Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems

Jakob Harzer, Jochem De Schutter, Per Rutquist and Moritz Diehl

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Optimization
Simulation
Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems
Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems
Setting

\[ x(\tau) \quad \text{is a trajectory of} \quad \frac{dx}{d\tau} = F(X) \quad (1) \]

But unfortunately we don’t know the dynamics \( F \).

At some \( X^* = X(\tau^*) \) approximate \( F(X^*) \).

Tool: Solution Operator

\[ X(\tau + 1) = \Phi_F(X^*) \quad (2) \]
\( X(\tau) \) is a trajectory of

\[
\frac{d}{d\tau} X = F(X)
\]  

(1)
Setting

\[ x(\tau) \]

\[ X(\tau) \]

\[ \tau \]

\[ \tau^{\ast} \]

\[ X^{\ast} \]

\[ F \]

\[ d_{\tau} X = F(X) \quad (1) \]

\[ \text{But unfortunately we don’t know the dynamics } F \]
Setting

- $X(\tau)$ is a trajectory of
  \[
  \frac{d}{d\tau} X = F(X) \tag{1}
  \]

- But unfortunately we don’t know the dynamics $F$

- At some $X^* = X(\tau^*)$ approximate $F(X^*)$

Implicit Central Difference Approximations

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\[ \tau^* - 1 \leq \tau \leq \tau^* \leq \tau^* + 1 \]

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  \[ X(\tau + 1) = \Phi_1^F(X(\tau)) \] (2)
Setting

$\tau^* - 1 \quad \tau^* \quad \tau^* + 1$

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$$X(\tau + 1) = \Phi_1^F(X(\tau))$$  \hspace{1cm} (2)

$$= \Psi^1(X(\tau))$$  \hspace{1cm} (3)
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Setting

$\tau^* - 1 \rightarrow \tau^* \rightarrow \tau^* + 1$

$\nabla \ n(\tau)$

$X(\tau)$ is a trajectory of

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$$X(\tau + 1) = \Phi_1^F(X(\tau)) \quad (2)$$

$$= \Psi_1^F(X(\tau)) \quad (3)$$
Dynamics Approximations

\[ \mathbf{F}(\mathbf{X}^*) \approx \Psi_1(\mathbf{X}^*) - \mathbf{X}^*_1(4) \]
Dynamics Approximations

\[ \tau^* - 1 \quad \tau^* \quad \tau^* + 1 \]

\[ x(\tau) \quad X(\tau) \]

\[ F(\tau) \approx \Psi(\tau) \]

Implicit Central Difference Approximations

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Dynamics Approximations

\[
x(\tau) X^{*} \Psi(\Psi^{*}) \Phi F \tau^{*} \approx \Psi^{1}(\Psi^{*}) - X^{*}^{1}(4)
\]
Dynamics Approximations

\[ F(X^*) \approx \frac{\Psi^1(X^*) - X^*}{1} \]  (4)

 Implicit Central Difference Approximations

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Dynamics Approximations

\[ F(X^*) \approx \frac{\Psi^1(X^*) - X^*}{1} \quad (4) \]
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\[ F(X^*) \approx \frac{\Psi^1(X^*) - X^*}{1} \quad \text{(4)} \]

\[ F(X^*) \approx \frac{\Psi^1(X^*) - \Psi^{-1}(X^*)}{2} \quad \text{(5)} \]
Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems
Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems
Implicit Dynamics Approximation - 2 Points

Two points \((X'_1, X'_2)\) at times \((\tau^* - 0.5, \tau^* + 0.5)\).

Interpolating polynomial \(P\).

Solve for \(X'_1, X'_2\):

\[
0 = X'_2 - \Psi(X'_1) \quad (6a)
\]

\[
0 = P(\tau^*) - X^* \quad (6b)
\]

Approximate the dynamics as

\[
F(X^*) \approx X'_2 - X'_1 \quad (7)
\]
Implicit Dynamics Approximation - 2 Points

\[
\begin{align*}
\tau^* - 1 & \quad \tau^* & \quad \tau^* + 1 \\
X(\tau) & \quad X_1' & \quad X_2' = \Psi(X_1')
\end{align*}
\]

Implicit Central Difference Approximations

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Implicit Dynamics Approximation - 2 Points

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\]

- Approximate the dynamics as

\[
F(X^*) \approx \frac{X'_2 - X'_1}{1} \quad (7)
\]
We need to solve a nonlinear system of equations.努力:

\[ \Psi(1) \times X'_{1} \]

Points \( X'_{1}, X'_{2} \) lie on a solution of the system \( X'(\tau) \), that is not \( X(\tau) \).

Implicit Dynamics Approximation - Observations
We need to solve a nonlinear system of equations.
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Effort: \( \Psi \times 1 \)
We need to solve a nonlinear system of equations.

- Effort: $\Psi \times 1$
- Points $X'_1, X'_2$ lie on a solution of the system $X' (\tau)$, that is not $X (\tau)$.
Three points \((X'_1, X'_2, X'_3)\) at times \((\tau^* - 1, \tau^*, \tau^* + 1)\).
Implicit Dynamics Approximation - 3 Points

Three points \((X'_1, X'_2, X'_3)\) at times \((\tau^* - 1, \tau^*, \tau^* + 1)\).
Implicit Dynamics Approximation - 3 Points

Three points \((X'_1, X'_2, X'_3)\) at times \((\tau^* - 1, \tau^*, \tau^* + 1)\).

The points satisfy

\[
X'_2 = \Psi(X'_1) \tag{8}
\]

\[
X'_3 = \Psi(X'_2) \tag{9}
\]

\[
P(\tau^*) = X(\tau^*) \tag{10}
\]
Implicit Dynamics Approximation - 3 Points

▶ **Three** points \((X'_1, X'_2, X'_3)\) at times \((\tau^* - 1, \tau^*, \tau^* + 1)\).

▶ The points satisfy

\[
X'_2 = \Psi(X'_1) \quad (8)
\]

\[
X'_3 = \Psi(X'_2) \quad (9)
\]

\[
P(\tau^*) = X(\tau^*) \quad (10)
\]
Three points \((X_1', X_2', X_3')\) at times \((\tau^* - 1, \tau^*, \tau^* + 1)\).

The points satisfy

\[
X_2' = \Psi(X_1') \\
X_3' = \Psi(X_2') \\
P(\tau^*) = X(\tau^*)
\]

Approximate the dynamics as

\[
F(X^*) \approx \frac{X_3' - X_1'}{2}
\]
Implicit Dynamics Approximation - 3 Points

- **Three points** \((X'_1, X'_2, X'_3)\) at times \((\tau^*-1, \tau^*, \tau^*+1)\).
- The points satisfy
  \[
  X'_2 = \Psi(X'_1) \tag{8}
  
  X'_3 = \Psi(X'_2) \tag{9}
  
  P(\tau^*) = X(\tau^*) \tag{10}
  
- Approximate the dynamics as
  \[
  F(X^*) \approx \frac{X'_3 - X'_1}{2} \tag{11}
  
- We recover the explicit central difference scheme from before!
Let $\tau = \tau^* + \Delta \tau$, $K$ stroboscopic points $X'_1, \ldots, X'_K$ at equidistant times

$$\Delta \tau_k = k - \frac{K + 1}{2}, \quad k = 1, \ldots, K$$  \hspace{1cm} (12)
Let $\tau = \tau^* + \Delta \tau$, $K$ stroboscopic points $X'_1, \ldots, X'_K$ at equidistant times

$$\Delta \tau_k = k - \frac{K + 1}{2}, \quad k = 1, \ldots, K$$  \hfill (12)

i.e.

$$\Delta \tau_k \in \begin{cases} \{\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\} & \text{for } K \text{ even} \\ \{\ldots, -1, 0, 1, \ldots\} & \text{for } K \text{ odd} \end{cases}$$  \hfill (13)
Implicit Dynamics Approximation - K Points

- Let \( \tau = \tau^* + \Delta \tau \), \( K \) stroboscopic points \( X'_1, \ldots, X'_K \) at equidistant times

\[
\Delta \tau_k = k - \frac{K + 1}{2}, \quad k = 1, \ldots, K \tag{12}
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\Delta \tau_k \in \begin{cases} 
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\{ \ldots, -1, 0, 1, \ldots \} & \text{for } K \text{ odd} 
\end{cases} \tag{13}
\]

- Interpolating polynomial

\[
P(\Delta \tau) = \sum_{k=1}^{K} \ell_k(\Delta \tau) X'_k, \quad \text{where } \ell_k(\Delta \tau) = \prod_{n=1, k \neq n}^{K} \frac{(\Delta \tau - \Delta \tau_n)}{(\Delta \tau_k - \Delta \tau_n)} \tag{14}
\]
Let $\tau = \tau^* + \Delta\tau$, $K$ stroboscopic points $X'_1, \ldots, X'_K$ at equidistant times
\begin{equation}
\Delta\tau_k = k - \frac{K + 1}{2}, \quad k = 1, \ldots, K
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Interpolating polynomial
\begin{equation}
P(\Delta\tau) = \sum_{k=1}^{K} \ell_k(\Delta\tau)X'_k, \quad \text{where} \quad \ell_k(\Delta\tau) = \prod_{n=1, k\neq n}^{K} \frac{\Delta\tau - \Delta\tau_n}{\Delta\tau_k - \Delta\tau_n}
\end{equation}

with
\begin{equation}
P(0) = \sum_{k=1}^{K} \ell_k(0)X'_k, \quad \dot{P}(0) = \sum_{k=1}^{K} \dot{\ell}_k(0)X'_k.
\end{equation}
Implicit Dynamics Approximation - K Points

Let $\tau = \tau^* + \Delta \tau$, $K$ stroboscopic points $X_1', \ldots, X_K'$ at equidistant times

$$\Delta \tau_k = k - \frac{K + 1}{2}, \quad k = 1, \ldots, K$$

i.e.

$$\Delta \tau_k \in \left\{ \ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots \right\} \quad \text{for } K \text{ even}$$

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Interpolating polynomial

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with

$$P(0) = \sum_{k=1}^{K} b_k X_k', \quad \dot{P}(0) = \sum_{k=1}^{K} c_k X_k'.$$

Implicit Central Difference Approximations

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Implicit Dynamics Approximation - K Points

- Solve

\[ 0 = X_2' - \Psi(X_1') \]  \hspace{1cm} (16a)

\[ \vdots \]  \hspace{1cm} (16b)

\[ 0 = X_K' - \Psi(X_{K-1}') \]  \hspace{1cm} (16c)

\[ 0 = X^* - \sum_{k=1}^{K} b_k X_k' , \]  \hspace{1cm} (16d)

Approximate the dynamics as

\[ F(X^*) \approx \sum_{k=1}^{K} c_k X_k' \]

Implicit Central Difference Approximations

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Implicit Dynamics Approximation - K Points

Solve

\[ 0 = X'_2 - \Psi(X'_1) \]  
\[ \vdots \]  
\[ 0 = X'_K - \Psi(X'_{K-1}) \]  
\[ 0 = X^* - \sum_{k=1}^{K} b_k X'_k, \]  

Approximate the dynamics as

\[ F(X^*) \approx \sum_{k=1}^{K} c_k X'_k \]
Example: $K = 4$
Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems
Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems
Highly Oscillatory Systems

- Highly Oscillatory Systems with \( \epsilon \ll 1 \)

\[
\dot{x} = f_0(x) + \epsilon f_1(x, \tau)
\]
Highly Oscillatory Systems with $\epsilon \ll 1$

$$\dot{x} = f_0(x) + \epsilon f_1(x, \tau)$$

Oscillatory Dynamics

$$\dot{x} = f_0(x)$$

with 1-periodic solution $x_0(\tau)$.
Highly Oscillatory Systems with $\epsilon \ll 1$

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Implicit Central Difference Approximations

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Highly Oscillatory Systems with $\epsilon \ll 1$

$$\dot{x} = f_0(x) + \epsilon f_1(x, \tau)$$

Oscillatory Dynamics

$$\dot{x} = f_0(x)$$

with 1-periodic solution $x_0(\tau)$.

The perturbed solution $x(\tau)$ and unperturbed $x_0(\tau)$ differ by

$$\|x_0(\tau) - x(\tau)\| = \mathcal{O}(\epsilon)$$

on a timescale of 1.
Averaging Methods for Highly Oscillatory Systems

\[ \dot{x} = f(x) = f_0(x) + \epsilon f_1(x, t) \]
Averaging Methods for Highly Oscillatory Systems

\[ \dot{x} = f(x) = f_0(x) + \epsilon f_1(x, t) \]

Averaged system on timescale \( O(1/\epsilon) \)

If \( x(0) = X(0) \) then the solution to the averaged system satisfies \( x(k) = X(k) \), \( k \in \mathbb{Z} \)

Implicit Central Difference Approximations

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Averaging Methods for Highly Oscillatory Systems

\[ \dot{x} = f(x) = f_0(x) + \epsilon f_1(x, t) \]

\[ \dot{X} = \epsilon F_1(x) \]

If \( x(0) = X(0) \) then the solution to the averaged system satisfies
\[ x(k) = X(k), k \in \mathbb{Z} \]
Averaging Methods for Highly Oscillatory Systems

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Averaging

\[ \dot{X} = \epsilon F_1(x) \]

If \( x(0) = X(0) \) then the solution to the averaged system satisfies \( x(k) = X(k) \), \( k \in \mathbb{Z} \)
Averaging Methods for Highly Oscillatory Systems

\[ \dot{x} = f(x) = f_0(x) + \epsilon f_1(x, t) \]

High Order Stroboscopic Averaging

\[ \dot{x} = f(x) = f_0(x) + \epsilon f_1(x, t) \]

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High Order Stroboscopic Averaging

\[ \dot{X} = F(x) = \epsilon F_1(x) + \epsilon^2 F_2(x) + \ldots \]
Averaging Methods for Highly Oscillatory Systems

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High Order Stroboscopic Averaging

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- If \( x(0) = X(0) \) then the solution to averaged system satisfies
  \[ x(k) = X(k), \quad k \in \mathbb{Z} \]

- Original system \( f \) on timescale \( O(1) \)

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Averaging Methods for Highly Oscillatory Systems

\[ \dot{x} = f(x) = f_0(x) + \epsilon f_1(x, t) \]

High Order Stroboscopic Averaging

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- If \( x(0) = X(0) \) then the solution to averaged system satisfies

\[ x(k) = X(k), \quad k \in \mathbb{Z} \]

- Averaged system \( F \) on timescale \( O(1/\epsilon) \)

Implicit Central Difference Approximations

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From before:

\[ \Psi(X) = \Phi^F_1(X) \]
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\[ = \Phi^f_1(X) \]

Micro-integration

\[ \Psi(X) \approx \tilde{\Phi}^f_1(X) \]

by f.e. multiple RK steps.
From before:

\[ \Psi(X) = \Phi^F_1(X) = \Phi^f_1(X) \]

Micro-integration

\[ \Psi(X) \approx \tilde{\Phi}^f_1(X) \]

by f.e. multiple RK steps.

We can use this 'one-cycle' map to approximate the average dynamics!
Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems
Implicit Central Difference Approximations of Averaged Dynamics of Oscillatory Systems
Average Dynamics Approximation [1, 3]

\[ F(X^*) \approx \frac{\Psi^1(X^*) - X^*}{1} \]

\[ F(X^*) \approx \frac{\Psi^1(X^*) - \Psi^{-1}(X^*)}{2} \]
Average Dynamics Approximation \[1, 3\]

\[
F(X^*) \approx \frac{\Psi^1(X^*) - X^*}{1}
\]

\[
F(X^*) \approx \frac{\Psi^1(X^*) - \Psi^{-1}(X^*)}{2}
\]
Implicit Averaged Dynamics Approximation

Solve

\[ 0 = X'_{k+1} - \Psi(X'_k), \quad k = 1, \ldots, N - 1 \]

\[ 0 = X^* - \sum_{k=1}^{K} b_k X'_k, \]

Approximate the dynamics as

\[ F(X^*) \approx \sum_{k=1}^{K} c_k X'_k \]
Implicit Averaged Dynamics Approximation

<table>
<thead>
<tr>
<th>Solve</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 = X'_k + 1 - Ψ(X'_k), k = 1, ..., N - 1</td>
</tr>
<tr>
<td>0 = X^* - \sum_{k=1}^{K} b_k X'_k,</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Approximate the dynamics as</th>
</tr>
</thead>
<tbody>
<tr>
<td>F(X^*) ≈ \sum_{k=1}^{K} c_k X'_k</td>
</tr>
</tbody>
</table>
Implicit Averaged Dynamics Approximation

\[ \tau \ast - 1 - \tau \ast \tau \ast + 1 \]

\[ X(\tau) \]
\[ \Phi_f (X \ast) \]

\[ x(\tau) \]
\[ \tau^* - 1 \tau^* \tau^* + 1 \]

\[ X'_{k+1} - \Psi(X'_k), \quad k = 1, \ldots, N - 1 \]

\[ 0 = X^* - \sum_{k=1}^{K} b_k X'_k, \]

\[ F(X^*) \approx \sum_{k=1}^{K} c_k X'_k \]

\[ \| F(X) - F_{CD,K}(X) \| = \begin{cases} O(\epsilon^{K+1}) & \text{for } K \text{ even} \\ O(\epsilon^K) & \text{for } K \text{ odd} \end{cases} \]
Implicit Central Difference Approximations

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Numerical Method for Efficient Simulation [1]

At some point \((\tau^*, X^*)\):

(a) perform one or more micro-integrations
(b) approximate the averaged dynamics

Macro-integrate the averaged dynamics

Integration horizon of integer size

Three sources of error:
(a) errors in the micro-integration
(b) errors in the approximation of the dynamics
(c) errors in the macro-integration
At some point \((\tau^*, X^*)\):
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Macro-integrate the averaged dynamics

Integration horizon of integer size \(N\) cycles since

\[
x(N) = X(N)
\]
At some point ($\tau^*, X^*$):

(a) perform one or more micro-integrations to evaluate the one-cycle map

(b) approximate the averaged dynamics

Macro-integrate the averaged dynamics

Integration horizon of integer size $N$ cycles since

$$x(N) = X(N)$$

Three sources of error:
At some point \((\tau^*, X^*)\):

(a) perform one or more micro-integrations to evaluate the one-cycle map

(b) approximate the averaged dynamics

Macro-integrate the averaged dynamics

Integration horizon of integer size \(N\) cycles since

\[ x(N) = X(N) \]

Three sources of error:

(a) errors in the micro-integration
At some point \((\tau^*, X^*)\):

(a) perform one or more micro-integrations to evaluate the one-cycle map

(b) approximate the averaged dynamics

Macro-integrate the averaged dynamics

Integration horizon of integer size \(N\) cycles since

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x(N) = X(N)
\]

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(c) errors in the macro-integration
Integration Experiment

Linear Oscillator, $\epsilon = -10^{-3}$

$$\frac{d}{d\tau} x = \begin{bmatrix} \epsilon & -2\pi \\ 2\pi & \epsilon \end{bmatrix} x$$
Integration Experiment

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- Integrate over interval \( \tau \in [0, 100] \)
Integration Experiment

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Micro integration error is very dominant
Integration Experiment

- Micro integration error is very dominant
- Little sense in using high-order average dynamics approximations methods

\[ \epsilon = -1 \cdot 10^{-3}, \; N = 100, \; H = 20.0 \]
Integration Experiment

- Micro integration error is very dominant
- Little sense in using high-order average dynamics approximations methods
- Gain of the implicit methods is in the reduced effort to compute the approximation
Micro integration error is very dominant

Little sense in using high-order average dynamics approximations methods

Gain of the implicit methods is in the reduced effort to compute the approximation
We derived implicit $K$-point methods to approximate the average dynamics.
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The implicit methods ($K$ even) are just as good as the explicit ones ($K$ odd), but require less effort.
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The implicit methods ($K$ even) are just as good as the explicit ones ($K$ odd), but require less effort.

We can integrate highly oscillatory systems very efficiently.

\[
\frac{d}{d\tau} x = f_0(x) + \epsilon f_1(x, u, \tau)
\]  

(19)
Thank you for your attention!
Useful Sources

Mari Paz Calvo, Philippe Chartier, Ander Murua, and Jesús María Sanz-Serna.
A stroboscopic numerical method for highly oscillatory problems.

Bengt Fornberg.
Generation of finite difference formulas on arbitrarily spaced grids.

U. Kirchgraber.
An ode-solver based on the method of averaging.

Jan Sanders, Ferdinand Verhulst, and J.B. Murdoch.
Averaging methods in nonlinear dynamical systems, 2d ed.
01 2007.
### Coefficients Implicit Approximation

<table>
<thead>
<tr>
<th>$\Delta \tau$</th>
<th>-2</th>
<th>$-\frac{3}{2}$</th>
<th>-1</th>
<th>$-\frac{1}{2}$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>1</th>
<th>$\frac{3}{2}$</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = 2$</td>
<td></td>
<td></td>
<td></td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>$K = 3$</td>
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<td>0</td>
<td></td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K = 4$</td>
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<td>$\frac{9}{16}$</td>
<td>$\frac{9}{16}$</td>
<td>$-\frac{1}{16}$</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$K = 5$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

**Table:** Coefficients $b_k$ to relate the stroboscopic points $X'_k$ to the integration point $X(\tau^*)$ via the interpolating polynomial. The lighter rows correspond to the introduced implicit method, the darker rows correspond to the existing explicit method.
<table>
<thead>
<tr>
<th>$\Delta \tau$</th>
<th>-2</th>
<th>$-\frac{3}{2}$</th>
<th>-1</th>
<th>$-\frac{1}{2}$</th>
<th>0</th>
<th>$\frac{1}{2}$</th>
<th>1</th>
<th>$\frac{3}{2}$</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K' = 2$</td>
<td></td>
<td></td>
<td></td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K' = 3$</td>
<td></td>
<td></td>
<td>$-\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$K' = 4$</td>
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<td></td>
<td>$-\frac{9}{8}$</td>
<td>$\frac{9}{8}$</td>
<td>$-\frac{1}{24}$</td>
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<td></td>
</tr>
<tr>
<td>$K' = 5$</td>
<td>$\frac{1}{12}$</td>
<td>$-\frac{2}{3}$</td>
<td>0</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{12}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table: Coefficients $c_k$ of the (implicit) central difference approximation, c.f. [2]. The lighter rows correspond to the introduced implicit method, the darker rows correspond to the existing explicit method.