Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2026-2027

Exercise 3: Optimality Conditions and Linear Least Squares
(to be returned before November 16th, 8:15)

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The aim of this sheet is to strengthen your knowledge in optimality conditions and least squares estimation.

Exercise Tasks

1. On Paper: Given the function $f(x) : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^\top Q\, x + c^\top x$ and fixed $c \in \mathbb{R}^n$.

   (a) Consider the not necessarily symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and compute the gradient $\nabla f(x) \in \mathbb{R}^n$ and the Hessian $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ of this function for any x.

   *(b) If Q is symmetric, what properties does it have to fulfil such that the unique minimizer x^* can be computed?

   *(c) Compute the unique minimizer and the minimum function value $f(x^*)$ under the correct assumptions. (3 points)

2. On Paper: We would like to find the parameters $\hat{\theta}_{\text{LS}}$ of a linear model $y_k = {\phi_k}^\top \theta + \epsilon_k$, where $\epsilon_k \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is an additive i.i.d. zero-mean Gaussian noise that perturbed a series of N scalar measurements $y = [y_1, \ldots, y_N]^\top \in \mathbb{R}^N$. From the lecture we know that $\hat{\theta}_{\text{LS}}$ can be computed using least-squares:

$$\hat{\theta}_{\text{LS}} = \arg\min_\theta \frac{1}{2}\|y - \Phi\theta\|_2^2$$

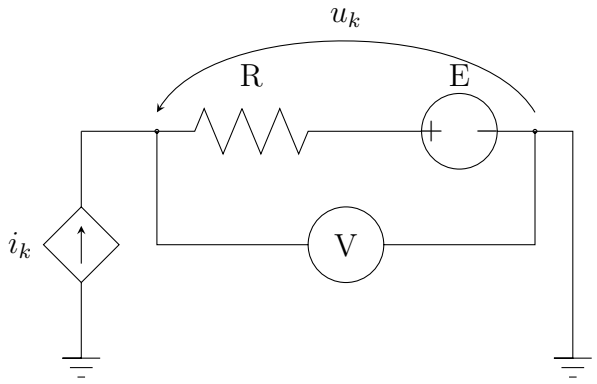
   where $\Phi \in \mathbb{R}^{N\times d}$. Assume that σ_ϵ^2 is known.

   (a) State the matrix Φ and the solution of least squares problem $\hat{\theta}_{\text{LS}}$.

   *(b) Calculate the covariance of the least squares estimate $\text{cov}(\hat{\theta}_{\text{LS}})$.

   Hint: Recall from Exercise 2 that the covariance matrix of a vector-valued variable $Y = AX+b$ for a constant $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is given by $\text{cov}(Y) = A\ \text{cov}(X)A^\top$. (2 points)

3. Consider the following experimental set up to estimate the values of E and R.



You obtain a dataset containing N measurements of the voltage u_k for different values of i_k. We assume that the input measurements i_k is not affected by noise, but that the measurements u_k are affected by i.i.d. additive noise $n_{\mathrm{u},k}$. Under these assumptions the measurement model is given by:

$$u_k = m_k + n_{\mathrm{u},k} \quad \text{where} \quad m_k = E + R \cdot i_k.$$

(a) Code: Load the dataset containing the measurements. Plot the data with 'x' markers and add labels to both axis. (0.5 points)

(b) On Paper: Formulate the problem of estimating the parameters E and R as a least squares problem where $\theta = [E\ R]^\top \in \mathbb{R}^2$ by defining $\Phi \in \mathbb{R}^{N\times 2}$ and $y \in \mathbb{R}^N$ such that the optimizer is given by $\hat{\theta} = \arg\min_\theta \|y - \Phi\theta\|_2^2$. (1 points)

(c) Code: Use the least squares estimator formulated in the previous subtask to find the experimental values of R and E for the dataset. Plot the linear fit through the measurement data. (1 points)

(d) Code: Plot a histogram of the residuals defined as $r_k = u_k - m_k$, where $m_k = E + R \cdot i_k$ is the voltage determined by the model, and u_k are the obtained measurements. (0.5 points)

(e) On Paper: Give an educated guess for the type of noise distribution. (0.5 points)

As we will learn later in the lecture, for i.i.d. measurement noise, the covariance matrix of the d estimated parameters $\hat{\theta}$ from a single experiment can be approximated by

$$\hat{\Sigma}_{\hat{\theta}} = \frac{\|y - \Phi\hat{\theta}\|_2^2}{N - d}(\Phi^\top \Phi)^{-1}$$

From this estimated covariance matrix, the estimated variances for the single components of the parameter vector can be extracted as the respective diagonal entry.

(f) Code: Calculate the estimated variance of both the R and E estimate and print their values along with a 1-σ confidence estimate in the format of

$$(\text{quantity}) = (\text{value}) \pm (1 \text{ standard deviation})$$

(1.5 points)