Exercises for Lecture Course on Numerical Optimal Control (NOC)
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Exercise 9: Pontryagin's Minimum Principle

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Consider the following optimal control problem:

$$
\begin{aligned}
\min_{x(\cdot),\,u(\cdot)} \quad & \int_0^T x_1(t)^2 + x_2(t)^2 + u(t)^2 \,\mathrm{d}t && \text{(1a)}\\
\text{s.t.} \quad & x_1(0) = 0, && \text{(1b)}\\
& x_2(0) = 1, && \text{(1c)}\\
& \dot{x}_1(t) = (1 - x_2(t)^2)\,x_1(t) - x_2(t) + u(t), \quad t \in [0, T], && \text{(1d)}\\
& \dot{x}_2(t) = x_1(t), \quad t \in [0, T], && \text{(1e)}\\
& -1 \le u(t) \le 1 \quad t \in [0, T], && \text{(1f)}
\end{aligned}
$$

where $T = 10$.
In this exercise, we will apply Pontryagin's maximum principle to this problem and solve it with the indirect single shooting method.

1. (a) Introduce the costate $\lambda(t)$ and write down the Hamiltonian $H(x(t), \lambda(t), u(t))$ of (1).

$$H(x, \lambda, u) = \lambda_1\,((1 - x_2^2)\,x_1 - x_2 + u) + \lambda_2\,x_1 + x_1^2 + x_2^2 + u^2$$

(b) Use Pontryagin's maximum principle to derive an expression for the optimal control $u^\star(t)$ as a function of $x(t)$ and $\lambda(t)$. Note that $u(t)$ may be a piecewise smooth function.
Hint: How does u enter in the Hamiltonian?

$$u^\star = \max\left(\min\left(-\frac{\lambda_1}{2}, 1\right), -1\right)$$

(c) Derive the costate equations, i.e., $\dot{\lambda}(t) = \ldots$

$$
\begin{aligned}
\dot{\lambda}_1 &= -\lambda_1\,(1 - x_2^2) - 2\,x_1 - \lambda_2,\\
\dot{\lambda}_2 &= \lambda_1\,x_1\,2\,x_2 - 2\,x_2 + \lambda_1.
\end{aligned}
$$

(d) Derive the terminal condition for the costate.

$$\lambda_1(T) = 0, \quad \lambda_2(T) = 0.$$

(e) Collect all conditions necessary to obtain the corresponding two-point boundary-value problem (TPBVP).

$$
\begin{aligned}
\dot{x}_1 &= (1 - x_2^2)\,x_1 - x_2 + u, & x_1(0) &= 0,\\
\dot{x}_2 &= x_1, & x_2(0) &= 1,\\
\dot{\lambda}_1 &= -\lambda_1\,(1 - x_2^2) - 2\,x_1 - \lambda_2, & \lambda_1(T) &= 0,\\
\dot{\lambda}_2 &= \lambda_1\,x_1\,2\,x_2 - 2\,x_2 + \lambda_1, & \lambda_2(T) &= 0
\end{aligned}
$$

(f) Using the provided template, solve the TPBVP with indirect single shooting. Use $[0, 0]$ as your initial guess for the initial costate. To integrate the system, our best chance is to use a variable stepsize integrator for stiff systems, such as the CVODES integrator from the SUNDIALS suite, available in CasADi. Note that the system is only piecewise smooth, which could potentially cause problems in the integrator, but we will ignore this and hope for the best. The resulting nonlinear system of equations is challenging to solve, and in CasADi, our best bet is to use IPOPT with a dummy objective function ("minimize 0, subject to $g(x) = 0$").