MPC Solutions and Stability of the Disturbance Attenuation Regulator

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LQR problem assumptions

Assumptions:

- (A, B) stabilizable and (A, Q) detectable
- $Q \succeq 0,\ R \succ 0$

Multi-stage LQR problem

$$\min_{\boldsymbol{u}} V(x_0, \boldsymbol{u}) = \frac{1}{2} \sum_{k=0}^{N-1} \left(x_k' Q x_k + u_k' R u_k \right) + \frac{1}{2} x_N' P_f x_N$$

subject to

$$x_{k+1} = Ax_k + Bu_k \quad k = 0, 1, \ldots, N-1$$

where $\boldsymbol{u} = (u_0, u_1, \ldots, u_{N-1})$ is the input sequence

MPC approach: Since backward DP is intractable with input constraints, solve for the *entire* input sequence in one shot

Model solution for entire sequence

The linear model gives us the complete state trajectory:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{\mathcal{A}} x_0 + \underbrace{\begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

Compactly: $\boldsymbol{x} = \mathcal{A}x_0 + \mathcal{B}\,\boldsymbol{u}$
where $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$

Objective function formulation

The objective function becomes

$$V(x_0, \boldsymbol{u}) = \frac{1}{2}\left(x_0' Q x_0 + \boldsymbol{x}' \mathcal{Q} \boldsymbol{x} + \boldsymbol{u}' \mathcal{R} \boldsymbol{u}\right)$$

where

$$\begin{aligned} \mathcal{Q} &= \operatorname{diag}\left(\begin{bmatrix} Q & Q & \dots & P_f \end{bmatrix}\right) \in \mathbb{R}^{Nn \times Nn} \\ \mathcal{R} &= \operatorname{diag}\left(\begin{bmatrix} R & R & \dots & R \end{bmatrix}\right) \in \mathbb{R}^{Nm \times Nm} \end{aligned}$$

Eliminating the state sequence

Substitute $\boldsymbol{x} = \mathcal{A}x_0 + \mathcal{B}\,\boldsymbol{u}$ into the objective

$$V(x_0, \boldsymbol{u}) \;=\; \frac{1}{2}x_0'(Q + \mathcal{A}'\mathcal{Q}\mathcal{A})x_0 + \boldsymbol{u}'(\mathcal{B}'\mathcal{Q}\mathcal{A})x_0 + \frac{1}{2}\boldsymbol{u}'(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})\boldsymbol{u}$$

This is quadratic in $\boldsymbol{u}$ with $\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R} \succ 0$
Therefore, solution to $\min_{\boldsymbol{u}} V$ exists and is unique

Multi-stage LQR solution

$$\boldsymbol{u}^*(x_0) = \mathcal{K}x_0 \qquad \text{(optimal input sequence is linear in } x_0\text{)}$$

$$V^*(x_0) = \frac{1}{2}x_0'\Pi x_0 \qquad \text{(optimal cost is quadratic in } x_0\text{)}$$

with

$$\mathcal{K} = -(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})^{-1}\mathcal{B}'\mathcal{Q}\mathcal{A} \qquad \text{(optimal gain matrix)}$$

$$\Pi = Q + \mathcal{A}'\mathcal{Q}\mathcal{A} - \mathcal{A}'\mathcal{Q}\mathcal{B}(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})^{-1}\mathcal{B}'\mathcal{Q}\mathcal{A}$$

The first control action is:

$$u_0^*(x_0) = K_0 x_0 \quad \text{where} \quad K_0 = \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} \mathcal{K}$$

DAR problem assumptions

Assumptions:

- (A, B) stabilizable and (A, Q) detectable
- $Q \succeq 0, \ R \succ 0$

Multi-stage DAR problem

$$\min_{\boldsymbol{u}} \max_{\boldsymbol{w}} V(x_0, \boldsymbol{u}, \boldsymbol{w}) = \frac{1}{2} \sum_{k=0}^{N-1} \left(x_k' Q x_k + u_k' R u_k \right) + \frac{1}{2} x_N' P_f x_N$$

subject to

$$x_{k+1} = A x_k + B u_k + G w_k \quad k = 0, 1, \ldots, N-1$$

where $\boldsymbol{u} = (u_0, u_1, \ldots, u_{N-1})$ and $\boldsymbol{w} = (w_0, w_1, \ldots, w_{N-1})$ are the input and disturbance sequences

Key challenge: Unlike LQR, we should not solve directly for $\boldsymbol{u}$ because the minmax problem requires feedback to achieve reasonable performance

Control parameterization for DAR

Key insight: Parameterize the control with feedback

$$u_k = \bar{K} x_k + v_k$$

where

- $\bar{K}$: feedback gain (chosen as infinite horizon DAR gain for $x_0 = 0$ *or* optimization variable)
- v_k: auxiliary control variable (optimization variable)

The DAR problem becomes

$$\min_{\boldsymbol{v}} \max_{\color{red}\boldsymbol{w}} V(x_0, \boldsymbol{v}, {\color{red}\boldsymbol{w}})$$

where $\boldsymbol{v} = (v_0, v_1, \ldots, v_{N-1})$ and $u_k = \bar{K} x_k + v_k$

Model solution with control parameterization

With $u_k = \bar{K} x_k + v_k$ and $A_K = A + B\bar{K}$, the system becomes

$$x_{k+1} = A_K x_k + B v_k + G w_k$$

The complete state trajectory

$$\boldsymbol{x} = \mathcal{A}_K x_0 + \mathcal{B}\boldsymbol{v} + \mathcal{G}\boldsymbol{w} \qquad \boldsymbol{x} = (x_1, x_2, \ldots, x_N)$$

where

$$\mathcal{A}_K = \begin{bmatrix} A_K \\ A_K^2 \\ \vdots \\ A_K^N \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} B & 0 & \cdots & 0 \\ A_K B & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_K^{N-1} B & A_K^{N-2} B & \cdots & B \end{bmatrix}$$

$$\mathcal{G} = \begin{bmatrix} G & 0 & \cdots & 0 \\ A_K G & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_K^{N-1} G & A_K^{N-2} G & \cdots & G \end{bmatrix}$$

Objective function formulation

The control sequence: $\boldsymbol{u} = \mathcal{K}_1 x_0 + \mathcal{K}_2 \boldsymbol{x} + \boldsymbol{v}$

$$\mathcal{K}_1 = \begin{bmatrix} \bar{K} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathcal{K}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \bar{K} & 0 & 0 & \cdots & 0 \\ 0 & \bar{K} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{K} & 0 \end{bmatrix}$$

Combined matrices

$$\mathcal{A}_u = \mathcal{K}_1 + \mathcal{K}_2 \mathcal{A}_K \quad \mathcal{B}_u = \mathcal{K}_2 \mathcal{B} + I \quad \mathcal{G}_u = \mathcal{K}_2 \mathcal{G}$$

Stacked form $\boldsymbol{x} = \bar{\mathcal{A}} x_0 + \bar{\mathcal{B}} \boldsymbol{v} + \bar{\mathcal{G}} \boldsymbol{w}$

$$\bar{\mathcal{A}} = \begin{bmatrix} I \\ \mathcal{A}_K \\ \mathcal{A}_u \end{bmatrix} \quad \bar{\mathcal{B}} = \begin{bmatrix} 0 \\ \mathcal{B} \\ \mathcal{B}_u \end{bmatrix} \quad \bar{\mathcal{G}} = \begin{bmatrix} 0 \\ \mathcal{G} \\ \mathcal{G}_u \end{bmatrix}$$

Eliminating the state and control sequences

Define the weighting matrix

$$\bar{\mathcal{D}} = \text{diag}(Q, \mathcal{Q}, \mathcal{R}) \quad \mathcal{Q} = \text{diag}(Q, Q, \ldots, P_f) \quad \mathcal{R} = \text{diag}(R, R, \ldots, R)$$

The objective function becomes

$$V(x_0, \boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2} \left| \bar{\mathcal{A}} x_0 + \bar{\mathcal{B}} \boldsymbol{v} + \bar{\mathcal{G}} \boldsymbol{w} \right|^2_{\bar{\mathcal{D}}}$$

This is quadratic in $(\boldsymbol{v}, \boldsymbol{w})$

$$V(x_0, \boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}' M \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}' d + \frac{1}{2} x_0' \bar{\mathcal{E}} x_0$$

where

$$M = \begin{bmatrix} \bar{\mathcal{B}}' \bar{\mathcal{D}} \bar{\mathcal{B}} & \bar{\mathcal{B}}' \bar{\mathcal{D}} \bar{\mathcal{G}} \\ \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{B}} & \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} \end{bmatrix} \quad d = \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}} \bar{\mathcal{A}} x_0 \quad \bar{\mathcal{E}} = \bar{\mathcal{A}}' \bar{\mathcal{D}} \bar{\mathcal{A}}$$

Two disturbance paradigms

The optimization problem

$$\min_{\boldsymbol{v}} \max_{\boldsymbol{w}} V(x_0, \boldsymbol{v}, \boldsymbol{w})$$

Two different disturbance paradigms

1. Signal-bounded disturbances

$$\|\boldsymbol{w}\|_2^2 = \sum_{k=0}^{N-1} |w_k|^2 \leq \alpha \quad \text{(single constraint)}$$

2. Stage-bounded disturbances

$$|w_k|^2 \leq \alpha_k \quad k = 0, 1, \ldots, N-1 \quad \text{(multiple constraints)}$$

This leads to two different optimization approaches with different Lagrange multiplier structures

Signal-bounded DAR problem

Define the Lagrange multiplier $\lambda > 0$ and Lagrangian function

$$L(x_0, \boldsymbol{v}, \boldsymbol{w}, \lambda) = V(x_0, \boldsymbol{v}, \boldsymbol{w}) - \frac{\lambda}{2}(\boldsymbol{w}'\boldsymbol{w} - \alpha)$$

The constrained problem becomes

$$\min_{\lambda} \min_{\boldsymbol{v}} \max_{\boldsymbol{w}} L(x_0, \boldsymbol{v}, \boldsymbol{w}, \lambda)$$

This is quadratic in $(\boldsymbol{v}, \boldsymbol{w})$:

$$L(x_0, \boldsymbol{v}, \boldsymbol{w}, \lambda) = \frac{1}{2} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}' M(\lambda) \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}' d + \frac{1}{2} x_0' \bar{\mathcal{E}} x_0 + \frac{1}{2} \lambda \alpha$$

where

$$M(\lambda) = \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \lambda I \end{bmatrix}$$

Signal-bounded DAR scalar minimization

Solve $\min_{\boldsymbol{v}} \max_{\boldsymbol{w}} L(x_0, \boldsymbol{v}, \boldsymbol{w}, \lambda)$ to obtain saddle point $(\boldsymbol{v}^*(\lambda), \boldsymbol{w}^*(\lambda))$, which requires:

$$\bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} \succeq 0 \qquad \text{convex in } \boldsymbol{v} \tag{1}$$

$$\bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \lambda I \preceq 0 \qquad \text{concave in } \boldsymbol{w} \tag{2}$$

From Theorem 9 (minmax fundamentals lecture) the remaining optimization is

$$\min_{\lambda} L(x_0, \boldsymbol{v}^*(\lambda), \boldsymbol{w}^*(\lambda), \lambda) \quad \text{s.t. } \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \lambda I \preceq 0$$

The optimal $\lambda^*(x_0, \alpha)$ solves

$$\min_{\lambda \geq |\bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}}|} \left(\frac{1}{2} x_0' \Pi(\lambda) x_0 + \frac{1}{2}\lambda\alpha \right)$$

where

$$\Pi(\lambda) = \bar{\mathcal{E}} - \bar{\mathcal{A}}'\bar{\mathcal{D}} \begin{bmatrix} \bar{\mathcal{B}} & \bar{\mathcal{G}} \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \lambda I \end{bmatrix}^{\dagger} \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}}\bar{\mathcal{A}}$$

Signal-bounded DAR solutions

From Theorem 9

$$\boldsymbol{v}^*(x_0, \alpha) = \mathcal{K}_v(\lambda^*(x_0, \alpha))x_0 \qquad \text{(optimal auxiliary control)}$$

$$V^*(x_0, \alpha) = \frac{1}{2}x_0' \Pi(\lambda^*(x_0, \alpha))x_0 + \frac{1}{2}\lambda\alpha \qquad \text{(optimal cost)}$$

with optimal auxiliary gain

$$\mathcal{K}_v(x_0, \alpha) = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \lambda^*(x_0, \alpha)I \end{bmatrix}^\dagger \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}}\bar{\mathcal{A}}$$

and cost

$$\Pi(x_0, \alpha) = \bar{\mathcal{E}} - \bar{\mathcal{A}}'\bar{\mathcal{D}} \begin{bmatrix} \bar{\mathcal{B}} & \bar{\mathcal{G}} \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \lambda^*(x_0, \alpha)I \end{bmatrix}^\dagger \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}}\bar{\mathcal{A}}$$

The first control action is

$$u_0^*(x_0) = \bar{K}x_0 + v_0^*(x_0) = K_0(\lambda^*(x_0, \alpha))x_0$$

where $K_0(\lambda) = \bar{K} + \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} \mathcal{K}_v(x_0, \alpha)$

Stage-bounded DAR problem

Define $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{N-1})$ and Lagrangian function

$$L(x_0, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda}) = V(x_0, \boldsymbol{v}, \boldsymbol{w}) - \frac{\lambda_0}{2}(|w_0|^2 - \alpha_0) - \cdots - \frac{\lambda_{N-1}}{2}(|w_{N-1}|^2 - \alpha_{N-1})$$

The constrained problem becomes

$$\min_{\boldsymbol{\lambda}} \min_{\boldsymbol{v}} \max_{\boldsymbol{w}} L(x_0, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda})$$

This is quadratic in $(\boldsymbol{v}, \boldsymbol{w})$

$$L(x_0, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda}) = \frac{1}{2} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}' M(\Lambda(\boldsymbol{\lambda})) \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}' d + \frac{1}{2} x_0' \bar{\mathcal{E}} x_0 + \frac{1}{2} \boldsymbol{\lambda}' \boldsymbol{\alpha}$$

where

$$\Lambda(\boldsymbol{\lambda}) = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) \qquad \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{N-1})$$

$$M(\Lambda(\boldsymbol{\lambda})) = \begin{bmatrix} \bar{\mathcal{B}}' \bar{\mathcal{D}} \bar{\mathcal{B}} & \bar{\mathcal{B}}' \bar{\mathcal{D}} \bar{\mathcal{G}} \\ \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{B}} & \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \end{bmatrix}$$

Stage-bounded DAR vector minimization

Solve $\min_{\boldsymbol{v}} \max_{\boldsymbol{w}} L(x_0, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\lambda})$ to obtain saddle point $(\boldsymbol{v}^*(\boldsymbol{\lambda}), \boldsymbol{w}^*(\boldsymbol{\lambda}))$, which requires:

$$\bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} \succeq 0 \qquad \text{term associated with } \boldsymbol{v}'\boldsymbol{v}$$

$$\bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \preceq 0 \qquad \text{term associated with } \boldsymbol{w}'\boldsymbol{w}$$

From Theorem 9 the remaining optimization is

$$\min_{\boldsymbol{\lambda}} L(x_0, \boldsymbol{v}^*(\boldsymbol{\lambda}), \boldsymbol{w}^*(\boldsymbol{\lambda}), \boldsymbol{\lambda}) \quad \text{s.t. } \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \preceq 0$$

The optimal $\boldsymbol{\lambda}^*(x_0, \boldsymbol{\alpha})$ solves

$$\min_{\boldsymbol{\lambda}} \left(\frac{1}{2} x_0' \Pi(\Lambda(\boldsymbol{\lambda})) x_0 + \frac{1}{2} \boldsymbol{\lambda}' \boldsymbol{\alpha} \right) \quad \text{s.t. } \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \preceq 0$$

where

$$\Pi(\Lambda(\boldsymbol{\lambda})) = \bar{\mathcal{E}} - \bar{\mathcal{A}}'\bar{\mathcal{D}} \begin{bmatrix} \bar{\mathcal{B}} & \bar{\mathcal{G}} \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \end{bmatrix}^{\dagger} \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}}\bar{\mathcal{A}}$$

Stage-bounded DAR solutions

From Theorem 9

$$\boldsymbol{v}^*(x_0, \boldsymbol{\alpha}) = \mathcal{K}_v(\boldsymbol{\lambda}^*(x_0, \boldsymbol{\alpha}))x_0 \qquad \text{(optimal auxiliary control)}$$

$$V^*(x_0, \boldsymbol{\alpha}) = \frac{1}{2}x_0'\Pi(\Lambda(\boldsymbol{\lambda}^*(x_0, \boldsymbol{\alpha})))x_0 + \frac{1}{2}\boldsymbol{\lambda}'\boldsymbol{\alpha} \qquad \text{(optimal cost)}$$

with optimal auxiliary gain

$$\mathcal{K}_v(x_0, \boldsymbol{\alpha}) = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}^*) \end{bmatrix}^{\dagger} \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}}\bar{\mathcal{A}}$$

and cost

$$\Pi(x_0, \boldsymbol{\alpha}) = \bar{\mathcal{E}} - \bar{\mathcal{A}}'\bar{\mathcal{D}} \begin{bmatrix} \bar{\mathcal{B}} & \bar{\mathcal{G}} \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}^*) \end{bmatrix}^{\dagger} \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}}\bar{\mathcal{A}}$$

The first control action is

$$u_0^*(x_0) = \bar{K}x_0 + v_0^*(x_0) = K_0(\boldsymbol{\lambda}^*(x_0, \boldsymbol{\alpha}))x_0$$

where $K_0(\boldsymbol{\lambda}) = \bar{K} + \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} \mathcal{K}_v(\boldsymbol{\lambda})$

Stage-bounded DAR: SDP to NLP conversion

The stage-bounded DAR optimization is a semidefinite program

$$\min_{\boldsymbol{\lambda}} \left(\frac{1}{2} x_0' \Pi(\Lambda(\boldsymbol{\lambda})) x_0 + \frac{1}{2} \boldsymbol{\lambda}' \boldsymbol{\alpha} \right) \quad \text{s.t. } \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \preceq 0$$

Proposition 1 ((Burer, Monteiro, and Zhang, 2002; Kuntz and Rawlings, 2024))

Let

$$\Theta := \{\boldsymbol{\lambda} \in \mathbb{R}^N \mid \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \preceq 0\}$$

$$\Phi_\epsilon := \{\boldsymbol{\lambda} \mid \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \prec 0,\ diag[chol(\bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}))] \leq \epsilon\}$$

Let $h(\cdot)$ be continuous and attain its minimum on Θ, then

$$\min_{\boldsymbol{\lambda} \in \Theta} h(\boldsymbol{\lambda}) = \lim_{\epsilon \downarrow 0} \min_{\boldsymbol{\lambda} \in \Phi_\epsilon} h(\boldsymbol{\lambda}) = \inf_{\epsilon > 0} \min_{\boldsymbol{\lambda} \in \Phi_\epsilon} h(\boldsymbol{\lambda})$$

If a minimizer $\boldsymbol{\lambda}^*$ is strictly feasible ($\bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}^*) \prec 0$), then there exists $\epsilon_0 > 0$ and for all $0 \leq \epsilon \leq \epsilon_0$

$$\min_{\boldsymbol{\lambda} \in \Theta} h(\boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda} \in \Phi_\epsilon} h(\boldsymbol{\lambda})$$

Stage-bounded DAR: finite-dimensional NLP

The stage-bounded DAR problem becomes a finite-dimensional nonlinear program

$$\min_{\boldsymbol{\lambda}} \left(\frac{1}{2} x_0' \Pi(\Lambda(\boldsymbol{\lambda})) x_0 + \frac{1}{2} \boldsymbol{\lambda}' \boldsymbol{\alpha} \right) \quad \text{s.t. } \operatorname{diag}[\mathcal{C}(\boldsymbol{\lambda})] \leq \epsilon$$

where $\mathcal{C}(\boldsymbol{\lambda})$ is the Cholesky factor of $\bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda})$

Key insight

The matrix inequality constraint $\bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}) \preceq 0$ is *approximated by* scalar constraints via Cholesky; *exact in the limit* $\epsilon \downarrow 0$ *or* if the optimizer is strictly feasible

This converts the semidefinite program to a standard nonlinear program that can be solved with conventional optimization algorithms

Summary: one-shot approaches

Objectives

LQR: $\min_{\boldsymbol{u}} \; V(x_0, \boldsymbol{u})$

Signal DAR: $\min_{\boldsymbol{v}} \; \max_{\boldsymbol{w}} \; V(x_0, \boldsymbol{v}, \boldsymbol{w}) \quad \|\boldsymbol{w}\|^2 = \alpha$

$$\lambda^*(x_0, \alpha) = \arg \min_{\lambda \,\geq\, |\bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}}|} \; \tfrac{1}{2} x_0' \Pi(\lambda) x_0 + \tfrac{1}{2} \lambda \alpha$$

Stage DAR: $\min_{\boldsymbol{v}} \; \max_{\boldsymbol{w}} \; V(x_0, \boldsymbol{v}, \boldsymbol{w}) \quad |w_k|^2 = \alpha_k$

$$\boldsymbol{\lambda}^*(x_0, \boldsymbol{\alpha}) = \arg \min_{\boldsymbol{\lambda}} \; \tfrac{1}{2} x_0' \Pi(\Lambda(\boldsymbol{\lambda})) x_0 + \tfrac{1}{2} \boldsymbol{\lambda}' \boldsymbol{\alpha} \quad \text{s.t. } \operatorname{diag}[\mathcal{C}(\boldsymbol{\lambda})] \leq \epsilon$$

Optimal gains

LQR: $\mathcal{K} = -(\mathcal{B}'\mathcal{Q}\mathcal{B} + \mathcal{R})^{-1} \mathcal{B}'\mathcal{Q}\mathcal{A}$

Signal DAR:

$$\mathcal{K}_v(x_0, \alpha) = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \lambda^* I \end{bmatrix}^{\dagger} \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}}\bar{\mathcal{A}}$$

Stage DAR:

$$\mathcal{K}_v(x_0, \boldsymbol{\alpha}) = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{B}}'\bar{\mathcal{D}}\bar{\mathcal{G}} \\ \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{B}} & \bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}^*) \end{bmatrix}^{\dagger} \begin{bmatrix} \bar{\mathcal{B}}' \\ \bar{\mathcal{G}}' \end{bmatrix} \bar{\mathcal{D}}\bar{\mathcal{A}}$$

Summary: DAR solution methods

Signal-bounded DAR

$$\min_{\lambda \geq |\bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}}|} \left(\frac{1}{2} x_0' \Pi(\lambda) x_0 + \frac{1}{2} \lambda \alpha \right)$$

Stage-bounded DAR (SDP)

$$\min_{\boldsymbol{\lambda}} \left(\frac{1}{2} x_0' \Pi(\Lambda) x_0 + \frac{1}{2} \boldsymbol{\lambda}' \boldsymbol{\alpha} \right) \quad \text{s.t. } \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \Lambda \preceq 0$$

Stage-bounded DAR (Cholesky NLP)

$$\min_{\boldsymbol{\lambda}} \left(\frac{1}{2} x_0' \Pi(\Lambda) x_0 + \frac{1}{2} \boldsymbol{\lambda}' \boldsymbol{\alpha} \right) \quad \text{s.t. } \text{diag}[\mathcal{C}(\boldsymbol{\lambda})] \leq \epsilon$$

Summary: control-constrained DAR solution methods

$\mathcal{V}(x_0)$ enforces $u \in \mathbb{U}$ robustly for all admissible disturbances

$$\mathcal{V}(x_0) := \{\boldsymbol{v} \mid (x_k, u_k) \in \mathbb{X} \times \mathbb{U}, \forall k \in \{0, 1, \ldots, N-1\}, \forall \boldsymbol{w} \in \mathbb{W}\}$$

After maxing over $\boldsymbol{w}$ we obtain
Signal-bounded DAR

$$\min_{\lambda \geq \left|\bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}}\right|,\ \boldsymbol{v} \in \mathcal{V}(x_0)} V(x(0); \boldsymbol{v}, \lambda)$$

Stage-bounded DAR (Cholesky NLP)

$$\min_{\boldsymbol{\lambda},\ \boldsymbol{v} \in \mathcal{V}(x_0)} V(x(0); \boldsymbol{v}, \Lambda(\boldsymbol{\lambda})) \quad \text{s.t. } \operatorname{diag}[\mathcal{C}(\boldsymbol{\lambda})] \leq \epsilon$$

Signal-bounded DAR with control bounds

Evaluating the value function using Theorem 10

$$\min_{\lambda \in \mathbb{R},\ \boldsymbol{v} \in \mathbb{R}^{Nm}} \quad \frac{1}{2} \left| \bar{\mathcal{B}} \boldsymbol{v} + \bar{\mathcal{A}} x_0 \right|^2_{\Sigma(\lambda)} + |x_0|^2_{\bar{\mathcal{E}}} + \frac{1}{2} \lambda \alpha$$

$$\Sigma(\lambda) := \bar{\mathcal{D}} + \bar{\mathcal{D}} \bar{\mathcal{G}} \left(\bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} - \lambda I \right)^{\dagger} \bar{\mathcal{G}}' \bar{\mathcal{D}}$$

$$\text{s.t.} \quad \lambda \geq \left| \bar{\mathcal{G}}' \bar{\mathcal{D}} \bar{\mathcal{G}} \right|$$

$$\bar{\boldsymbol{u}}(\boldsymbol{v}) + \sqrt{\alpha}\, \boldsymbol{d} \leq \overline{\boldsymbol{u}}$$

$$\bar{\boldsymbol{u}}(\boldsymbol{v}) - \sqrt{\alpha}\, \boldsymbol{d} \geq \underline{\boldsymbol{u}}$$

where $\bar{\boldsymbol{u}}(\boldsymbol{v}) = \mathcal{A}_u x_0 + \mathcal{B}_u \boldsymbol{v}$ and $\boldsymbol{d} = \sqrt{\text{diag}(\mathcal{G}_u \mathcal{G}_u')} \in \mathbb{R}^{Nm}$

Stage-bounded DAR with control bounds

$$\min_{\boldsymbol{\lambda}\in\mathbb{R}^N,\ \boldsymbol{v}\in\mathbb{R}^{Nm}} \quad \frac{1}{2}\left|\bar{\mathcal{B}}\boldsymbol{v} + \bar{\mathcal{A}}x_0\right|^2_{\Sigma(\Lambda)} + |x_0|^2_{\bar{\mathcal{E}}} + \frac{1}{2}\boldsymbol{\lambda}'\boldsymbol{\alpha}$$

$$\Sigma(\Lambda(\boldsymbol{\lambda})) := \bar{\mathcal{D}} + \bar{\mathcal{D}}\bar{\mathcal{G}}\left(\bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda})\right)^{\dagger}\bar{\mathcal{G}}'\bar{\mathcal{D}}$$

$$\text{s.t.} \quad \operatorname{diag}[\operatorname{chol}(\bar{\mathcal{G}}'\bar{\mathcal{D}}\bar{\mathcal{G}} - \Lambda(\boldsymbol{\lambda}))] \leq \varepsilon$$

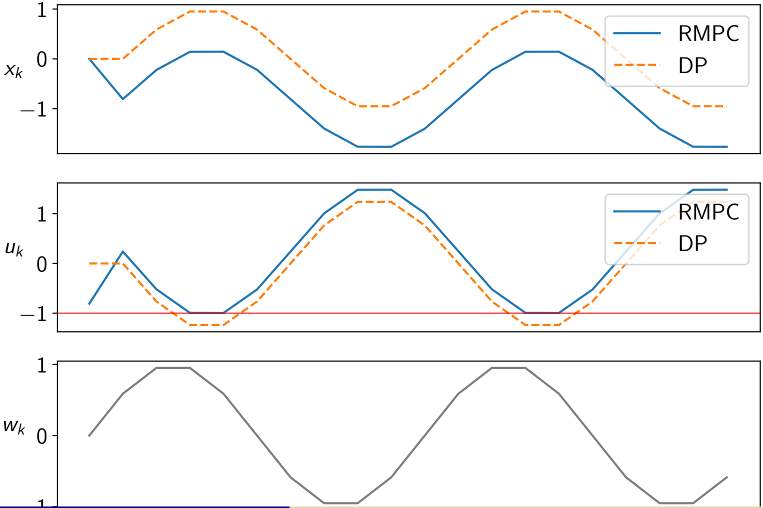
$$\bar{\boldsymbol{u}}(\boldsymbol{v}) + \sum_{k=0}^{N-1}\sqrt{\alpha_k}\,\boldsymbol{d}_k \leq \overline{\boldsymbol{u}}$$

$$\bar{\boldsymbol{u}}(\boldsymbol{v}) - \sum_{k=0}^{N-1}\sqrt{\alpha_k}\,\boldsymbol{d}_k \geq \underline{\boldsymbol{u}}$$

where $\boldsymbol{d}_k = \sqrt{\operatorname{diag}(\mathcal{G}_{u,k}\mathcal{G}'_{u,k})} \in \mathbb{R}^{Nm}$ for $k \in \{0, 1, \ldots, N-1\}$

Example: RMPC vs dynamic programming

$$x^+ = 1.3x + u + w \qquad w_k = \sin(2\pi k/10) \qquad u \geq -1$$



Nonlinear MPC

$$V_N^0(x) = \min_{\boldsymbol{u} \in \mathcal{U}(x)} V(x, \boldsymbol{u})$$

where

$$V(x, \boldsymbol{u}) = \sum_{k=0}^{N-1} \ell\big(x(k), u(k)\big) + V_f\big(x(N)\big)$$

subject to $x^+ = f(x, u, 0), \quad u \in \mathbb{U}, \quad x(N) \in \mathbb{X}_f$

- **Feedback law:** $\kappa(x) = u^*(0; x)$
- **Domain:** $\mathcal{X}$ is the set of feasible initial conditions
- **Closed-loop dynamics:**

$$x^+ = f_{cl}(x, w) := f(x, \kappa(x), w)$$

The closed-loop trajectory is denoted by $\phi(k; x, \boldsymbol{w}_k)$, where

$$\boldsymbol{w}_k = \big(w(0), w(1), \dots, w(k-1)\big)$$

Nonlinear robust MPC (RMPC)

$$V_N^0(x) = \min_{\boldsymbol{u} \in \mathcal{U}^r(x)} \max_{\boldsymbol{w} \in \mathbb{W}^N} V(x, \boldsymbol{u}, \boldsymbol{w})$$

where

$$V(x, \boldsymbol{u}, \boldsymbol{w}) = \sum_{k=0}^{N-1} \ell\big(x(k), u(k)\big) + V_f\big(x(N)\big)$$

subject to $x^+ = f(x, u, w), \quad u \in \mathbb{U}, \quad x(N) \in \mathbb{X}_f, \quad w \in \mathbb{W}$

- **Feedback law:** $\kappa^r(x) = u^*(0; x)$
- **Domain:** $\mathcal{X}^r$ is the set of feasible initial conditions
- **Closed-loop dynamics:**

$$x^+ = f_{cl}(x, w) := f(x, \kappa^r(x), w)$$

The closed-loop trajectory is denoted by $\phi(k; x, \boldsymbol{w}_k)$, where

$$\boldsymbol{w}_k = \big(w(0), w(1), \ldots, w(k-1)\big)$$

Robust positive invariance

Once a state enters this set $\mathcal{X}$, it can never escape it, regardless of what disturbances occur

Robust positive invariance

A set $\mathcal{X} \subset \mathbb{R}^n$ is *robustly positively invariant* (RPI) for

$$x^+ = f_{cl}(x, w), \quad w \in \mathbb{W},$$

if $x \in \mathcal{X}$ implies that $f_{cl}(x, w) \in \mathcal{X}$ for all $w \in \mathbb{W}$

Asymptotic stability

The system state converges to zero with a guaranteed decay rate

Asymptotic stability

The origin of the nominal system $x^+ = f_{cl}(x, 0)$ is *asymptotically stable* (AS) if there exists a function $\beta \in \mathcal{KL}$ such that

$$|\phi(k; x)| \;\leq\; \beta\big(|x|, k\big)$$

for all $x \in \mathcal{X}$ and all $k \geq 0$, where $\phi(k; x)$ denotes the closed-loop state at time k with $w \equiv 0$

Robust asymptotic stability

The system converges toward zero within a bound proportional to the size of disturbances

Robust asymptotic stability

The origin of $x^+ = f_{cl}(x, w)$, $w \in \mathbb{W}$ is *robustly asymptotically stable* (RAS) in the RPI set $\mathcal{X}$ if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|\phi(k; x, \boldsymbol{w}_k)| \ \leq \ \beta\big(|x|, k\big) + \gamma\Big(\|\boldsymbol{w}_k\|\Big)$$

for all $x \in \mathcal{X}$, all $k \geq 0$, and all disturbance sequences $\boldsymbol{w}_k \in \mathbb{W}^k$, where $\|\boldsymbol{w}_k\| := \max\{|w(0)|, \ldots, |w(k-1)|\}$

Robust asymptotic stability in the worst-case sense

The system converges toward zero within a bound set by the uniform disturbance bound $r_{\mathbb{W}}$

Robust asymptotic stability in the worst-case sense

The origin of $x^+ = f_{cl}(x, w)$, $w \in \mathbb{W}$ is *robustly asymptotically stable in the worst-case sense* (RASiW) on an RPI set $\mathcal{X}$ if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for all $x \in \mathcal{X}$, all $k \geq 0$, and all disturbance sequences $\boldsymbol{w}_k \in \mathbb{W}^k$,

$$|\phi(k; x, \boldsymbol{w}_k)| \ \leq \ \beta\big(|x|, k\big) + \gamma(r_{\mathbb{W}})$$

where $r_{\mathbb{W}} := \sup_{w \in \mathbb{W}} \|w\|$

Robustness with respect to stage cost: ℓ-RASiW

The system's performance cost converges within a bound set by $r_{\mathbb{W}}$

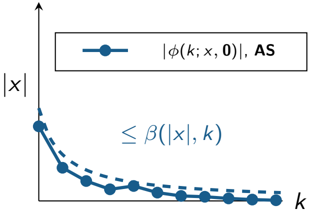
Robustness with respect to stage cost: ℓ-RASiW

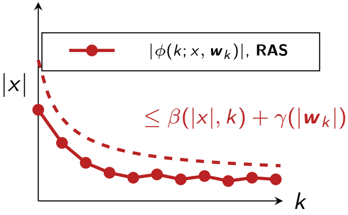
The closed-loop system $x^+ = f_{cl}(x, w)$, $w \in \mathbb{W}$ satisfies ℓ-RASiW on an RPI set $\mathcal{X}$ if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that, for all $x \in \mathcal{X}$, all $k \geq 0$, and all $\boldsymbol{w}_k \in \mathbb{W}^k$,

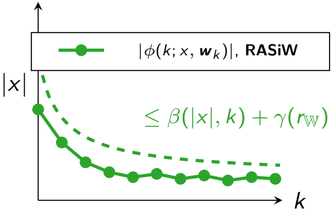
$$\ell\Big(\phi(k; x, \boldsymbol{w}_k),\ \kappa\big(\phi(k; x, \boldsymbol{w}_k)\big)\Big) \ \leq\ \beta\big(|x|, k\big) + \gamma(r_{\mathbb{W}})$$

where $r_{\mathbb{W}} := \sup_{w \in \mathbb{W}} \|w\|$

Asymptotic stability properties

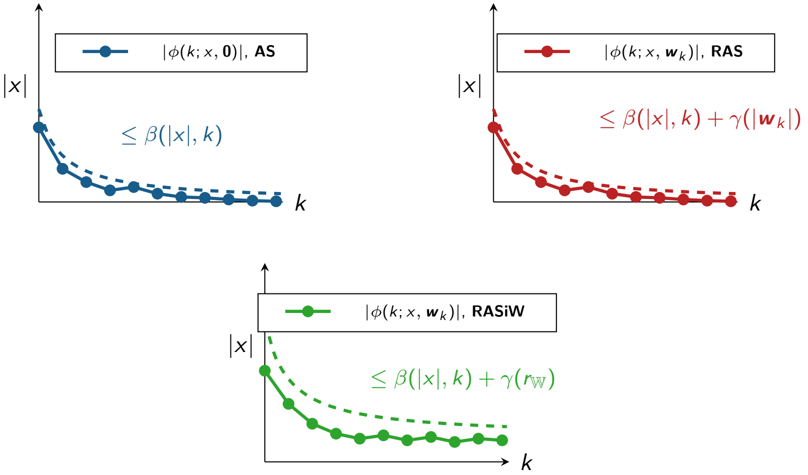






Asymptotic stability properties: $\|\boldsymbol{w}_k\| \to 0$

$\|\boldsymbol{w}_k\| \to 0$



MPC stability

Theorem 2 (Nominal MPC)

For every $\rho > 0$, there exists $\delta > 0$ such that for the set $\mathbb{W} \subseteq \{w \in \mathbb{R}^p : \|w\| \leq \delta\}$, the system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$, and the set $\mathcal{S} := \{x \in \mathbb{R}^n : V_N^0(x) \leq \rho\} \cap \mathcal{X}$ we have that:

1. *$\mathcal{S}$ is RPI (Allan, Bates, Risbeck, and Rawlings, 2017)*
2. *The origin is RAS in $\mathcal{S}$ (Allan et al., 2017)*
3. *The origin is RASiW in $\mathcal{S}$*
4. *The system is ℓ-RASiW in $\mathcal{S}$*

RMPC stability

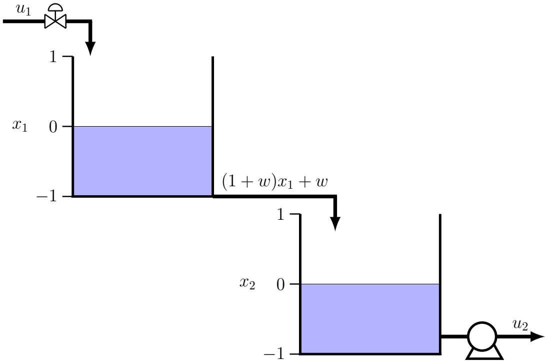
Theorem 3 (RMPC)

For the system $x^+ = f(x, \kappa^r(x), w)$, $w \in \mathbb{W}$, we have that:

1. $\mathcal{X}^r$ *is RPI*
2. *The origin is RASiW in* $\mathcal{X}^r$
3. *The system is* ℓ*-RASiW in* $\mathcal{X}^r$

Note: RMPC cannot handle arbitrarily large disturbances due to terminal constraints

Liquid level control (McAllister and Rawlings, 2022)



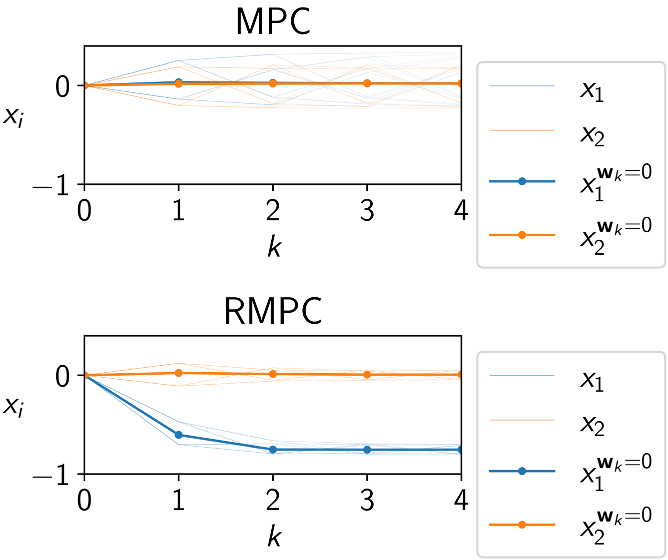
$$\frac{dx_1}{dt} = -(1+w)x_1 + u_1 - w$$
$$\frac{dx_2}{dt} = (1+w)x_1 - u_2 + w$$

Disturbance: $w = \pm 0.3$

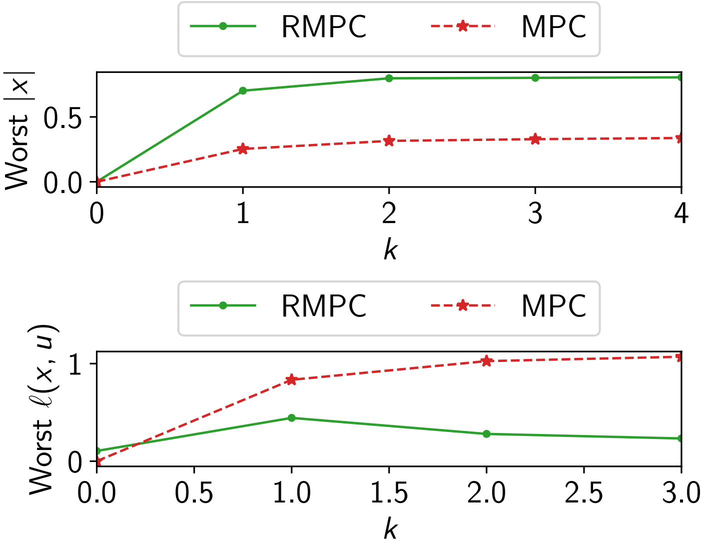
Stage cost: $\ell(x,u) = x'Qx + u'Ru, \quad Q = \mathrm{diag}([0.1, 20]), \; R = \mathrm{diag}([0.1, 0.1])$

Discrete time implementation:

Liquid level control example



Liquid level control example



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