Homework 7: Convexity

Hand in: 02.12.2025 (Tuesday)
Please follow the submission instructions from the webpage of the course.

Correction: tutorial session on 04.12.2025 (Thursday)

Exercise 1: Convex sets (8 points)

Which of the following sets $\mathcal{X} \subset \mathbb{R}^n$ or $\mathcal{A} \subset \mathbb{R}^{n \times n}$ are convex? Justify your answer.

1. $\mathcal{X} = \left\{ x \in \mathbb{R}^n \text{ such that } \sum_{i=1}^{n} |x_i| \leq 1 \right\}$
2. $\mathcal{X} = \left\{ x \in \mathbb{R}^3 \text{ such that } x_1 = x_2 \cdot x_3 \right\}$
3. $\mathcal{A} = \left\{ A \in \mathbb{R}^{n \times n} \text{ such that } A = A^\top \right\}$
4. $\mathcal{A} = \left\{ A \in \mathbb{R}^{n \times n} \text{ such that } A = A^\top \text{ and } \forall x \in \mathbb{R}^n,\ x^\top A x \geq 0 \right\}$

Exercise 2: Convex functions (8 points)

Which of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ are convex? Justify your answer.

1. $f(x, y) = xy$
2. $f(x, y) = e^{2x-3y} + 4y$
3. $f(x, y) = \sin(x) + \cos(y)$
4. $f(x, y) = \max\{x, y\} = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } y > x \end{cases}$

Exercise 3: Jensen Inequality (8 points)

1. Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex set. Let $x_1, \ldots, x_m$ be some elements of $\mathcal{X}$. Show that the average point $\frac{x_1 + \cdots + x_m}{m}$ is also an element of $\mathcal{X}$.

   Hint: *prove this property via induction.*

   Hint: *find some α such that:* $\frac{x_1 + \cdots + x_{m+1}}{m+1} = (1 - \alpha)\frac{x_1 + \cdots + x_m}{m} + \alpha x_{m+1}$.

2. Now let $f : \mathcal{X} \to \mathbb{R}$ be a convex function. Show the following inequality:

$$f\left(\frac{x_1 + \cdots + x_m}{m}\right) \leq \frac{f(x_1) + \cdots + f(x_m)}{m} \tag{1}$$

3. Now show the following generalization:

$$f\left(\sum_{j=1}^{m} \alpha_j x_j\right) \leq \sum_{j=1}^{m} \alpha_j f(x_j) \tag{2}$$

   for any $\alpha_1, \ldots, \alpha_m \geq 0$ such that $\sum_{j=1}^{m} \alpha_j = 1$.

Exercise 4: Minimizer of the Cross Entropy (8 points)

Let $\mathcal{P}_m$ be the set of probability distributions over the set $\{1, \ldots, m\}$, i.e.:

$$\mathcal{P}_m = \left\{ p \in \mathbb{R}^m \text{ such that } \forall j,\ p_j \geq 0, \text{ and } \sum_{j=1}^{m} p_j = 1 \right\} \tag{3}$$

1. Prove that for any $p, q \in \mathcal{P}_m$, the inequality holds:

$$\sum_{j=1}^{m} p_j \log\left(\frac{q_j}{p_j}\right) \leq 0 \tag{4}$$

   Hint: Apply the Jensen inequality (2) *to the function* $f(x) = -\log(x)$ *(after proving that this is a convex function).*

2. We define the cross entropy between two distributions as follows:

$$L(p, q) = -\sum_{j=1}^{m} p_j \log(q_j) \tag{5}$$

   Let p be an element of $\mathcal{P}_m$. Show that $q = p$ is a minimizer of:

$$\underset{q \in \mathcal{P}_m}{\text{minimize}}\ L(p, q) \tag{6}$$

   Remark: You do not have to show that it is the unique minimizer.

Programming tasks (4 bonus points)

Open the jupyter notebook `programming_exercise2.ipynb`, and fill in the missing parts of the code.

If you are struggling with downloading Jupyter notebook, you can also use it online via

`https://jupyter.org/try-jupyter/lab`.