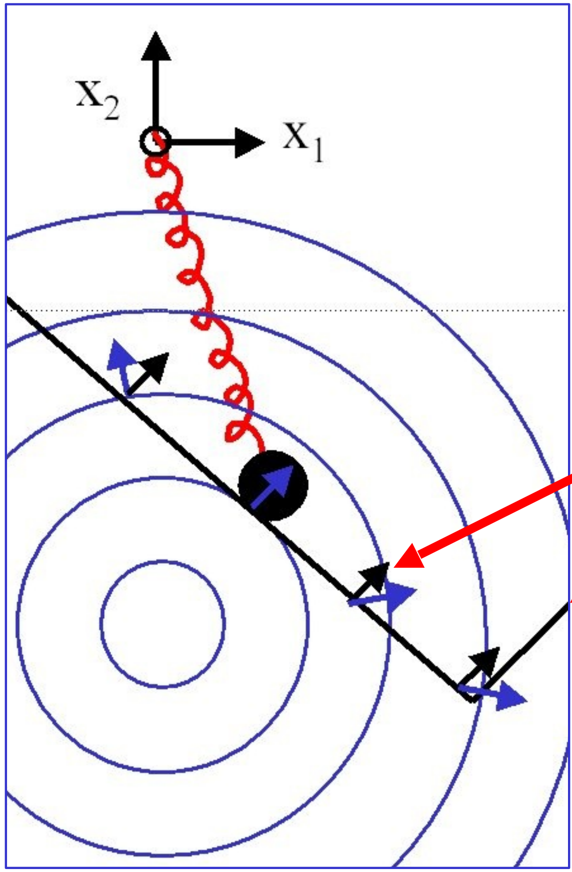
gradient vector

$$\nabla f(x) = (2x_1, 2x_2 + m)$$

unconstrained minimum:

$$0 = \nabla f(x^*) \Leftrightarrow (x_1^*, x_2^*) = (0, -\frac{m}{2})$$

Now: ball on a spring with constraints



$$\min f(x)$$
$$h_1(x) := 1 + x_1 + x_2 \quad \geq \quad 0$$
$$h_2(x) := 3 - x_1 + x_2 \quad \geq \quad 0$$

gradient ∇h_1 of active constraint

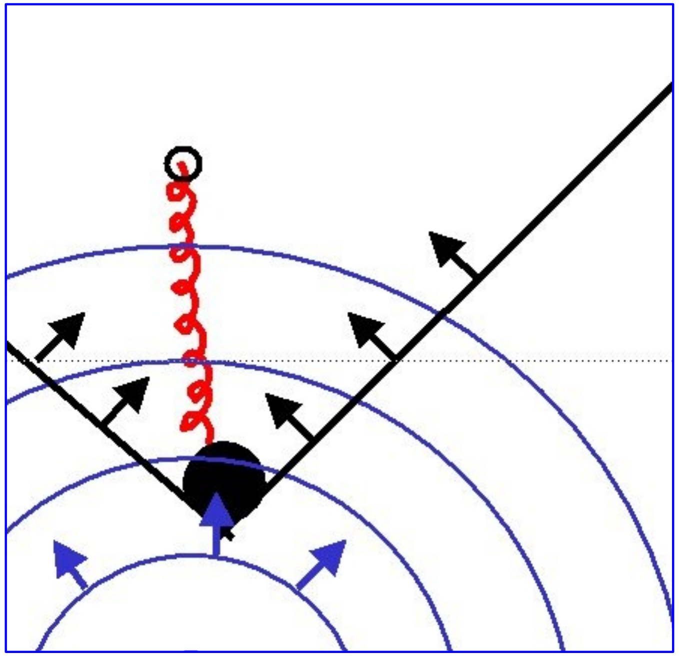
inactive constraint h_2

constrained minimum:

$$\boxed{\nabla f(x^*) = \mu_1 \nabla h_1(x^*)}$$

Lagrange multiplier

Ball on a spring with two active constraints



$$\min f(x)$$

$$h_1(x) := 1 + x_1 + x_2 \quad \geq \quad 0$$

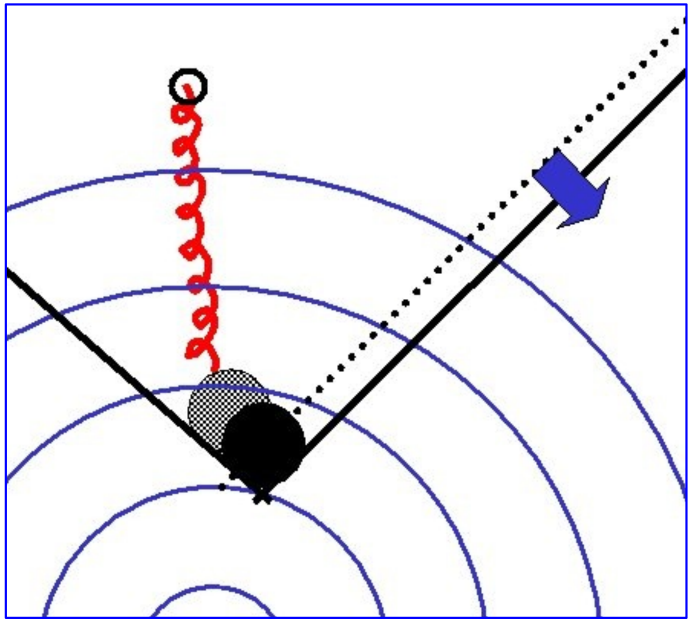
$$h_2(x) := 3 - x_1 + x_2 \quad \geq \quad 0$$

„equilibrium of forces“

$$\boxed{\nabla f(x^*) = \mu_1 \nabla h_1(x^*) + \mu_2 \nabla h_2(x^*) \qquad \mu_1, \mu_2 \geq 0}$$

„constraint forces“

Multipliers as „shadow prices“



What happens if we relax a constraint?
Feasible set becomes bigger,
so new minimum $f(x_\varepsilon{}^*)$ becomes smaller.
How much would we gain?

$$\boxed{f(x_\varepsilon^*) \approx f(x^*) - \mu\varepsilon}$$

old constraint: $h(x) \geq 0$

new constraint: $h(x) + \varepsilon \geq 0$

Multipliers show the hidden cost of constraints.

The Lagrangian Function

For constrained problems, introduce modification of objective function:

$$\boxed{L(x,\lambda,\mu) := f(x^*) - \sum \lambda_i g_i(x) - \sum \mu_i h_i(x)}$$

- equality multipliers λ_i may have both signs in a solution
- inequality multipliers μ_i cannot be negative (cf. shadow prices)
- for inactive constraints, multipliers μ_i are zero

"Equilibrium of forces" can now be written as:

$$\boxed{\nabla_x L(x^*,\lambda^*,\mu^*) = 0}$$

Necessary optimality conditions

Karush-Kuhn-Tucker (KKT) conditions:

If x^* a local minimum, then

- x^* feasible, i.e., $\boxed{g(x^*) = 0}$ and $\boxed{h(x^*) \geq 0}$
- there exist λ^*, μ^* such that $\boxed{\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0}$
- $\boxed{\mu^* \geq 0}$
- and $\boxed{h(x^*)^\top \mu^* = 0}$

(i.e., $\mu_i^* = 0$ or $h_i(x^*) = 0$ for each i)

Overview Constrained Optimization

- Necessary Optimality Conditions (KKT-conditions)
- **Newton-type methods for equality constrained optimisation**
- How to treat inequalites? (example: Interior Point Methods)

...

- ...

IPOPT: an Interior Point Optimization Algorithm

Math. Program., Ser. A 106, 25–57 (2006)

Andreas Wächter · Lorenz T. Biegler

On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming

Received: March 12, 2004 / Accepted: September 2, 2004
Published online: April 28, 2005 –

Abstract. We present a primal-dual interior-point algorithm with a filter line-search method for nonlinear programming. Local and global convergence properties of this method were analyzed in previous work. Here we provide a comprehensive description of the algorithm, including the feasibility restoration phase for the filter method, second-order corrections, and inertia correction of the KKT matrix. Heuristics are also considered that allow faster performance. This method has been implemented in the IPOPT code, which we demonstrate in a detailed numerical study based on 954 problems from the CUTEr test set. An evaluation is made of several line-search options, and a comparison is provided with two state-of-the-art interior-point codes for nonlinear programming.

CasADi optimisation environment

CasADi – A software framework for nonlinear optimization and optimal control

Joel A. E. Andersson · Joris Gillis · Greg Horn · James B. Rawlings · Moritz Diehl (submitted)

existing reference:

S. Forth et al. (eds.), *Recent Advances in Algorithmic Differentiation*, Lecture Notes in Computational Science and Engineering 87, DOI 10.1007/978-3-642-30023-3_27, 297

CasADi: A Symbolic Package for Automatic Differentiation and Optimal Control

Joel Andersson, Johan Åkesson, and Moritz Diehl

Summary

- Lagrangian function plays important role in constrained optimization
- KKT conditions are necessary optimality conditions
- Newton-type methods try to find a KKT point by successive linearizations
- Inequality constraints can be addressed by Interior Point (IP) methods, e.g. in IPOPT code
- Derivatives of problem functions can be automatically provided e.g. by CasADi optimisation environment