

Modeling and System Identification – Microexam 1

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Surname:

Name:

Matriculation number:

Study:

Programm: Bachelor Master

Please fill in your name above and tick exactly **ONE** box for the right answer of each question below.
You can get a maximum of **10 points** on this microexam.

1. What is the probability density function (PDF) $p_X(x)$ for a normally distributed random variable X with mean -3 and standard deviation 3 ? The answer is $p_X(x) = \frac{1}{\sqrt{2\pi}9} \dots$

(a) <input type="checkbox"/> $e^{-\frac{(x+3)^2}{6}}$	(b) <input checked="" type="checkbox"/> $e^{-\frac{(x+3)^2}{18}}$	(c) <input type="checkbox"/> $e^{-\frac{(x-3)^2}{3}}$	(d) <input type="checkbox"/> $e^{-\frac{(x-3)^2}{9}}$
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2. What does the term $\frac{1}{\sqrt{2\pi}9}$ in $p_X(x)$ ensure? (The term **dx** was missing on letter (a). So (a) and (d) are correct for microexam 1.

(a) <input checked="" type="checkbox"/> $\int_{-\infty}^{\infty} p(x) dx = 1$	(b) <input type="checkbox"/> $p(x) > 0$	(c) <input type="checkbox"/> $p(x) \geq 0$	(d) <input type="checkbox"/> Nothing
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3. Which of the following functions is NOT convex on $x \in [-1, 1]$

(a) <input type="checkbox"/> $x + 42$	(b) <input type="checkbox"/> $\exp(-x)$	(c) <input checked="" type="checkbox"/> $\sin^{-1}(x)$	(d) <input type="checkbox"/> $-\cos(x)$
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4. Which of the following statements does NOT hold for all PDFs $p(x)$ of a scalar random variable?

(a) <input type="checkbox"/> $\int_{-\infty}^{\infty} p(x) dx = 1$	(b) <input type="checkbox"/> $p(x) \geq 0$	(c) <input checked="" type="checkbox"/> $p(x) < 1$	(d) <input type="checkbox"/> $\int_{-1}^1 p(x) dx \geq 0$
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5. What is the PDF of a random variable Y with uniform distribution on the interval $[5, 7]$? For $z \in [5, 7]$ it has the value:

(a) <input type="checkbox"/> $p_z(Y) = \frac{1}{5}$	(b) <input type="checkbox"/> $p_z(Y) = \frac{1}{7}$	(c) <input type="checkbox"/> $p_Y(z) = \frac{1}{\sqrt{2}}$	(d) <input checked="" type="checkbox"/> $p_Y(z) = \frac{1}{2}$
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6. Regard a random variable $X \in \mathbb{R}^n$ with mean $d \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. For a fixed $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$, regard another random variable Y defined by $Y = b + AX$. What is the mean μ_Y of Y ? The answer is $\mu_Y = \dots$

(a) <input type="checkbox"/> $b + AX$	(b) <input type="checkbox"/> $AXX^T A^T$	(c) <input checked="" type="checkbox"/> $b + Ad$	(d) <input type="checkbox"/> $b^T Ad^T$
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7. Regard the random variable Y in the above Question, what is the covariance matrix of Y ?

(a) <input type="checkbox"/> $d^T \Sigma d$	(b) <input checked="" type="checkbox"/> $A \Sigma A^T$	(c) <input type="checkbox"/> $A^T \Sigma^{-1} A$	(d) <input type="checkbox"/> $A \Sigma^{-1} A^T$
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8. Consider a multi-dimensional random variable $X \in \mathbb{R}^n$ with mean value μ . What is the covariance? $\text{cov}(X) = \dots$

(a) <input type="checkbox"/> $\mathbb{E}\{(X - \mu)\}^2$	(b) <input type="checkbox"/> $\mathbb{E}\{(X - \mu)^2\}$
(c) <input checked="" type="checkbox"/> $\mathbb{E}\{(X - \mu)(X - \mu)^T\}$	(d) <input type="checkbox"/> $\mathbb{E}\{(X - \mu)^T(X - \mu)\}$

9. Consider a multi-dimensional random variable $X \in \mathbb{R}^d$. What are the dimensions of the covariance? $\text{cov}(X) \in \dots$

(a) <input type="checkbox"/> $\mathbb{R}^{1 \times d}$	(b) <input type="checkbox"/> $\mathbb{R}^{d \times 2d}$	(c) <input checked="" type="checkbox"/> $\mathbb{R}^{d \times d}$	(d) <input type="checkbox"/> $\mathbb{R}^{d \times 1}$
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10. Regard a zero mean random variable $X \in \mathbb{R}^n$ with covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. Given a vector $c \in \mathbb{R}^n$, what is the mean of $Z = c^T X X^T c$?

(a) <input type="checkbox"/> Σ^T	(b) <input type="checkbox"/> Σc	(c) <input checked="" type="checkbox"/> $c^T \Sigma c$	(d) <input type="checkbox"/> $c^T c \Sigma^{-1}$
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11. What is the minimizer x^* of the convex function $f : \mathbb{R}_{++} \rightarrow \mathbb{R}, f(x) = -\log(x) + 5x$?

(a) <input type="checkbox"/> $x^* = -5$	(b) <input checked="" type="checkbox"/> $x^* = 1/5$	(c) <input type="checkbox"/> $x^* = e^5 - 1$	(d) <input type="checkbox"/> $x^* = 5$
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12. What is the minimizer x^* of $f : \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = \frac{1}{2} \|Ax - b\|_2^2$ if $\text{rank}(A) = n$? The solution is $x^* = \dots$

(a) <input type="checkbox"/> $A + b$	(b) <input checked="" type="checkbox"/> $(A^T A)^{-1} A^T b$
(c) <input type="checkbox"/> $A^{-1} + b$	(d) <input type="checkbox"/> $(A^T A)^{-1} Ab$

13. For a matrix $\Phi \in \mathbb{R}^{N \times d}$ with rank d (and $N \geq d$), what is its pseudo-inverse Φ^+ ?

(a) <input type="checkbox"/> $(\Phi \Phi^T)^{-1} \Phi^T$	(b) <input type="checkbox"/> $(\Phi^T \Phi)^{-1} \Phi$	(c) <input type="checkbox"/> $\Phi(\Phi^T \Phi)^{-1}$	(d) <input checked="" type="checkbox"/> $(\Phi^T \Phi)^{-1} \Phi^T$
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14. What is the gradient of $f : \mathbb{R}^n \rightarrow \mathbb{R}, f(x) = \frac{1}{2} \| -b + Dx \|_W^2$ (with D of rank n and W positive definite)?

(a) <input type="checkbox"/> Wb	(b) <input type="checkbox"/> $b + Wb$
(c) <input checked="" type="checkbox"/> $(D^T W D)x - D^T W b$	(d) <input type="checkbox"/> $(D W D^T + D)^{-1}$

15. Given a sequence of numbers $y(1), \dots, y(N)$, what is the minimizer θ^* of the function $f(\theta) = \sum_{k=1}^N (y(k) - 6\theta)^2$?
The answer is $\theta^* = \dots$

(a) <input type="checkbox"/> $\frac{1}{2} \sum_{k=1}^N y(k)^2$	(b) <input checked="" type="checkbox"/> $\frac{\sum_{k=1}^N y(k)}{6N}$	(c) <input type="checkbox"/> $\frac{1}{3N} \sum_{k=1}^N y(k)^2$	(d) <input type="checkbox"/> $\sum_{k=1}^N \frac{y(k)}{6}$
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16. Given a sequence of i.i.d. scalar random variables $X(1), \dots, X(N)$, each with mean μ and variance σ^2 , what is the expected value of Y defined by $Y = \sum_{k=1}^N X(k)$?

(a) <input checked="" type="checkbox"/> $N\mu$	(b) <input type="checkbox"/> $\frac{\mu}{N}$	(c) <input type="checkbox"/> $\frac{\mu}{\sigma^2}$	(d) <input type="checkbox"/> $\frac{\mu}{\sqrt{\sigma^2}}$
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17. Regard the random variable Y in the above Question, what is the variance of the variable Y ? The answer is $\text{var}(Y) = \dots$

(a) <input type="checkbox"/> $N\sigma$	(b) <input type="checkbox"/> $\frac{\sigma^2}{N-1}$	(c) <input checked="" type="checkbox"/> $N\sigma^2$	(d) <input type="checkbox"/> $\frac{\sigma^2}{N}$
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18. Consider the model $y(k) = \theta_1 + \frac{\theta_2}{3} x(k)^2 + \frac{\theta_3}{4} x(k)^3 + \epsilon(k)$ and the vector of unknown parameters $\theta = (\theta_1, \theta_2, \theta_3)^T$. The additive noise $\epsilon(k)$ is assumed to have zero mean and to be i.i.d. For a given sequence of N scalar input and output measurements $x(1), \dots, x(N)$ and $y(1), \dots, y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}_N$ by minimizing the function $f(\theta) = \|y_N - \Phi_N \theta\|_2^2$. If $y_N = [y(1), \dots, y(N)]^T$, how do we need to choose the matrix $\Phi_N \in \mathbb{R}^{N \times 3}$? $\Phi_N = \dots$

(a) <input type="checkbox"/> $\begin{bmatrix} 2 & 3x(1)^2 & 4x(1)^3 \\ \vdots & \vdots & \vdots \\ 1 & 3x(N)^2 & 4x(N)^3 \end{bmatrix}$	(b) <input checked="" type="checkbox"/> $\begin{bmatrix} 1 & \frac{x(1)^2}{3} & \frac{x(1)^3}{4} \\ \vdots & \vdots & \vdots \\ 1 & \frac{x(N)^2}{3} & \frac{x(N)^3}{4} \end{bmatrix}$	(c) <input type="checkbox"/> $\begin{bmatrix} \frac{x(1)^2}{3} & 2 & \frac{x(1)^3}{4} \\ \vdots & \vdots & \vdots \\ \frac{x(N)^2}{3} & 2 & \frac{x(N)^3}{4} \end{bmatrix}$	(d) <input type="checkbox"/> $\begin{bmatrix} 2 & x(1)^2 & x(1)^3 \\ \vdots & \vdots & \vdots \\ 2 & x(N)^2 & x(N)^3 \end{bmatrix}$
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19. Which of the following formulas computes the covariance for a least squares estimator and a single experiment with i.i.d. noise components $\epsilon_N = [\epsilon(1), \dots, \epsilon(N)]^T$? $\hat{\Sigma}_{\hat{\theta}} = \dots$

(a) <input checked="" type="checkbox"/> $\frac{\ y_N - \Phi_N \hat{\theta}\ _2^2}{N-d} (\Phi_N^T \Phi_N)^{-1}$	(b) <input type="checkbox"/> $\frac{\ y_N - \Phi_N \hat{\theta}\ _2}{N-d} (\Phi_N \Phi_N^T)^{-1}$
(c) <input type="checkbox"/> $\frac{\ y_N - \Phi_N \hat{\theta}\ _2}{N-d} (\Phi_N^T + \Phi_N)$	(d) <input type="checkbox"/> $\Phi_N^+ \sigma_{\epsilon_N}$

20. Given a set of measurements $y(1), \dots, y(N)$ following the model $y(k) = \phi(k)^T \theta + \epsilon(k)$, where $\phi(k)$ are the regression vectors with $\Phi_N = [\phi(1)^T \dots \phi(N)^T]^T$, θ the unknown parameters and $\epsilon(k) \sim \mathcal{N}(0, \sigma_\epsilon^2)$ the i.i.d. noise contribution for $k = 1, \dots, N$, we can compute the LLS estimator of the parameters θ as $\hat{\theta}_{LS}$. Defining the covariance of $\hat{\theta}_{LS}$ as $\Sigma_{\hat{\theta}}$, which of the following is NOT true?

(a) <input type="checkbox"/> $\hat{\theta}_{LS}$ is a random variable	(b) <input type="checkbox"/> $\hat{\theta}_{LS} = \Phi_N^+ y_N$
(c) <input checked="" type="checkbox"/> $\hat{\theta}_{LS} \sim \mathcal{N}(0, \Sigma_{\hat{\theta}})$	(d) <input type="checkbox"/> $\Sigma_{\hat{\theta}} = \sigma_\epsilon^2 (\Phi_N^+ \Phi_N^+{}^T)$