Object-Oriented Interface to OCP QPs in `tmpc`

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Table of Contents

1 Introduction to `tmpc`

2 Obect-Oriented Interface to OCP QPs

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- `tmpc` widely uses static polymorphism. Therefore, it can be seen as a code-generation tool, with the code-generation done by a C++ compiler.
- `tmpc` does not rely on a specific matrix arithmetics implementation. Algorithms[1] are parameterized by a class that defines an implementation of matrix arithmetics.

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Definition of OCP QP

The OCP QP is a QP in the form[2]

$$\begin{aligned} \underset{x,u}{\text{minimize}} \quad & \sum_{n=0}^{N} \frac{1}{2} \begin{bmatrix} x_k \\ u_k \\ 1 \end{bmatrix}^\top \begin{bmatrix} Q_k & S_k & q_k \\ S_k^\top & R_k & r_k \\ q_k^\top & r_k^\top & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ 1 \end{bmatrix} \\ \text{subject to} \quad & x_{k+1} = A_k x_k + B_k u_k + b_k, \, n = 0, \dots, N-1, \\ & \begin{bmatrix} \underline{x}_k \\ \underline{u}_k \end{bmatrix} \leq \begin{bmatrix} x_k \\ u_k \end{bmatrix} \leq \begin{bmatrix} \overline{x}_k \\ \overline{u}_k \end{bmatrix}, \qquad n = 0, \dots, N, \\ & \underline{d}_k \leq \begin{bmatrix} C_k & D_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \leq \overline{d}_k, \quad n = 0, \dots, N \end{aligned}$$

where u_k are the control inputs, x_k are the states.

[2] Slightly changed Gianluca's notation; x comes before u.

Definition of OCP QP Stage

- The OCP QP consists of 16 elements: Q, R, S, q, r, A, B, b, $\underline{x}$, $\underline{u}$, $\overline{x}$, $\overline{u}$, C, D, $\underline{d}$, $\overline{d}$.

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Definition (QP stage)

A OCP QP stage is a combination of elements corresponding to the same time index k:

$$\mathcal{S}_k = (Q_k, R_k, S_k, q_k, r_k, A_k, B_k, b_k, \underline{x}_k, \underline{u}_k, \overline{x}_k, \overline{u}_k, C_k, D_k, \underline{d}_k, \overline{d}_k) \ .$$

Definition of OCP QP Stage Size

Within one stage, matrices and vectors have consistent dimensions:

$$Q_k \in \mathbb{R}^{n_{x,k} \times n_{x,k}},\ R \in \mathbb{R}^{n_{u,k} \times n_{u,k}},\ S \in \mathbb{R}^{n_{x,k} \times n_{u,k}},\ q \in \mathbb{R}^{n_{x,k}},\ r \in \mathbb{R}^{n_{u,k}},$$
$$A \in \mathbb{R}^{n_{x,k+1} \times n_{x,k}},\ B \in \mathbb{R}^{n_{x,k+1} \times n_{u,k}},\ b \in \mathbb{R}^{n_{x,k+1}},\ \underline{x}, \overline{x} \in \mathbb{R}^{x_k},\ \underline{u}, \overline{u} \in \mathbb{R}^{u_k},$$
$$C \in \mathbb{R}^{n_{c,k} \times n_{x,k}},\ D \in \mathbb{R}^{n_{c,k} \times n_{u,k}},\ \underline{d}, \overline{d} \in \mathbb{R}^{n_{c,k}} \quad (1)$$

Definition (stage size)

The k-th stage size is

$$\mathcal{N}_k = (n_{x,k}, n_{u,k}, n_{c,k}, n_{x,k+1}) \ .$$

QP Stage Sequence Operations

- An OCP QP can be seen as a collection of stages:

$$\mathcal{QP} = (\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_N)\ .$$

- Any stage subsequence $(\mathcal{S}_m, \mathcal{S}_{m+1}, \ldots, \mathcal{S}_n)$, $0 \leq m < n \leq N$ of a QP is also a QP.
- Note that A_N,B_N,b_N do not enter the minimization problem formulation. . .
- . . . but they are useful if you concatenate two QPs:

$$(\mathcal{QP}^{(1)}, \mathcal{QP}^{(2)}) = (\mathcal{S}_0^{(1)}, \mathcal{S}_1^{(1)}, \ldots, \mathcal{S}_N^{(1)}, \mathcal{S}_0^{(2)}, \mathcal{S}_1^{(2)}, \ldots, \mathcal{S}_N^{(2)})$$

  provided that the matrix sizes are consistent, i.e. the number of rows in $A_N^{(1)}$, $B_N^{(1)}$, $b_N^{(1)}$ is equal to $n_{x,0}^{(2)}$.
- By eliminating the equality constraints and intermediate state variables (condensing), a new QP can be obtained, which consists of a single stage of size

$$\left(n_{x,0}, \sum_{k=0}^{N} n_{u,k}, \sum_{k=0}^{N} n_{d,k}, n_{+,N} \right)\ .$$

Constructing and Initializing a QP Stage

```
// Declare matrix math kernel type
using Kernel = BlazeKernel<double>;

// Construct a QpStage object with specified dimensions
QpStage<Kernel> stage {QpSize {3, 2, 0}, 0};

// Fill the values
stage
.Q({
    {1., 0., 0.},
    {0., 2., 0.},
    {0., 0., 3.}
})
.R({
    {5., 0.},
    {0., 6.}
})
.S({
    {7., 8.},
    {9., 10.},
    {11., 12.}
})
.q({13., 14., 15.})
.r({16., 17.});        // ... A, B, b, lx, lu, ux, uu and so on
```

Constructing a Multiple Stage OCP QP

```cpp
// An alias for QpStage<Kernel>
using Stage = QpStage<Kernel>;

// Init stages
Stage stage0 = createStage0();
Stage stage1 = createStage1();
Stage stage2 = createStage2();

// An OCP QP is just a collection of QpStage
std::vector<Stage> qp;
qp.push_back(stage0);
qp.push_back(stage1);
qp.push_back(stage2);
```

Using C++ Standard Algorithms on QP Stage Sequences

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Example 1: print all stages of a QP

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std::copy(qp.begin(), qp.end(), std::ostream_iterator<Stage>(std::cout, "\n"));
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std::copy(qp.begin(), qp.end(), std::ostream_iterator<Stage>(std::cout, "\n"));
```

Example 2: find a stage with a Hessian which is not positive-definite

```
auto bad_stage = std::find_if(qp.begin(), qp.end(),
  [] (Stage const& s) { return !s.isPositiveDefinite(); });
```

More Fancy Operations: Gauss-Newton Approximation

- Consider the Gauss-Newton cost Hessian approximation of a quadratic cost function

$$H = \begin{bmatrix} J_{yx} & J_{yu} \end{bmatrix}^\top \begin{bmatrix} J_{yx} & J_{yu} \end{bmatrix} = \begin{bmatrix} J_{yx}^\top J_{yx} & J_{yx}^\top J_{yu} \\ J_{yu}^\top J_{yx} & J_{yu}^\top J_{yu} \end{bmatrix} = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}$$

and the cost gradient

$$g = \begin{bmatrix} J_{yx} & J_{yu} \end{bmatrix}^\top y = \begin{bmatrix} J_{yx}^\top y \\ J_{yu}^\top y \end{bmatrix} = \begin{bmatrix} q \\ r \end{bmatrix} ,$$

where $J_{yx} = \frac{\mathrm{d}y}{\mathrm{d}x}$, $J_{yu} = \frac{\mathrm{d}y}{\mathrm{d}u}$ and y is the residual.

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  where $J_{yx} = \frac{\mathrm{d}y}{\mathrm{d}x}$, $J_{yu} = \frac{\mathrm{d}y}{\mathrm{d}u}$ and y is the residual.
- This corresponds to setting elements of a QP stage like following:

procedure $\textsc{GaussNewtonCostApproximation}(y, J_{yx}, J_{yu})$

$Q \leftarrow J_{yx}^\top J_{yx}$

$R \leftarrow J_{yu}^\top J_{yu}$

$S \leftarrow J_{yx}^\top J_{yu}$

$q \leftarrow J_{yx}^\top y$

$r \leftarrow J_{yu}^\top y$

end procedure

More Fancy Operations: Linearized Shooting Equality

- Consider the shooting constraint of the form

$$x_{k+1} = f(x_k, u_k)$$

and its linearized version

$$\Delta x_{k+1} = \underbrace{\frac{\mathrm{d}f}{\mathrm{d}x}(x_k, u_k)}_{A_k} \Delta x_k + \underbrace{\frac{\mathrm{d}f}{\mathrm{d}u}(x_k, u_k)}_{B_k} \Delta u_k + \underbrace{f(x_k, u_k) - x_{k+1}}_{b_k}$$

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- **procedure** $\textsc{LinearizedShootingEquality}(f, J_{fx}, J_{fu}, x^+)$

  $A \leftarrow J_{fx}$

  $B \leftarrow J_{fu}$

  $b \leftarrow f - x^+$

  end procedure

More Fancy Operations: Linearized General Constraints

- Consider general constraints of the form

$$\underline{g} \leq g(x_k, u_k) \leq \overline{g}$$

and its linearized version

$$\underbrace{\underline{g} - g(x_k, u_k)}_{\underline{d}_k} \leq \underbrace{\frac{\mathrm{d}g}{\mathrm{d}x}(x_k, u_k)}_{C_k} \Delta x_k + \underbrace{\frac{\mathrm{d}g}{\mathrm{d}u}(x_k, u_k)}_{D_k} \Delta u_k \leq \underbrace{\overline{g} - g(x_k, u_k)}_{\overline{d}_k}$$

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$$\underline{g} \leq g(x_k, u_k) \leq \overline{g}$$

and its linearized version

$$\underbrace{\underline{g} - g(x_k, u_k)}_{\underline{d}_k} \leq \underbrace{\frac{\mathrm{d}g}{\mathrm{d}x}(x_k, u_k)}_{C_k} \Delta x_k + \underbrace{\frac{\mathrm{d}g}{\mathrm{d}u}(x_k, u_k)}_{D_k} \Delta u_k \leq \underbrace{\overline{g} - g(x_k, u_k)}_{\overline{d}_k}$$

- **procedure** $\textsc{LinearizedGeneralConstraints}(g, J_{gx}, J_{gu}, \underline{g}, \overline{g})$
  - $C \leftarrow J_{gx}$
  - $D \leftarrow J_{gu}$
  - $\underline{d} \leftarrow \underline{g} - g$
  - $\overline{d} \leftarrow \overline{g} - g$

  end procedure

More Fancy Operations: Initial Value Embedding

- Consider the initial value constraint

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- Substituting it into the linearized shooting equality and the linearized general equalities gives

$$\Delta x_1 = \underbrace{\frac{\mathrm{d}f}{\mathrm{d}u}(x_0, u_0)}_{B_0} \Delta u_0 + \underbrace{f(x_0, u_0) - x_1 + \frac{\mathrm{d}f}{\mathrm{d}x}(x_0, u_0)(\tilde{x}_0 - x_0)}_{b_0}$$

$$\underbrace{\underline{g} - g(x_0, u_0) - \frac{\mathrm{d}g}{\mathrm{d}x}(x_0, u_0)(\tilde{x}_0 - x_0)}_{\underline{d}_0} \leq \underbrace{\frac{\mathrm{d}g}{\mathrm{d}u}(x_0, u_0)}_{D_0} \Delta u_0$$

$$\leq \underbrace{\overline{g} - g(x_0, u_0) - \frac{\mathrm{d}g}{\mathrm{d}x}(x_0, u_0)(\tilde{x}_0 - x_0)}_{\overline{d}_0}$$

More Fancy Operations: Initial Value Embedding

procedure $\textsc{InitialValueEmbedding}(\tilde{x}_0, x_0, x_1, f, J_{fx}, J_{fu}, g, J_{gx}, J_{gu}, g_l, g_u)$

Require: $n_{x,0} = 0$

$B_0 \leftarrow J_{fu}$

$b_0 \leftarrow f - x_1 + J_{fx}(\tilde{x}_0 - x_0)$

$D_0 \leftarrow J_{gu}$

$\underline{d}_0 \leftarrow g_l - g - J_{gx}(\tilde{x}_0 - x_0)$

$\overline{d}_0 \leftarrow g_u - g - J_{gx}(\tilde{x}_0 - x_0)$

end procedure

More Fancy Operations: Relative Bounds

- Consider the bound constraints

$$\begin{bmatrix} \underline{\underline{x}}_k \\ \underline{\underline{u}}_k \end{bmatrix} \leq \begin{bmatrix} x_k \\ u_k \end{bmatrix} \leq \begin{bmatrix} \overline{\overline{x}}_k \\ \overline{\overline{u}}_k \end{bmatrix}$$

which in the case of SQP transforms to

$$\underbrace{\begin{bmatrix} \underline{\underline{x}}_k - x_k \\ \underline{\underline{u}}_k - u_k \end{bmatrix}}_{\begin{bmatrix} \underline{x}_k \\ \underline{u}_k \end{bmatrix}} \leq \begin{bmatrix} \Delta x_k \\ \Delta u_k \end{bmatrix} \leq \underbrace{\begin{bmatrix} \overline{\overline{x}}_k - x_k \\ \overline{\overline{u}}_k - u_k \end{bmatrix}}_{\begin{bmatrix} \overline{x}_k \\ \overline{u}_k \end{bmatrix}}$$

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- **procedure** $\textsc{RelativeBounds}(x, u, x_l, u_l, x_u, u_u)$

  $\underline{x} \leftarrow x_l - x$

  $\underline{u} \leftarrow u_l - u$

  $\overline{x} \leftarrow x_u - x$

  $\overline{u} \leftarrow u_u - u$

  end procedure

Code Example: Setting Up an OCP QP Stage

```
// Alias for the matrix math kernel
using K = BlazeKernel<double>;

// Construct the QpStage
QpStage<K> stage { QpSize {NX, NU, 0}, NX };

// Variables
extern K::StaticVector lx, ux;      // absolute state bounds
extern K::StaticVector lu, uu;      // absolute control bounds
K::StaticVector<NX> x, x_plus, f;  // current state, next state, next calculated state
K::StaticVector<NU> u;              // current input
K::StaticVector<NY> y;              // residual (the cost function is y^T * y)
K::StaticMatrix<NX, NX> df_dx;      // sensitivities
K::StaticMatrix<NX, NU> df_du;
K::StaticMatrix<NY, NX> dy_dx;
K::StaticMatrix<NY, NU> dy_du;

// Set x and u, calculate x_plus, y and the corresponding sensitivities:
//
// ...

// Set up the QP stage. Isn't it expressive?
stage.gaussNewtonCostApproximation(y, dy_dx, dy_du);
stage.linearizedShootingEquality(f, df_dx, df_du, x_plus);
stage.relativeBounds(x, u, lx, lu, ux, uu);
```

Solving an OCP QP: the Holy Trinity

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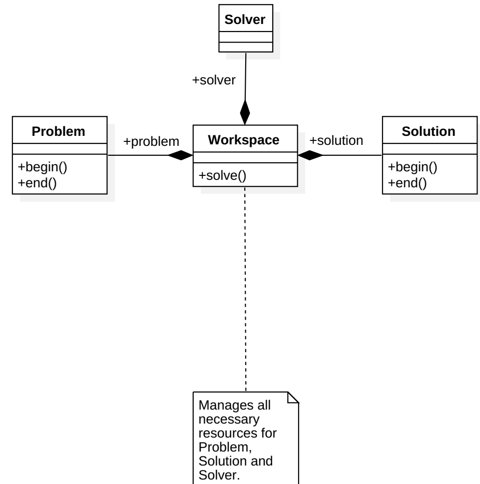

Code Example: Solving a QP

```
// A type to use for real numbers
using Real = double;
// Alias for Workspace. Want qpOASES? Just change to QpOasesWorkspace!
using Workspace = HpmpcWorkspace<Real>;

// Create a QP Workspace for 2 stages with specified dimensions.
// All the data structures for the solver will be created here.
Workspace workspace {QpSize {3, 0, 0}, QpSize {0, 0, 0}};
// Reference to stage 0. It is not a QpStage object, although it has the same interface.
// Modifiers will write directly to solver data structures - no memory overhead!
auto& stage0 = workspace.problem()[0];

// Set cost
stage0.gaussNewtonCostApproximation(
    DynamicVector<Real> {1., 2., 42.},
    IdentityMatrix<Real> {3u},
    DynamicMatrix<Real> {3u, 0u}
);
// Set bounds
stage0.bounds(-infinity<Real>(), -infinity<Real>(), infinity<Real>(), infinity<Real>());

// Solve the problem
workspace.solve();
// Output the solution
std::cout << workspace.solution()[0].x() << std::endl;
```

Code Example: Solving a QP

What will it print out?

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What will it print out?

```
 -1
 -2
-42
```

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  - Allows different solvers to be used interchangeably.
- The matrix math interface is separated from its implementation.
  - One can switch between different implementations of matrix math (single precision, double precision, `Eigen3`, `Blaze`, any custom).
- Tight coupliing between problem, solver and solution is resolved by introducing workspaces.

Final Slide

Questions? Comments?