Model Predictive Control and Reinforcement Learning
– Lecture 5: Actor-Critic Methods –

Joschka Boedecker

University of Freiburg

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Policy Gradient Methods

- Up to this point, we represented a model or a value function by some parameterized function approximator and extracted the policy implicitly
- Now, we are going to talk about *Policy Gradient Methods*: methods which consider a parameterized *policy*

$$\pi_{\boldsymbol{\theta}}(a \mid s) = \mathsf{Pr}\{A_t = a \mid S_t = s, \boldsymbol{\theta}_t = \boldsymbol{\theta}\},$$

  with parameters $\boldsymbol{\theta}$
- Policy Gradient Methods are able to represent stochastic policies and scale naturally to very large or continuous action spaces

Overview

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Policy Gradient Methods

- Remember, we consider a parameterized *policy*

$$\pi_{\boldsymbol{\theta}}(a \mid s) = \Pr\{A_t = a \mid S_t = s, \boldsymbol{\theta}_t = \boldsymbol{\theta}\},$$

  with parameters $\boldsymbol{\theta}$

- We update these parameters based on the gradient of some performance measure $J(\boldsymbol{\theta})$ that we want to maximize, i.e. via *gradient ascent*:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \alpha \widehat{\nabla J(\boldsymbol{\theta}_t)},$$

  where $\widehat{\nabla J(\boldsymbol{\theta}_t)} \in \mathbb{R}^d$ is a stochastic estimate whose expectation approximates the gradient of the performance measure w.r.t. $\boldsymbol{\theta}_t$

Policy Gradient Theorem

Policy Objective Functions:

- We consider episodic problems where we define performance as: $J(\boldsymbol{\theta}) = V^{\pi_{\boldsymbol{\theta}}}(S_0)$

Policy Gradient Theorem

For any differentiable policy $\pi(a|s, \boldsymbol{\theta})$ and any of the above policy objective functions, the policy gradient is:

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \mathbb{E}_\pi[\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(A \mid S) Q^{\pi_{\boldsymbol{\theta}}}(S, A)]$$

Reminder: $V^{\pi_{\boldsymbol{\theta}}} = \sum_a \pi_{\boldsymbol{\theta}}(a|s) Q^{\pi_{\boldsymbol{\theta}}}(s, a)$

Policy Gradient Theorem

Proof (episodic case):

$$\nabla_{\boldsymbol{\theta}} V^{\pi_{\boldsymbol{\theta}}}(s) = \nabla_{\boldsymbol{\theta}} \left[\sum_a \pi_{\boldsymbol{\theta}}(a \mid s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) \right], \quad \text{for all } s \in \mathcal{S}$$

$$= \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) + \pi_{\boldsymbol{\theta}}(a \mid s) \nabla_{\boldsymbol{\theta}} Q^{\pi_{\boldsymbol{\theta}}}(s, a) \right] \text{ (product rule of calculus)}$$

$$= \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) + \pi_{\boldsymbol{\theta}}(a \mid s) \nabla_{\boldsymbol{\theta}} \sum_{s', r} P(s', r \mid s, a) \left(r + V^{\pi_{\boldsymbol{\theta}}}(s') \right) \right]$$

$$= \sum_a \left[\nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) + \pi_{\boldsymbol{\theta}}(a \mid s) \sum_{s'} P(s' \mid s, a) \nabla_{\boldsymbol{\theta}} V^{\pi_{\boldsymbol{\theta}}}(s') \right]$$

$$= \sum_a \Bigg[\nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) + \pi_{\boldsymbol{\theta}}(a \mid s) \sum_{s'} P(s' \mid s, a) \sum_{a'} [\nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a' \mid s') Q^{\pi_{\boldsymbol{\theta}}}(s', a') +$$

$$\pi_{\boldsymbol{\theta}}(a' \mid s') \sum_{s''} P(s'' \mid s', a') \nabla_{\boldsymbol{\theta}} V^{\pi_{\boldsymbol{\theta}}}(s'')] \Bigg]$$

$$= \sum_{x \in \mathcal{S}} \sum_{k=0}^{\infty} \Pr(s \to x, k, \pi_{\boldsymbol{\theta}}) \sum_a \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid x) Q^{\pi_{\boldsymbol{\theta}}}(x, a)$$

Policy Gradient Theorem

Proof (episodic case):

$$
\begin{aligned}
\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) &= \nabla_{\boldsymbol{\theta}} V^{\pi_{\boldsymbol{\theta}}}(s_0) \\
&= \sum_s \left(\sum_{k=0}^{\infty} \Pr(s_0 \to s, k, \pi_{\boldsymbol{\theta}}) \right) \sum_a \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a|s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) \\
&= \sum_s \eta(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a|s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) \\
&= \sum_{s'} \eta(s') \sum_s \frac{\eta(s)}{\sum_{s'} \eta(s')} \sum_a \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a|s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) \\
&= \sum_{s'} \eta(s') \sum_s \rho^{\pi}(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a|s) Q^{\pi_{\boldsymbol{\theta}}}(s, a) \\
&\propto \sum_s \rho^{\pi}(s) \sum_a \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a|s) Q^{\pi_{\boldsymbol{\theta}}}(s, a)
\end{aligned}
$$

(Q.E.D.)

Score Function

- Likelihood ratios exploit the following identity:

$$\overbrace{\nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s)}^{\text{We want the expectation of this}} = \pi_{\boldsymbol{\theta}}(a \mid s) \frac{\nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s)}{\pi_{\boldsymbol{\theta}}(a \mid s)}$$

$$= \underbrace{\pi_{\boldsymbol{\theta}}(a \mid s) \nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(a \mid s)}_{\text{Easy to take the expectation because we can sample from } \pi!}$$

- $\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(a \mid s)$ is called the **score function**

Score Function: Example

Consider a Gaussian policy, where the mean is a linear combination of state features:
$\pi_{\boldsymbol{\theta}}(a \mid s) \sim \mathcal{N}(s^\top \boldsymbol{\theta}, \sigma^2)$, i.e.

$$\pi_{\boldsymbol{\theta}}(a \mid s) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2}\frac{(s^\top \boldsymbol{\theta} - a)^2}{\sigma^2})$$

Exercise (5min)

Derive the score function.

Score Function: Example

Consider a Gaussian policy, where the mean is a linear combination of state features: $\pi_{\boldsymbol{\theta}}(a \mid s) \sim \mathcal{N}(s^\top \boldsymbol{\theta}, \sigma^2)$, i.e.

$$\pi_{\boldsymbol{\theta}}(a \mid s) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(s^\top \boldsymbol{\theta} - a)^2}{\sigma^2}\right)$$

Solution

The log yields

$$\log \pi_{\boldsymbol{\theta}}(a \mid s) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(s^\top \boldsymbol{\theta} - a)^2$$

and the gradient

$$\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(a \mid s) = -\frac{1}{2\sigma^2}(s^\top \boldsymbol{\theta} - a)2s = \frac{(a - s^\top \boldsymbol{\theta})s}{\sigma^2}.$$

REINFORCE

- REINFORCE: Monte Carlo Policy Gradient
- Builds upon Monte Carlo returns as an unbiased sample of Q^{π}
- However, therefore REINFORCE can suffer from high variance

REINFORCE

REINFORCE: Monte-Carlo Policy-Gradient Control (episodic) for π_*

Input: a differentiable policy parameterization $\pi(a|s,\boldsymbol{\theta})$
Algorithm parameter: step size $\alpha > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):
  Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot,\boldsymbol{\theta})$
  Loop for each step of the episode $t = 0, 1, \ldots, T-1$:
    $G \leftarrow \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$ $\qquad (G_t)$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\gamma^t G \nabla \ln \pi(A_t|S_t,\boldsymbol{\theta})$

Variance Reduction with Baselines

- Vanilla REINFORCE provides *unbiased* estimates of the gradient $\nabla J(\theta)$, but it can suffer from high variance
- Goal: reduce variance while remaining unbiased
- Observation: we can generalize the policy gradient theorem by including an arbitrary *action-independent baseline* $b(s)$, i.e.

$$\begin{aligned}
\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) &\propto \sum_s \rho^\pi(s) \sum_a (Q^\pi(s,a) - b(s)) \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s) \\
&= \sum_s \rho^\pi(s) \left[\sum_a Q^\pi(s,a) \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s) - b(s) \underbrace{\nabla_{\boldsymbol{\theta}} \sum_a \pi_{\boldsymbol{\theta}}(a \mid s)}_{=0} \right] \\
&= \sum_s \rho^\pi(s) \sum_a Q^\pi(s,a) \nabla_{\boldsymbol{\theta}} \pi_{\boldsymbol{\theta}}(a \mid s)
\end{aligned}$$

- Baselines can reduce the variance of gradient estimates significantly!

Variance Reduction with Baselines

- A constant value can be used as a baseline
- The state-value function can be used as a baseline

Question

Is the Q-function a valid baseline?

Question

Assume an approximation of the state-value function as a baseline. Is REINFORCE then biased?

REINFORCE with Baselines

Indeed, an estimate of the state value function, $\hat{v}(S_t, w)$, is a very reasonable choice for $b(s)$:

REINFORCE with Baseline (episodic), for estimating $\pi_{\boldsymbol{\theta}} \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$
Input: a differentiable state-value function parameterization $\hat{v}(s,\mathbf{w})$
Algorithm parameters: step sizes $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\mathbf{w}} > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ and state-value weights $\mathbf{w} \in \mathbb{R}^d$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):
  Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot, \boldsymbol{\theta})$
  Loop for each step of the episode $t = 0, 1, \ldots, T-1$:
    $G \leftarrow \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$ (G_t)
    $\delta \leftarrow G - \hat{v}(S_t,\mathbf{w})$
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}} \delta \nabla \hat{v}(S_t,\mathbf{w})$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}} \gamma^t \delta \nabla \ln \pi(A_t|S_t, \boldsymbol{\theta})$

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Actor-Critic Methods

- Methods that learn approximations to both policy and value functions are called actor-critic methods
  actor: learned policy
  critic: learned value function (usually a state-value function)

Question

Is REINFORCE-with-baseline considered as an actor-critic method?

Actor-Critic Methods

- REINFORCE-with-baseline is unbiased, but tends to learn slowly and has high variance
- To gain from advantages of TD methods we use actor-critic methods with a bootstrapping critic

One-step actor-critic methods

Replace the full return of REINFORCE with one-step return as follows:

$$\begin{aligned}
\boldsymbol{\theta}_{t+1} &= \boldsymbol{\theta}_t + \alpha \left(G_{t:t+1} - \hat{v}(S_t, \boldsymbol{w}) \right) \frac{\nabla \pi(A_t \mid S_t, \boldsymbol{\theta}_t)}{\pi(A_t \mid S_t, \boldsymbol{\theta}_t)} \\
&= \boldsymbol{\theta}_t + \alpha \left(R_{t+1} + \gamma \hat{v}(S_{t+1}, \boldsymbol{w}) - \hat{v}(S_t, \boldsymbol{w}) \right) \frac{\nabla \pi(A_t \mid S_t, \boldsymbol{\theta}_t)}{\pi(A_t \mid S_t, \boldsymbol{\theta}_t)} \\
&= \boldsymbol{\theta}_t + \alpha \delta_t \frac{\nabla \pi(A_t \mid S_t, \boldsymbol{\theta}_t)}{\pi(A_t \mid S_t, \boldsymbol{\theta}_t)}
\end{aligned}$$

Actor-Critic Methods

One-step Actor–Critic (episodic), for estimating $\pi_\theta \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s,\boldsymbol{\theta})$
Input: a differentiable state-value function parameterization $\hat{v}(s,\mathbf{w})$
Parameters: step sizes $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\mathbf{w}} > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ and state-value weights $\mathbf{w} \in \mathbb{R}^{d}$ (e.g., to $\mathbf{0}$)
Loop forever (for each episode):
    Initialize S (first state of episode)
    $I \leftarrow 1$
    Loop while S is not terminal (for each time step):
        $A \sim \pi(\cdot|S,\boldsymbol{\theta})$
        Take action A, observe S', R
        $\delta \leftarrow R + \gamma\hat{v}(S',\mathbf{w}) - \hat{v}(S,\mathbf{w})$ (if S' is terminal, then $\hat{v}(S',\mathbf{w}) \doteq 0$)
        $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}}\delta\nabla\hat{v}(S,\mathbf{w})$
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}} I\delta\nabla\ln\pi(A|S,\boldsymbol{\theta})$
        $I \leftarrow \gamma I$
        $S \leftarrow S'$

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Proximal Policy Optimization

- Motivation: how can we take the biggest possible improvement step on a policy using the data we currently have, without stepping so far that we accidentally cause performance collapse?
- We collect data with $\pi_{\boldsymbol{\theta}_{\text{old}}}$
- And we want to optimize some objective to get a new policy $\pi_{\boldsymbol{\theta}}$
- In PPO, we *ignore* the change in state distribution and optimize a **surrogate objective**:

$$
\begin{aligned}
J_{\text{old}}(\theta) &= \mathbb{E}_{S\sim\rho^{\pi_{\boldsymbol{\theta}_{\text{old}}}},A\sim\pi_{\boldsymbol{\theta}}}[\mathcal{A}^{\pi_{\boldsymbol{\theta}_{\text{old}}}}(S,A)] \\
&= \mathbb{E}_{(S,A)\sim\pi_{\boldsymbol{\theta}_{\text{old}}}}\left[\frac{\pi_{\boldsymbol{\theta}}}{\pi_{\boldsymbol{\theta}_{\text{old}}}}\mathcal{A}^{\pi_{\boldsymbol{\theta}_{\text{old}}}}(S,A)\right]
\end{aligned}
$$

- Improvement Theory: $\eta(\pi_{\boldsymbol{\theta}}) \geq J_{\text{old}}(\theta) - c \cdot \max_s D_{\mathrm{KL}}(\pi_{\boldsymbol{\theta}_{\text{old}}}||\pi_{\boldsymbol{\theta}})$
- If we keep the KL-divergence between our old and new policies small, optimizing the surrogate is close to optmizing $\eta(\pi_{\boldsymbol{\theta}})$!

Proximal Policy Optimization

- Adaptive Penalty Surrogate Objective:

$$\mathbb{E}_{(S,A)\sim\pi_{\boldsymbol{\theta}_{\text{old}}}}\left[\frac{\pi_{\boldsymbol{\theta}}}{\pi_{\boldsymbol{\theta}_{\text{old}}}}\mathcal{A}^{\pi_{\boldsymbol{\theta}_{\text{old}}}}(S,A)-\beta\, D_{\text{KL}}(\pi_{\boldsymbol{\theta}_{\text{old}}}||\pi_{\boldsymbol{\theta}})\right]$$

- Clipped Surrogate Objective:

$$\mathbb{E}_{(S,A)\sim\pi_{\boldsymbol{\theta}_{\text{old}}}}\left[\min\left(\frac{\pi_{\boldsymbol{\theta}}}{\pi_{\boldsymbol{\theta}_{\text{old}}}}\mathcal{A}^{\pi_{\boldsymbol{\theta}_{\text{old}}}}(S,A),\ \text{clip}(\frac{\pi_{\boldsymbol{\theta}}}{\pi_{\boldsymbol{\theta}_{\text{old}}}},1-\epsilon,1+\epsilon)\mathcal{A}^{\pi_{\boldsymbol{\theta}_{\text{old}}}}(S,A)\right)\right]$$

Proximal Policy Optimization

Algorithm 1 PPO-Clip

Input: initial policy parameters θ_0, initial value function parameters ϕ_0

for $k = 0, 1, 2, \ldots$ **do**

  Collect set of trajectories $\mathcal{D}_k = \{\tau_i\}$ by running policy $\pi_k = \pi(\theta_k)$ in the environment.

  Compute rewards-to-go $\hat{R}_t$.

  Compute advantage estimates, $\hat{A}_t$ (using any method of advantage estimation) based on the current value function V_{ϕ_k}.

  Update the policy by maximizing the PPO-Clip objective:

$$\theta_{k+1} = \arg\max_{\theta} \frac{1}{|\mathcal{D}_k| T} \sum_{\tau \in \mathcal{D}_k} \sum_{t=0}^{T} \min\left(\frac{\pi_\theta(a_t|s_t)}{\pi_{\theta_k}(a_t|s_t)} A^{\pi_{\theta_k}}(s_t, a_t), \quad g(\epsilon, A^{\pi_{\theta_k}}(s_t, a_t)) \right),$$

  typically via stochastic gradient ascent with Adam.

  Fit value function by regression on mean-squared error:

$$\phi_{k+1} = \arg\min_{\phi} \frac{1}{|\mathcal{D}_k| T} \sum_{\tau \in \mathcal{D}_k} \sum_{t=0}^{T} \left(V_\phi(s_t) - \hat{R}_t \right)^2,$$

  typically via some gradient descent algorithm.

end for

credits: https://spinningup.openai.com/en/latest/algorithms/ppo.html

Deep Deterministic Policy Gradient

- DDPG is an actor-critic method (*Continuous DQN*)
- Recall the DQN-target: $Y_j = R_j + \gamma \max_a Q_{\mathbf{w}^-}(S_{j+1}, a)$
- In case of continuous actions, the maximization step is not trivial
- Therefore, we approximate deterministic actor μ representing the $\arg\max_a Q_{\mathbf{w}}(S_{j+1}, a)$ by a neural network and update its parameters following the

Deterministic Policy Gradient Theorem

$$\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \mathbb{E}_{\mu}\left[\nabla_{\boldsymbol{\theta}} \mu_{\boldsymbol{\theta}}(S) \nabla_a Q_{\mathbf{w}}(S, a)|_{a=\mu(S)}\right].$$

- Exploration by adding Gaussian noise to the output of μ

Deep Deterministic Policy Gradient

- The Q-function is fitted to the adapted TD-target:

$$Y_j = R_j + \gamma Q_{\mathbf{w}^-}(S_{j+1}, \mu_{\boldsymbol{\theta}^-}(S_{j+1}))$$

- The parameters of the target networks of the actor $\boldsymbol{\theta}^-$ and the critic $\mathbf{w}^-$ are then adjusted with a soft update

$$\mathbf{w}^- \leftarrow (1-\tau)\mathbf{w}^- + \tau\mathbf{w} \text{ and } \boldsymbol{\theta}^- \leftarrow (1-\tau)\boldsymbol{\theta}^- + \tau\boldsymbol{\theta}$$

  with $\tau \in (0, 1]$
- DDPG is very popular and builds the basis for more SOTA actor-critic algorithms
- However, it can be quite unstable and sensitive to its hyperparameters

Deep Deterministic Policy Gradient

Algorithm 1: DDPG

Initialize replay memory D to capacity N
Initialize critic Q and actor μ with random weights
for *episode* $i = 1, .., M$ **do**
    for $t = 1, .., T$ **do**
        select action $A_t = \mu(s_t, \boldsymbol{\theta}) + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma)$
        Store transition (S_t, A_t, S_{t+1}, R_t) in D
        Sample minibatch of transitions (S_j, A_j, R_j, S_{j+1}) from D
        Set $y_j = \begin{cases} R_j & \text{if } S_{j+1} \text{ is terminal} \\ R_j + \gamma \; Q(S_{j+1}, \mu(S_{j+1}, \boldsymbol{\theta}^-), \mathbf{w}^-) & \text{else} \end{cases}$
        Update the parameters of Q according to the TD-error
        Update the parameters of μ according to:

$$\nabla_\theta J \approx \frac{1}{N} \sum_j \nabla_a Q_{\mathbf{w}}(S_j, a)|_{a=\mu(S_j)} \nabla_{\boldsymbol{\theta}} \mu_{\boldsymbol{\theta}}(S_j)$$

        Adjust the parameters of the target networks via a soft update

Soft Actor-Critic

- Soft Actor-Critic: entropy-regularized value-learning
- The policy is trained to maximize a trade-off between expected return and entropy ($H(p) = \mathbb{E}_{x\sim p}[-\log p(x)]$), a measure of randomness in the policy:

$$\pi_* = \arg\max_{\pi} \mathbb{E}_\pi \left[\sum_{t=0}^{\infty} R_{t+1} + \alpha H(\pi(\cdot|S_t = s_t)) \right]$$

- The value functions are then defined as:

$$V^\pi(s) = \mathbb{E}_\pi[\sum_{t=0}^{\infty} R_{t+1} + \alpha H(\pi(\cdot \mid S_t = s_t))|S_0 = s, A_0 = a]$$

$$Q^\pi(s,a) = \mathbb{E}_\pi[\sum_{t=0}^{\infty} R_{t+1} + \alpha \sum_{t=1}^{\infty} \gamma^t H(\pi(\cdot \mid S_t = s_t))|S_0 = s, A_0 = a]$$

- And their relation as: $V^\pi(s) = \mathbb{E}_\pi[Q^\pi(s,a)] + \alpha H(\pi(\cdot \mid S_t = s))$

Soft Actor-Critic

- The corresponding Bellman equation for Q^π is

$$Q^\pi(s_t, a_t) = \mathbb{E}_\pi\Big[R_{t+1} + \gamma\big(Q^\pi(S_{t+1}, A_{t+1}) + \alpha H(\pi(\cdot \mid S_{t+1}))\big)\Big]$$
$$= \mathbb{E}_\pi[R_{t+1} + \gamma V^\pi(S_{t+1})].$$

- Loss for the Q-networks:

$$L(\mathbf{w}_i, \mathcal{D}) = \underset{(S,A,R,S')\sim\mathcal{D}}{\mathrm{E}}\left[\Big(Q_{\mathbf{w}_i}(S, A) - y(R, S')\Big)^2\right]$$

where the target is:

$$y(r, s') = r + \gamma\left(\min_{j=1,2} Q_{\mathbf{w}_j^-}\left(s', \tilde{A}'\right) - \alpha \log \pi_\theta\left(\tilde{A}' \mid s'\right)\right), \quad \tilde{A}' \sim \pi_\theta(\cdot \mid s')$$

Soft Actor-Critic

- We want to find a policy which maximizes expected future return and expected future entropy, i.e. which maximizes $V^\pi(s)$:

$$\begin{aligned} V^\pi(s) &= \mathbb{E}_{A\sim\pi(\cdot|s)}\left[Q^\pi(s, A)\right] + \alpha H(\pi(\cdot \mid s)) \\ &= \mathbb{E}_{A\sim\pi(\cdot|s)}\left[Q^\pi(s, A) - \alpha \log \pi(A \mid s)\right] \end{aligned}$$

- To optimize the policy despite the sampling of actions, we make use of the *reparameterization trick*:

$$\tilde{A}_\theta(s, \xi) = \tanh\left(\mu_\theta(s) + \sigma_\theta(s) \odot \xi\right), \quad \xi \sim \mathcal{N}(0, I)$$

- We can thus rewrite the expectation from above as:

$$\mathbb{E}_{A\sim\pi_\theta}\left[Q^{\pi_\theta}(s, A) - \alpha \log \pi_\theta(A \mid s)\right] = \mathbb{E}_{\xi\sim\mathcal{N}(0,I)}\left[Q^{\pi_\theta}\left(s, \tilde{A}_\theta(s,\xi)\right) - \alpha \log \pi_\theta\left(\tilde{A}_\theta(s,\xi) \mid s\right)\right]$$

- Final policy loss is then:

$$\max_\theta \mathbb{E}_{s,\xi}\left[\min_{j=1,2} Q_{\mathbf{w}_j^-}\left(s, \tilde{A}_\theta(s,\xi)\right) - \alpha \log \pi_\theta\left(\tilde{A}_\theta(s,\xi) \mid s\right)\right]$$

Soft Actor-Critic

Algorithm 1 Soft Actor-Critic

Input: initial policy parameters θ, Q-function parameters ϕ_1, ϕ_2, empty replay buffer $\mathcal{D}$
Set target parameters equal to main parameters $\phi_{\text{targ},1} \leftarrow \phi_1$, $\phi_{\text{targ},2} \leftarrow \phi_2$
repeat
  Observe state s and select action $a \sim \pi_\theta(\cdot|s)$
  Execute a in the environment
  Observe next state s', reward r, and done signal d to indicate whether s' is terminal
  Store (s, a, r, s', d) in replay buffer $\mathcal{D}$
  If s' is terminal, reset environment state.
  if it's time to update **then**
    for j in range(however many updates) **do**
      Randomly sample a batch of transitions, $B = \{(s, a, r, s', d)\}$ from $\mathcal{D}$
      Compute targets for the Q functions:

$$y(r, s', d) = r + \gamma(1-d)\left(\min_{i=1,2} Q_{\phi_{\text{targ},i}}(s', \tilde{a}') - \alpha \log \pi_\theta(\tilde{a}'|s')\right), \quad \tilde{a}' \sim \pi_\theta(\cdot|s')$$

      Update Q-functions by one step of gradient descent using

$$\nabla_{\phi_i} \frac{1}{|B|} \sum_{(s,a,r,s',d)\in B} \left(Q_{\phi_i}(s,a) - y(r, s', d)\right)^2 \qquad \text{for } i = 1, 2$$

      Update policy by one step of gradient ascent using

$$\nabla_\theta \frac{1}{|B|} \sum_{s\in B} \left(\min_{i=1,2} Q_{\phi_i}(s, \tilde{a}_\theta(s)) - \alpha \log \pi_\theta\left(\tilde{a}_\theta(s)|\, s\right)\right),$$

      where $\tilde{a}_\theta(s)$ is a sample from $\pi_\theta(\cdot|s)$ which is differentiable wrt θ via the reparametrization trick.
      Update target networks with

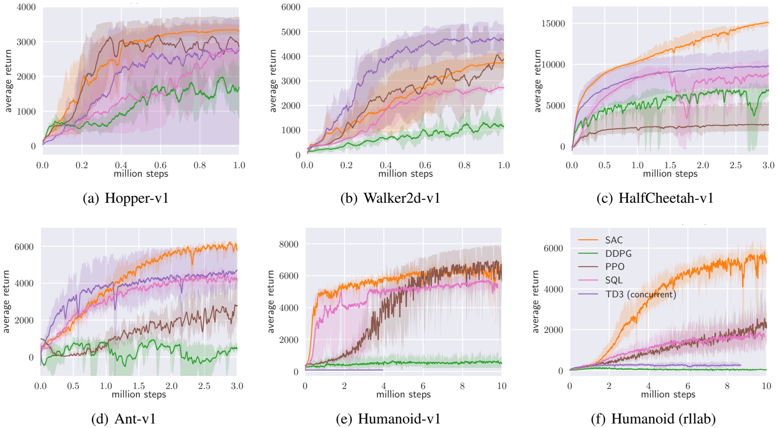
$$\phi_{\text{targ},i} \leftarrow \rho\phi_{\text{targ},i} + (1-\rho)\phi_i \qquad \text{for } i = 1, 2$$

    end for
  end if
until convergence

credits: `https://spinningup.openai.com/en/latest/algorithms/sac.html`

Soft Actor-Critic

- Performance comparison from (Haarnoja et al., 2018):



(a) Hopper-v1 (b) Walker2d-v1 (c) HalfCheetah-v1

(d) Ant-v1 (e) Humanoid-v1 (f) Humanoid (rllab)

Summary by Learning Goals

Having heard this lecture, you can now...

- understand policy gradient methods and derive the policy gradient theorem
- design and implement actor-critic methods that combine policy and value function learning
- apply state-of-the-art algorithms (PPO, DDPG, SAC) to continuous control problems

If you want to get an even more detailed overview about the current SOTA, you can have a look at `Stable Baselines3`, which is a good start for training your own RL agents:

`https://github.com/DLR-RM/stable-baselines3`