Modeling and System Identification – Microexam 1

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Surname: Name: Matriculation number:

Study: Programm: Bachelor ☐ Master ☐

Please fill in your name above and tick exactly **ONE** box for the right answer of each question below.
You can get a maximum of **10 points** on this microexam.

1. What is the probability density function (PDF) $p_X(x)$ for a normally distributed random variable X with mean -3 and standard deviation 3? The answer is $p_X(x) = \frac{1}{\sqrt{2\pi 9}} \dots$

(a) ☐ $e^{-\frac{(x+3)^2}{6}}$	(b) ☐ $e^{-\frac{(x+3)^2}{18}}$	(c) ☐ $e^{-\frac{(x-3)^2}{3}}$	(d) ☐ $e^{-\frac{(x-3)^2}{9}}$

2. What does the term $\frac{1}{\sqrt{2\pi 9}}$ in $p_X(x)$ ensure?

(a) ☐ $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$	(b) ☐ $p(x) > 0$	(c) ☐ $p(x) \geq 0$	(d) ☐ Nothing

3. Which of the following functions is NOT convex on $x \in [-1, 1]$

(a) ☐ $x + 42$	(b) ☐ $\exp(-x)$	(c) ☐ $\sin^{-1}(x)$	(d) ☐ $-\cos(x)$

4. Which of the following statements does NOT hold for all PDFs $p(x)$ of a scalar random variable?

(a) ☐ $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$	(b) ☐ $p(x) \geq 0$	(c) ☐ $p(x) < 1$	(d) ☐ $\int_{-1}^{1} p(x)\mathrm{d}x \geq 0$

5. What is the PDF of a random variable Y with uniform distribution on the interval $[5, 7]$? For $z \in [5, 7]$ it has the value:

(a) ☐ $p_z(Y) = \frac{1}{5}$	(b) ☐ $p_z(Y) = \frac{1}{7}$	(c) ☐ $p_Y(z) = \frac{1}{\sqrt{2}}$	(d) ☐ $p_Y(z) = \frac{1}{2}$

6. Regard a random variable $X \in \mathbb{R}^n$ with mean $d \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. For a fixed $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$, regard another random variable Y defined by $Y = b + AX$. What is the mean μ_Y of Y ?The answer is $\mu_Y = \dots$

(a) ☐ $b + AX$	(b) ☐ $AXX^\top A^\top$	(c) ☐ $b + Ad$	(d) ☐ $b^\top Ad^\top$

7. Regard the random variable Y in the above Question, what is the covariance matrix of Y?

(a) ☐ $d^\top \Sigma d$	(b) ☐ $A\Sigma A^\top$	(c) ☐ $A^\top \Sigma^{-1} A$	(d) ☐ $A\Sigma^{-1}A^\top$

8. Consider a multi-dimensional random variable $X \in \mathbb{R}^n$ with mean value μ. What is the covariance? $\mathrm{cov}(X) = \dots$

(a) ☐ $\mathbb{E}\{(X-\mu)\}^2$	(b) ☐ $\mathbb{E}\{(X-\mu)^2\}$
(c) ☐ $\mathbb{E}\{(X-\mu)(X-\mu)^\top\}$	(d) ☐ $\mathbb{E}\{(X-\mu)^\top(X-\mu)\}$

9. Consider a multi-dimensional random variable $X \in \mathbb{R}^d$. What are the dimensions of the covariance? $\mathrm{cov}(X) \in \dots$

(a) ☐ $\mathbb{R}^{1\times d}$	(b) ☐ $\mathbb{R}^{d\times 2d}$	(c) ☐ $\mathbb{R}^{d\times d}$	(d) ☐ $\mathbb{R}^{d\times 1}$

10. Regard a zero mean random variable $X \in \mathbb{R}^n$ with covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$. Given a vector $c \in \mathbb{R}^n$, what is the mean of $Z = c^\top XX^\top c$?

(a) ☐ $\Sigma^\top$	(b) ☐ Σc	(c) ☐ $c^\top \Sigma c$	(d) ☐ $c^\top c\Sigma^{-1}$

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11. What is the minimizer x^* of the convex function $f : \mathbb{R}_{++} \to \mathbb{R}$, $f(x) = -\log(x) + 5x$?

(a) ☐ $x^* = -5$	(b) ☐ $x^* = 1/5$	(c) ☐ $x^* = e^5 - 1$	(d) ☐ $x^* = 5$

12. What is the minimizer $x^\star$ of $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \frac{1}{2}\|Ax - b\|_2^2$ if $\text{rank}(A) = n$? The solution is $x^\star = \dots$

(a) ☐ $A + b$	(b) ☐ $(A^\top A)^{-1}A^\top b$
(c) ☐ $A^{-1} + b$	(d) ☐ $(A^\top A)^{-1}Ab$

13. For a matrix $\Phi \in \mathbb{R}^{N\times d}$ with rank d (and $N \geq d$), what is its pseudo-inverse Φ^+ ?

(a) ☐ $(\Phi\Phi^\top)^{-1}\Phi^\top$	(b) ☐ $(\Phi^\top\Phi)^{-1}\Phi$	(c) ☐ $\Phi(\Phi^\top\Phi)^{-1}$	(d) ☐ $(\Phi^\top\Phi)^{-1}\Phi^\top$

14. What is the gradient of $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \frac{1}{2}\| -b + Dx\|_W^2$ (with D of rank n and W positive definite)?

(a) ☐ Wb	(b) ☐ $b + Wb$
(c) ☐ $(D^\top WD)x - D^\top Wb$	(d) ☐ $(DWD^\top + D)^{-1}$

15. Given a sequence of numbers $y(1), \dots, y(N)$, what is the minimizer θ^* of the function $f(\theta) = \sum_{k=1}^{N}(y(k) - 6\theta)^2$? The answer is $\theta^* = \dots$

(a) ☐ $\frac{1}{2}\sum_{k=1}^{N} y(k)^2$	(b) ☐ $\frac{\sum_{k=1}^{N} y(k)}{6N}$	(c) ☐ $\frac{1}{3N}\sum_{k=1}^{N} y(k)^2$	(d) ☐ $\sum_{k=1}^{N}\frac{y(k)}{6}$

16. Given a sequence of i.i.d. scalar random variables $X(1), \dots, X(N)$, each with mean μ and variance σ^2, what is the expected value of Y defined by $Y = \sum_{k=1}^{N} X(k)$?

(a) ☐ $N\mu$	(b) ☐ $\frac{\mu}{N}$	(c) ☐ $\frac{\mu}{\sigma^2}$	(d) ☐ $\frac{\mu}{\sqrt{\sigma^2}}$

17. Regard the random variable Y in the above Question, what is the variance of the variable Y? The answer is $\text{var(Y)} = \dots$

(a) ☐ $N\sigma$	(b) ☐ $\frac{\sigma^2}{N-1}$	(c) ☐ $N\sigma^2$	(d) ☐ $\frac{\sigma^2}{N}$

18. Consider the model $y(k) = \theta_1 + \frac{\theta_2}{3}x(k)^2 + \frac{\theta_3}{4}x(k)^3 + \epsilon(k)$ and the vector of unknown parameters $\theta = (\theta_1, \theta_2, \theta_3)^\top$. The additive noise $\epsilon(k)$ is assumed to have zero mean and to be i.i.d. For a given sequence of N scalar input and output measurements $x(1), \dots, x(N)$ and $y(1), \dots, y(N)$, we want to compute the linear least squares (LLS) estimate $\hat{\theta}_N$ by minimizing the function $f(\theta) = \|y_N - \Phi_N\theta\|_2^2$. If $y_N = [y(1), \dots, y(N)]^\top$, how do we need to choose the matrix $\Phi_N \in \mathbb{R}^{N\times 3}$? $\Phi_N = \dots$

(a) ☐	(b) ☐	(c) ☐	(d) ☐
$\begin{bmatrix} 2 & 3x(1)^2 & 4x(1)^3 \\ \vdots & \vdots & \vdots \\ 1 & 3x(N)^2 & 4x(N)^3 \end{bmatrix}$	$\begin{bmatrix} 1 & \frac{x(1)^2}{3} & \frac{x(1)^3}{4} \\ \vdots & \vdots & \vdots \\ 1 & \frac{x(N)^2}{3} & \frac{x(N)^3}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{x(1)^2}{3} & 2 & \frac{x(1)^3}{4} \\ \vdots & \vdots & \vdots \\ \frac{x(N)^2}{3} & 2 & \frac{x(N)^3}{4} \end{bmatrix}$	$\begin{bmatrix} 2 & x(1)^2 & x(1)^3 \\ \vdots & \vdots & \vdots \\ 2 & x(N)^2 & x(N)^3 \end{bmatrix}$

19. Which of the following formulas computes the covariance for a least squares estimator and a single experiment with i.i.d. noise components $\epsilon_N = [\epsilon(1), \dots, \epsilon(N)]^T$? $\hat{\Sigma}_{\hat{\theta}} = \dots$

(a) ☐ $\frac{\|y_N - \Phi_N\hat{\theta}\|_2^2}{N-d}(\Phi_N^\top\Phi_N)^{-1}$	(b) ☐ $\frac{\|y_N - \Phi_N\hat{\theta}\|_2}{N-d}(\Phi_N\Phi_N^\top)^{-1}$
(c) ☐ $\frac{\|y_N - \Phi_N\hat{\theta}\|_2}{N-d}(\Phi_N^\top + \Phi_N)$	(d) ☐ $\Phi_N^+\sigma_{\epsilon_N}$

20. Given a set of measurements $y(1), \dots, y(N)$ following the model $y(k) = \phi(k)^\top\theta + \epsilon(k)$, where $\phi(k)$ are the regression vectors with $\Phi_N = [\phi(1)^\top \dots \phi(N)^\top]^\top$, θ the unknown parameters and $\epsilon(k) \sim \mathcal{N}(0, \sigma_\epsilon^2)$ the i.i.d. noise contribution for $k = 1, ..., N$, we can compute the LLS estimator of the parameters θ as $\hat{\theta}_{\text{LS}}$. Defining the covariance of $\hat{\theta}_{\text{LS}}$ as $\Sigma_{\hat{\theta}}$, which of the following is NOT true?

(a) ☐ $\hat{\theta}_{\text{LS}}$ is a random variable	(b) ☐ $\hat{\theta}_{\text{LS}} = \Phi_N^+ y_N$
(c) ☐ $\hat{\theta}_{\text{LS}} \sim \mathcal{N}(0, \Sigma_{\hat{\theta}})$	(d) ☐ $\Sigma_{\hat{\theta}} = \sigma_\epsilon^2(\Phi_N^+\Phi_N^{+\top})$

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