Exercises for Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2026-2027

Exercise 2: Statistics + Parameter Estimation

(to be returned before November 9th, 8:15)

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In this exercise, we repeat some tools from statistics and solve a simple estimation problem using linear least squares.

Exercise Tasks

1. * On Paper: The covariance matrix of a vector-valued random variable $X \in \mathbb{R}^n$ with mean $\mathbb{E}\{X\} = \mu_X$ is defined by

$$\mathrm{cov}(X) := \mathbb{E}\{(X - \mu_X)\,(X - \mu_X)^\top\}.$$

   Prove that the covariance matrix of a vector-valued variable $Y = AX + b$ with constant $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is given by

$$\mathrm{cov}(Y) = A\ \mathrm{cov}(X) A^\top.$$

2. *On Paper: Let $X \in \mathbb{R}^n$ be a vector-valued random variable with mean $\mu \in \mathbb{R}^n$. Show that the covariance matrix $\mathrm{cov}(X)$ can also be calculated by

$$\mathrm{cov}(X) = \mathbb{E}\{XX^\top\} - \mu_X {\mu_X}^\top$$

3. On Paper: Suppose we are measuring a constant $x_0 \in \mathbb{R}$ perturbed by random independent noise ϵ with mean $\mu_\epsilon = 0$ and variance $\sigma_\epsilon^2 > 0$, i.e. we have

$$x = x_0 + \epsilon.$$

   (a) State the mean μ_x and the variance σ_x^2 of the random variable x.

   (b) Let $(x_1, \ldots, x_n)$ denote a sample of n observations of x. The sample mean is given by $\bar{x}^{[n]} = \frac{1}{n}\sum_{i=1}^n x_i$ and it is an unbiased estimator of the mean μ_x.
   What is the variance of $\bar{x}^{[n]}$?

   (c) Prove that the Least Squares (LS) estimate for x_0 is the sample mean $\bar{x}^{[n]}$. State the minimization problem explicitly. Is it convex?

4. Consider the following experimental setup, where we measure the temperature-dependent expansion of a fluid in a long transparent pipe, such as in a traditional thermometer. We describe the length of the visible fluid with the affine model

$$m(T; \theta_1, \theta_2) = \theta_1 \cdot T + \theta_2. \tag{1}$$

where T is the temperature in Celsius, and the parameters θ_1 and θ_2 relate to the specific expansion coefficient of the material and the length of the fluid at temperature $T = 0\,^\circ\mathrm{C}$, respectively. Below, you find the measurements. Using the data, you will compute estimates for the parameters θ_1 and θ_2.

k	1	2	3	4
T_k [°C]	5	15	35	60
L_k [cm]	6.55	9.63	17.24	29.64

(a) Code: Plot the measurements T_k, L_k using 'x' markers.

(b) On Paper: Using the model from above, calculate the experimental values for the parameters θ_1 and θ_2 by minimizing the sum of squared distances, i.e.

$$\hat{\theta}_1, \hat{\theta}_2 = \underset{\theta_1,\, \theta_2}{\arg\min} \quad \sum_{k=1}^{4} \left(L_k - m(T_k; \theta_1, \theta_2)\right)^2,$$

Give an analytical expression for the values of $\hat{\theta}_1$ and $\hat{\theta}_2$ with respect to the measurements $T_1, \ldots, T_4$ and $L_1, \ldots, L_4$.

Hint: Compute the solution by setting the gradient of the objective function $f(\theta_1, \theta_2) = \sum_k (L_k - m(T_k, \theta_1, \theta_2))^2$ with respect to the parameters (θ_1, θ_2) to zero, i.e. $\nabla f(\theta_1, \theta_2) = 0$. This will give you a 2×2 linear system. Check if the objective function is convex!

Code: Calculate the values of $\hat{\theta}_1$ and $\hat{\theta}_2$ using the data. Plot the fit $m(T; \hat{\theta}_1, \hat{\theta}_2) = \hat{\theta}_1 T + \hat{\theta}_2$ over the range $[0, 100]$ in the same figure as before.

(c) Code: Now, use a third order polynomial and fit it to the data using `np.polyfit`. Again minimize the sum of squared distances to find optimal values for the coefficients of your model equation. Plot the fit in the same figure as before.

(d) Code: You take another measurement: At $T = 70\,^\circ\mathrm{C}$ you measure a length of $L = 32.89\,\mathrm{cm}$. You can use this additional datapoint to validate your fit. Add the measurement to the existing plot.

On Paper: Which fit looks more reasonable to you?

Hint: The phenomenon of fitting a model to a data set which then does not pass validation is called 'overfitting'.