technische universität
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process automation
systems

Monotonicity in propagating reachable sets

Scalable robust model predictive control for high dimensional systems

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Conceptual RMPC Problem

$$
\begin{aligned}
\underset{\mathbf{u},\mathbf{x}}{\text{minimize}} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k, d_k) + V_f(x_N) \\
\text{subject to:} \quad & x_{k+1} = f(x_k, u_k, d_k), x_0 = x_{\text{init}} \\
& 0 \geq g(x_k, u_k, d_k), \\
& 0 \geq g_f(x_N), \\
& k \in [0, N-1]
\end{aligned}
$$

1. Objective
2. Constraints
3. Performance
4. Stability

▶ Uncertainty entering in dynamics, constraints and cost

Two perspectives on the robust OCP

Open-loop min-max robust OCP (as in single shooting)

$$\begin{aligned} \min_{u} \max_{w \in \mathbb{W}} \; & \sum_{k=0}^{N-1} \ell(\tilde{x}_k(u,w), u_k) + V_{\mathrm{f}}(\tilde{x}_N(u,w)) \\ \text{s.t.} \; \max_{w \in \mathbb{W}} \; & h(\tilde{x}_k(u,w), u_k) \leq 0, \quad k = 0, \dots, N-1 \\ \max_{w \in \mathbb{W}} \; & r(\tilde{x}_N(u,w)) \leq 0 \end{aligned}$$

Set-based robust OCP

$$\begin{aligned} \min_{\mathbb{X}, \pi(\cdot)} \quad & \sum_{k=0}^{N-1} \mathcal{L}(\mathbb{X}_k, \pi_k(\cdot)) + \mathcal{L}_{\mathrm{f}}(\mathbb{X}_N) \\ \text{s.t.} \quad & \mathbb{X}_0 = \{\bar{x}_0\}, \\ & \mathbb{X}_{k+1} = \mathcal{F}(\mathbb{X}_k, \pi_k(\cdot)), \quad k = 0, \dots, N-1, \\ & 0 \geq h(x_k, \pi_k(\cdot)), \quad \forall x_k \in \mathbb{X}_k, \; k = 0, \dots, N-1, \\ & 0 \geq r(x_N), \quad \forall x_N \in \mathbb{X}_N. \end{aligned}$$

- Uncertainty gets convoluted through system function over multiple time steps
- Set-based robust OCP decouples the propagation
- Helpful to reduce complexity

Agenda

- Propagating reachable sets through nonlinear systems
  - General description
  - Simplifications/ Favorable Cases

- Monotonicity as favorable case

- Generalizing from there on

- When in doubt, try neural networks

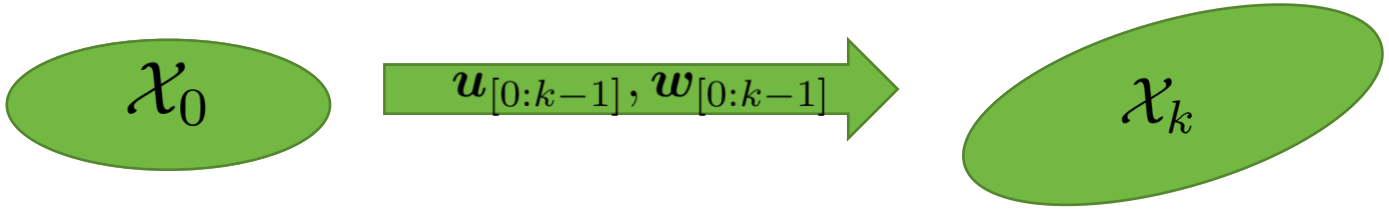
What is a reachable set?

- Consider systems

$$\dot{\boldsymbol{x}} = f_\mathrm{c}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)) \qquad \boldsymbol{x}_{k+1} = f_\mathrm{d}(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k)$$

- Consider fixed time discretization for continuous systems
- Given an initial set $\mathcal{X}_0$ and some inputs $\boldsymbol{u}_{[0:k-1]}$, the reachable set $\mathcal{X}_k$ at time t_k spans all the states, the system can reach

$$\mathcal{X}_k = \{\boldsymbol{x}(t_k, \boldsymbol{u}_{[0:k-1]}, \boldsymbol{w}_{[0:k-1]}, x_0) | \forall \boldsymbol{x}_0 \in \mathcal{X}_0, \forall \boldsymbol{w}_{[0:k-1]} \in \mathbb{W}^k\}$$



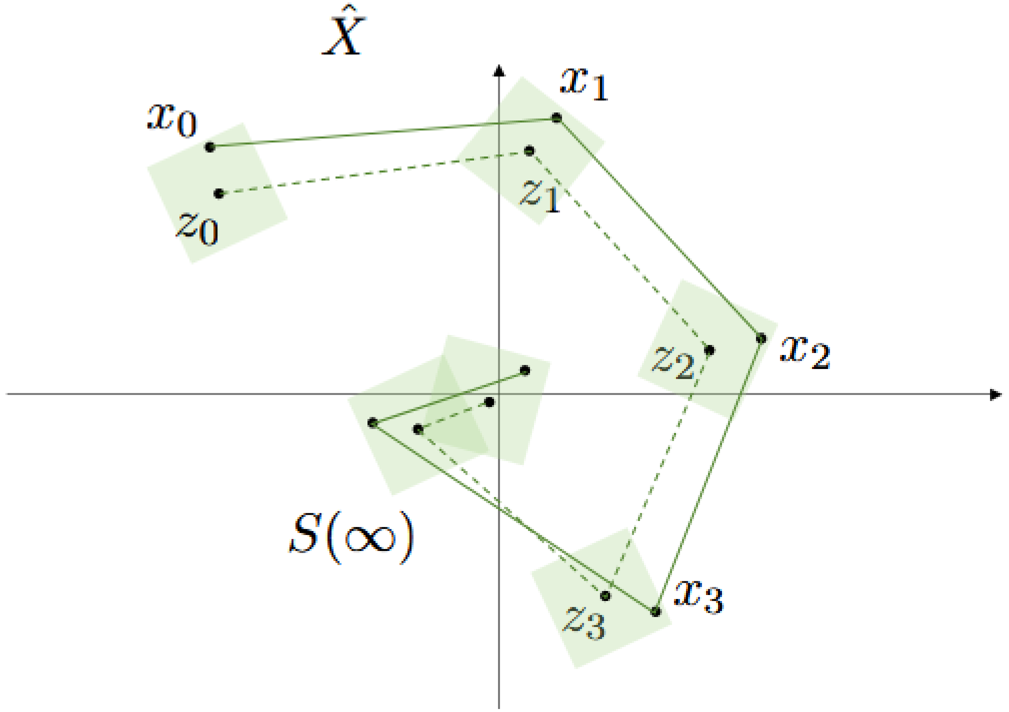
- Are important for verification and robust predictions

Tube-based model predictive control

- Reachable sets span the tube
- Linear system, additive uncertainty

→Tube dynamics independent from nominal trajectory

- More unfavorable cases require online computation of tubes or reachable sets



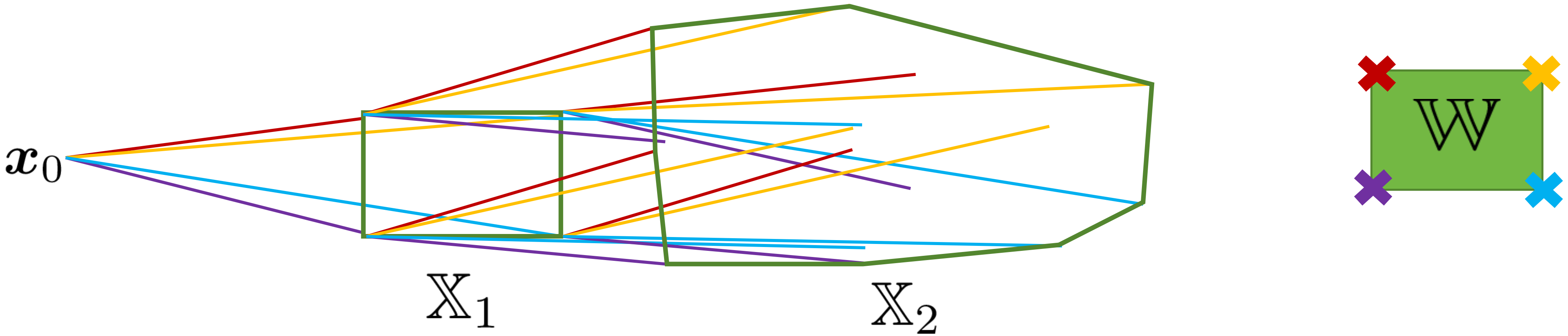
Computing exact reachable sets

- Exact reachability can be undecidable [1]
- Consider nonlinear system $x_{k+1} = x_k^2 + c,\ x_0 = 0,\ \forall c \in [-2, 0.25]$
- Given c and y,
  verifying $\exists k : y = x_k$ is hard
- Fun fact: the set of all complex c for which x_k remains small is the Mandelbrot set

- [1] N. Fijalkow, J. Ouaknine, A. Pouly, J. Sousa-Pinto, and J. Worrell, “On the decidability of reachability in linear time-invariant systems,” in *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control*, in HSCC ’19. New York, NY, USA: Association for Computing Machinery, Apr. 2019, pp. 77–86. doi: 10.1145/3302504.3311796.

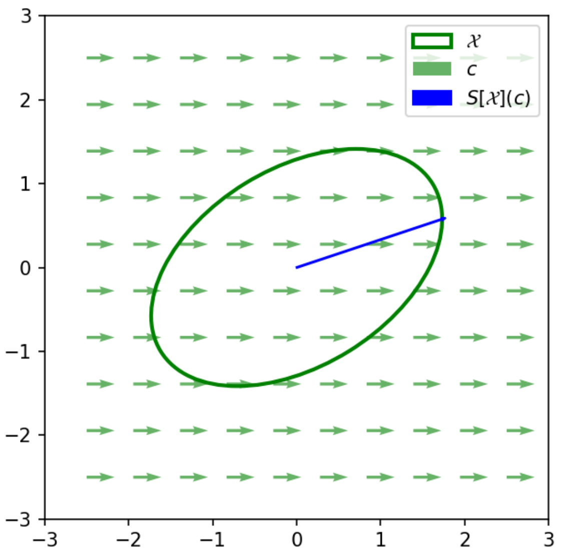
Approximations of reachable sets via scenarios

- In the linear case, outer approximation via scenario tree
- Consider all vertices of uncertainty set $\mathbb{W}$ as branches



- Reachable set encompassed by convex hull of all scenarios
- Number of scenarios grows exponentially
- No guarantees in the nonlinear case for non-discrete $\mathbb{W}$

Support function for general shapes



- Introduced by Villanueva[1]
- General convex set, utilizes support function

$$\forall \boldsymbol{c} \in \mathbb{R}^n, S\left[\mathcal{X}\right] := \max_{\boldsymbol{x}} \{\boldsymbol{c}^\mathsf{T}\boldsymbol{x} | \boldsymbol{x} \in \mathcal{X}\}$$

- Always points to the most outward point $\boldsymbol{x} \in \mathcal{X}$ in direction $\boldsymbol{c}$
- Set inclusion

$$S\left[\mathbb{X}(0)\right](c) \geq \max_{\boldsymbol{x}_0 \in \mathcal{X}_0} \boldsymbol{c}^\mathsf{T}\boldsymbol{x}_0\ , \ \forall \boldsymbol{c} \in \mathbb{R}^{n_x}$$

$$\Rightarrow \mathcal{X}_0 \subseteq \mathbb{X}(0)$$

[1] M. E. Villanueva, B. Houska, and B. Chachuat, "Unified framework for the propagation of continuous-time enclosures for parametric nonlinear ODEs," *J Glob Optim*, vol. 62, no. 3, pp. 575–613, Jul. 2015, doi: 10.1007/s10898-014-0235-6.

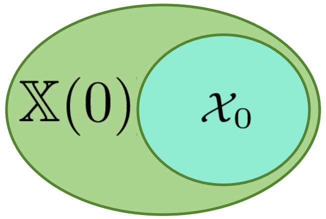
Generalized differential inequalities (GDIs)

- Support function $\forall \boldsymbol{c} \in \mathbb{R}^n, S\left[\mathcal{X}\right] := \max_{\boldsymbol{x}}\{\boldsymbol{c}^\intercal \boldsymbol{x} | \boldsymbol{x} \in \mathcal{X}\}$
- The set $\mathbb{X}(t, \boldsymbol{u})$ describes the evolution of a convex over-approximation of the reachable set via $S\left[\mathbb{X}(t, \boldsymbol{u})\right](c), \ \forall \boldsymbol{c} \in \mathbb{R}^{n_x}$, if:

$$S\left[\mathbb{X}(0)\right](c) \geq \max_{\boldsymbol{x}_0 \in \mathcal{X}_0} \boldsymbol{c}^\intercal \boldsymbol{x}_0 \qquad t = 0$$

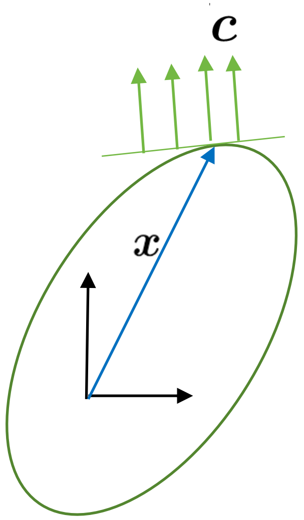
$$\dot{S}\left[\mathbb{X}(t, \boldsymbol{u})\right](c) \geq \max_{\boldsymbol{x} \in \mathbb{X}(t), \boldsymbol{c}^\intercal \boldsymbol{x} = S[\mathbb{X}(t, \boldsymbol{u})](c), \boldsymbol{w} \in \mathbb{W}} \boldsymbol{c}^\intercal f_{\mathrm{c}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w}) \qquad t > 0$$

Initial set



Change of boundary of $\mathbb{X}(t, \boldsymbol{u})$

Maximum change of system at the furthest point wrt. $\boldsymbol{c}$



- Then $\mathcal{X}(t, \boldsymbol{u}) \subseteq \mathbb{X}(t, \boldsymbol{u})$

M. E. Villanueva, B. Houska, and B. Chachuat, "Unified framework for the propagation of continuous-time enclosures for parametric nonlinear ODEs," *J Glob Optim*, vol. 62, no. 3, pp. 575–613, Jul. 2015, doi: 10.1007/s10898-014-0235-6.

Generalized difference inequalities - Discrete

- The sequence of sets $\mathbb{X}_k$ describes a convex over-approximation of the reachable sets via

$$S\left[\mathbb{X}_{k+1}(\boldsymbol{u}_k)\right](c) \geq \max_{\boldsymbol{x}\in\mathbb{X}_k,\boldsymbol{w}\in\mathbb{W}} \boldsymbol{c}^\mathsf{T} f_\mathrm{d}(\boldsymbol{x},\boldsymbol{u}_k,\boldsymbol{w})\ ,\ \forall \boldsymbol{c}\in\mathbb{R}^{n_x}$$

$$S\left[\mathbb{X}_0\right](c) \geq \max_{\boldsymbol{x}\in\mathcal{X}_0} \boldsymbol{c}^\mathsf{T}\boldsymbol{x},\ \forall \boldsymbol{c}\in\mathbb{R}^{n_x}$$

- The border of $\mathbb{X}_{k+1}$ must be larger than the maximum next state starting from $\mathbb{X}_k$ in every direction
- Then $\mathcal{X}_k \subseteq \mathbb{X}_k$

[1] M. E. Villanueva, B. Houska, and B. Chachuat, "Unified framework for the propagation of continuous-time enclosures for parametric nonlinear ODEs," *J Glob Optim*, vol. 62, no. 3, pp. 575–613, Jul. 2015, doi: 10.1007/s10898-014-0235-6.

Challenges of GDIs in predictive control

$$S\left[\mathbb{X}_{k+1}(\boldsymbol{u}_k)\right](c) \geq \max_{\boldsymbol{x}\in\mathbb{X}_k,\boldsymbol{w}\in\mathbb{W}} \boldsymbol{c}^\intercal f_\mathrm{d}(\boldsymbol{x},\boldsymbol{u}_k,\boldsymbol{w}) \quad \forall \boldsymbol{c} \in \mathbb{R}^n$$

- Max operator leads to bilevel optimization problem
- Difficult to solve in gradient based frameworks

- Open-loop propagation of uncertainty
- Optimizing over policies changes dynamics

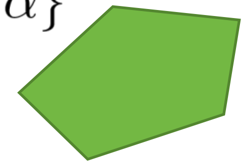
- Inequality must hold in all directions $\boldsymbol{c} \in \mathbb{R}^n$
- Infinite dimensional constraint
- Maximization in support function

- Find special cases for which complexity can be reduced

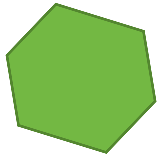
Fixed parameterizations of reachable sets

- Convex parameterizations
  - Intervals/Rectangles $\mathbb{X}^{\text{Int}} = \{\boldsymbol{x} | \boldsymbol{x}^{\text{min}} \leq \boldsymbol{x} \leq \boldsymbol{x}^{\text{max}}\}$
    - Only $2n_x$ parameters
    - Can only represent scaling and translation

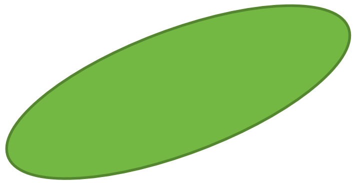
  - Polytopes $\mathbb{X}^{\text{Poly}} = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{V}\boldsymbol{x} \leq \alpha\}$
    - Generalizing rectangles
    - Arbitrary number of faces



  - Zonotopes $\mathbb{X}^{\text{zono}} = \{\boldsymbol{x} \in \mathbb{R}^{n_x} | \alpha^- \leq \boldsymbol{V}\boldsymbol{x} \leq \alpha^+\}$
    - Polytopes with parallel faces

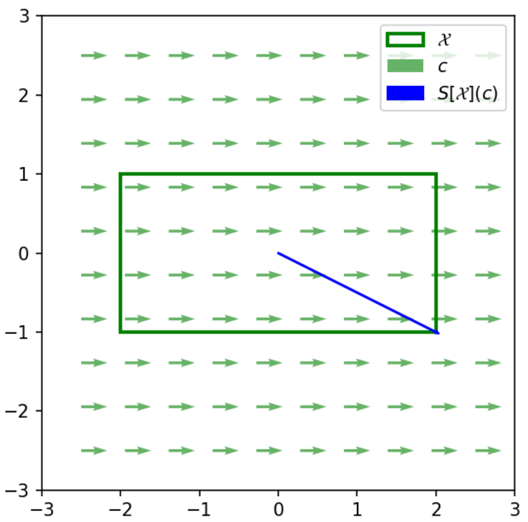


  - Ellipsoids $\mathbb{X}^{\text{Ell}} = \{\boldsymbol{x}_{\text{center},k} + \boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{v} | \boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{v}^\top \boldsymbol{v} \leq 1\}$



- How can reachable sets then be calculated?

Differential and Difference Inequalties



- To obtain the difference inequality for intervals, use[1]

$$S\left[\mathbb{X}^{\mathrm{Int}}\right](c) = \frac{1}{2}\boldsymbol{c}^{\intercal}(\boldsymbol{x}^{\min} + \boldsymbol{x}^{\max}) + \frac{1}{2}\mathrm{abs}(\boldsymbol{c})^{\intercal}(\boldsymbol{x}^{\max} - \boldsymbol{x}^{\min})$$

- Discrete case: Difference inequality for every state ***i***

$$\boldsymbol{x}_{k+1,i}^{\min} \leq \min_{\xi \in [\boldsymbol{x}_k^{\min}, \boldsymbol{x}_k^{\max}], \boldsymbol{w} \in \mathbb{W}} \boldsymbol{f}_{\mathrm{d},i}(\xi, \boldsymbol{u}_k, \boldsymbol{w}) \qquad \boldsymbol{x}_{k+1,i}^{\max} \geq \max_{\xi \in [\boldsymbol{x}_k^{\min}, \boldsymbol{x}_k^{\max}], \boldsymbol{w} \in \mathbb{W}} \boldsymbol{f}_{\mathrm{d},i}(\xi, \boldsymbol{u}_k, \boldsymbol{w})$$

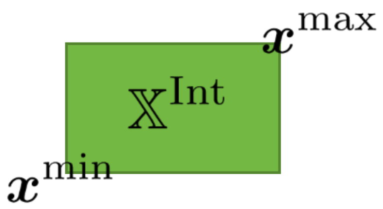
- Continuous case: Differential inequality

$$\dot{\boldsymbol{x}}_i^{\min} \leq \min_{\xi \in [\boldsymbol{x}^{\min}, \boldsymbol{x}^{\max}], \xi_i = \boldsymbol{x}_i^{\min}, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{f}_{\mathrm{c},i}(\xi, \boldsymbol{u}, \boldsymbol{w}) \qquad \dot{\boldsymbol{x}}_i^{\max} \geq \max_{\xi \in [\boldsymbol{x}^{\min}, \boldsymbol{x}^{\max}], \xi_i = \boldsymbol{x}_i^{\max}, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{f}_{\mathrm{c},i}(\xi, \boldsymbol{u}, \boldsymbol{w})$$

- In the continuous case only growth of the border considered

[1] M. E. Villanueva, B. Houska, and B. Chachuat, "Unified framework for the propagation of continuous-time enclosures for parametric nonlinear ODEs," *J Glob Optim*, vol. 62, no. 3, pp. 575–613, Jul. 2015, doi: 10.1007/s10898-014-0235-6.

Interval reachable sets



- Simple set $\mathbb{X}^{\mathrm{Int}} = \{\boldsymbol{x} | \boldsymbol{x}^{\min} \leq \boldsymbol{x} \leq \boldsymbol{x}^{\max}\}$
- Reachable set overapproximated by

$$\boldsymbol{x}^{\min}(t_k, \boldsymbol{u}_{[0:k-1]}, \mathbb{X}_0^{\mathrm{Int}}, \mathbb{W}) \leq \boldsymbol{x}(t_k, \boldsymbol{u}_{[0:k-1]}, \boldsymbol{w}_{[0:k-1]}, \boldsymbol{x}_0)$$
$$\leq \boldsymbol{x}^{\max}(t_k, \boldsymbol{u}_{[0:k-1]}, \mathbb{X}_0^{\mathrm{Int}}, \mathbb{W}), \forall \boldsymbol{w}_{[0:k-1]} \in \mathbb{W}^k, \forall \boldsymbol{x}_0 \in \mathbb{X}_0^{\mathrm{Int}}$$

With $\mathcal{X}_k \subseteq \mathbb{X}_k^{\mathrm{Int}}$ and $\mathcal{X}_0 \subseteq \mathbb{X}_0^{\mathrm{Int}}$

- How to obtain $\boldsymbol{x}^{\min}(t_k, \boldsymbol{u}_{[0:k-1]}, \mathbb{X}_0^{\mathrm{Int}}, \mathbb{W})$, $\boldsymbol{x}^{\max}(t_k, \boldsymbol{u}_{[0:k-1]}, \mathbb{X}_0^{\mathrm{Int}}, \mathbb{W})$?

Monotonicity for difference inequalities

- Just consider discrete case $\boldsymbol{x}_{k+1} = f_\mathrm{d}(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k)$
- The system is monotone in states $\boldsymbol{x} \in \mathbb{X}$ and uncertainty $\boldsymbol{w} \in \mathbb{W}$ if for $\hat{\boldsymbol{x}} \geq \tilde{\boldsymbol{x}}$

$$f_\mathrm{d}(\hat{\boldsymbol{x}}, \boldsymbol{u}, \boldsymbol{w}) \geq f_\mathrm{d}(\tilde{\boldsymbol{x}}, \boldsymbol{u}, \boldsymbol{w}), \quad \forall \boldsymbol{u} \in \mathbb{U},\ \forall \boldsymbol{w} \in \mathbb{W},$$

  and for $\hat{\boldsymbol{w}} \geq \tilde{\boldsymbol{w}}$

$$f_\mathrm{d}(\boldsymbol{x}, \boldsymbol{u}, \hat{\boldsymbol{w}}) \geq f_\mathrm{d}(\boldsymbol{x}, \boldsymbol{u}, \tilde{\boldsymbol{w}}), \quad \forall \boldsymbol{u} \in \mathbb{U},\ \forall \boldsymbol{x} \in \mathbb{X},$$

- Monotonicity can be shown by signs of Jacobian
  - All elements positive for discrete system
  - Off-diagonal positive for continuous system
- Temperature control, epidemic models, double integrator

D. Angeli und E. D. Sontag, „Monotone control systems“, *IEEE Trans. Automat. Contr.*, Bd. 48, Nr. 10, S. 1684–1698, Okt. 2003, doi: 10.1109/TAC.2003.817920.

Monotonicity for difference inequalities

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$$f_{\mathrm{d}}(\hat{\boldsymbol{x}}, \boldsymbol{u}, \boldsymbol{w}) \geq f_{\mathrm{d}}(\tilde{\boldsymbol{x}}, \boldsymbol{u}, \boldsymbol{w}), \quad \forall \boldsymbol{u} \in \mathbb{U},\ \forall \boldsymbol{w} \in \mathbb{W},$$

  and for $\hat{\boldsymbol{w}} \geq \tilde{\boldsymbol{w}}$

$$f_{\mathrm{d}}(\boldsymbol{x}, \boldsymbol{u}, \hat{\boldsymbol{w}}) \geq f_{\mathrm{d}}(\boldsymbol{x}, \boldsymbol{u}, \tilde{\boldsymbol{w}}), \quad \forall \boldsymbol{u} \in \mathbb{U},\ \forall \boldsymbol{x} \in \mathbb{X},$$

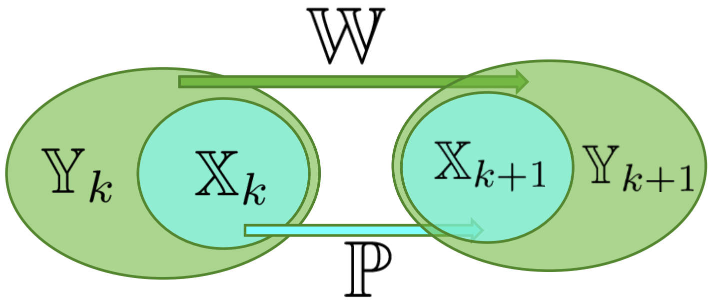
- Solution to difference inequality is trivial for $\boldsymbol{w} \in \mathbb{W} = [\boldsymbol{w}^{\min}, \boldsymbol{w}^{\max}]$

$$\boldsymbol{x}_{k+1,i}^{\max} \geq \max_{\xi \in [\boldsymbol{x}_k^{\min}, \boldsymbol{x}_k^{\max}], \boldsymbol{w} \in \mathbb{W}} \boldsymbol{f}_{\mathrm{d},i}(\xi, \boldsymbol{u}_k, \boldsymbol{w}) = \boldsymbol{f}_{\mathrm{d},i}(\boldsymbol{x}_k^{\max}, \boldsymbol{u}_k, \boldsymbol{w}^{\max})$$

$$\boldsymbol{x}_{k+1,i}^{\min} \leq \min_{\xi \in [\boldsymbol{x}_k^{\min}, \boldsymbol{x}_k^{\max}], \boldsymbol{w} \in \mathbb{W}} \boldsymbol{f}_{\mathrm{d},i}(\xi, \boldsymbol{u}_k, \boldsymbol{w}) = \boldsymbol{f}_{\mathrm{d},i}(\boldsymbol{x}_k^{\min}, \boldsymbol{u}_k, \boldsymbol{w}^{\min})$$

Monotonicity is important – General sets

- We have the sets $\mathbb{X}_k$ and $\mathbb{Y}_k$ with $\mathbb{X}_k \subseteq \mathbb{Y}_k$ and $\mathbb{P}$ and $\mathbb{W}$ with $\mathbb{P} \subseteq \mathbb{W}$
- Note that $S_{\mathbb{P}}\left[\mathbb{X}_k\right](\boldsymbol{c}) \leq S_{\mathbb{W}}\left[\mathbb{Y}_k\right](\boldsymbol{c}),\ \forall \boldsymbol{c} \in \mathbb{R}^{n_x}$
- Then $S_{\mathbb{P}}\left[\mathbb{X}_{k+1}\right](\boldsymbol{c}) \leq S_{\mathbb{W}}\left[\mathbb{Y}_{k+1}\right](\boldsymbol{c}),\ \forall \boldsymbol{c} \in \mathbb{R}^{n_x}$
- Because of the max operator, a larger set can only lead to an increase in



$$S\left[\mathbb{X}_{k+1}(\boldsymbol{u}_k)\right](c) \geq \max_{\boldsymbol{x} \in \mathbb{X}_k, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{c}^{\mathsf{T}} f_{\mathrm{d}}(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w})$$

- The notion of monotonicity is intertwined with the description of reachable sets

Robust MPC for monotone systems

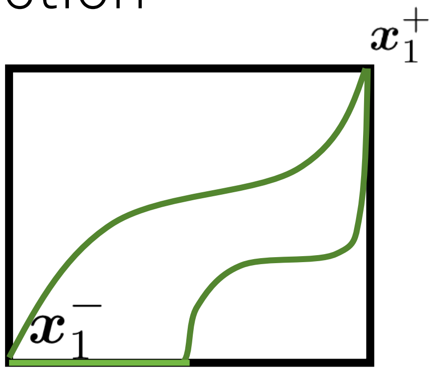
For discrete systems

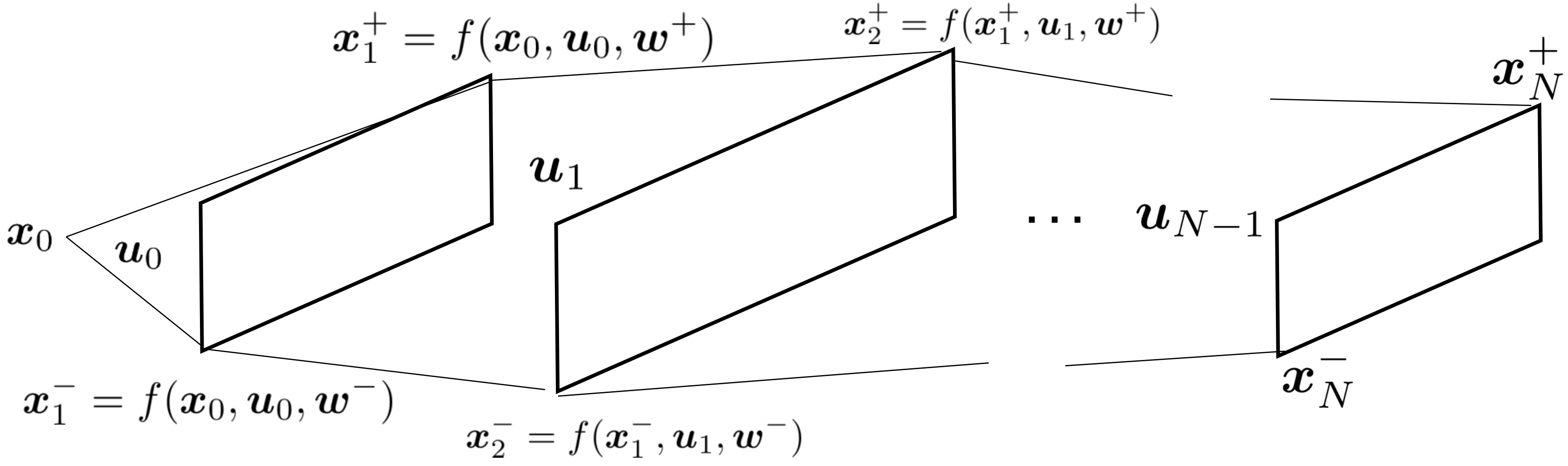
Can be easily extended for continuous systems

Follows [Heinlein, Subramanian, Molnar, and Lucia, „Robust MPC approaches for monotone systems“]

Reachable set propagation

- Monotonicity solves two of the three challenges in GDI
  - Interval sets can be calculated by evaluating the system function
  - No bilevel problem
  - No over-approximation error
- Directly applicable for open-loop robust MPC





Open-loop robust monotone MPC approach

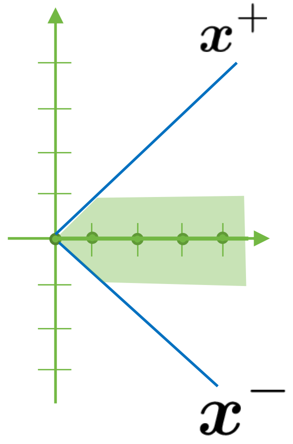
$$
\begin{aligned}
\min_{\boldsymbol{x}^{\pm}_{[0:N]},\boldsymbol{u}_{[0:N-1]}} \quad & J(\boldsymbol{x}^{\pm}_{[0:N]},\boldsymbol{u}_{[0:N-1]}) \\
\text{s.t} : \boldsymbol{x}^{\pm}_0 &= \boldsymbol{x}_0, \\
\boldsymbol{x}^{+}_{k+1} &= f(\boldsymbol{x}^{+}_k,\boldsymbol{u}_k,\boldsymbol{w}^{+}), \\
\boldsymbol{x}^{-}_{k+1} &= f(\boldsymbol{x}^{-}_k,\boldsymbol{u}_k,\boldsymbol{w}^{-}), \\
[\boldsymbol{x}^{-}_k,\boldsymbol{x}^{+}_k] &\subseteq \mathbb{X}, \\
\boldsymbol{u}_k &\in \mathbb{U}, \\
[\boldsymbol{x}^{-}_N,\boldsymbol{x}^{+}_N] &\subseteq \mathbb{X}_f,
\end{aligned}
$$

- Two uncertainty scenarios: $\boldsymbol{w}^{+}$and $\boldsymbol{w}^{-}$
- For box constraints no conservatism due to intervals
- One input for all uncertainty realizations

Problem with open-loop predictions

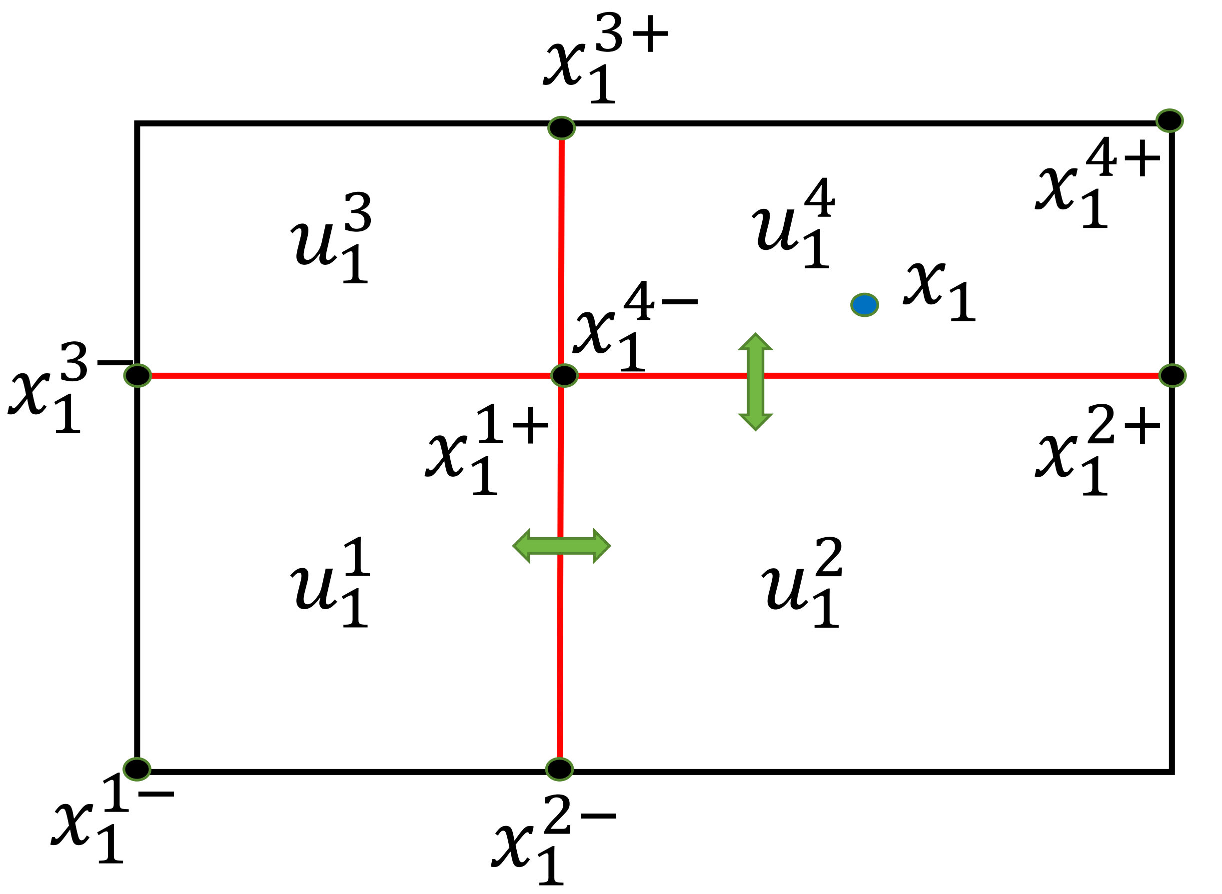
- Assume the system $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{u}_k + \boldsymbol{w}_k$, with $\boldsymbol{w}_k \in [-1, 1]$
- Regardless of chosen input $\boldsymbol{u}_k$, reachable sets will blow up
- MPC is no open-loop method → **Feedback**
- Include the information from the next measurement in the prediction (**Recourse**)
- Possible feedback policy here

  $\boldsymbol{u}_k = -\boldsymbol{x}_k$
- Feedback policy changes dynamics
  - Monotonicity may be lost $\boldsymbol{u} = -1.1\boldsymbol{x}$
  - May lead to tightening of input constraints



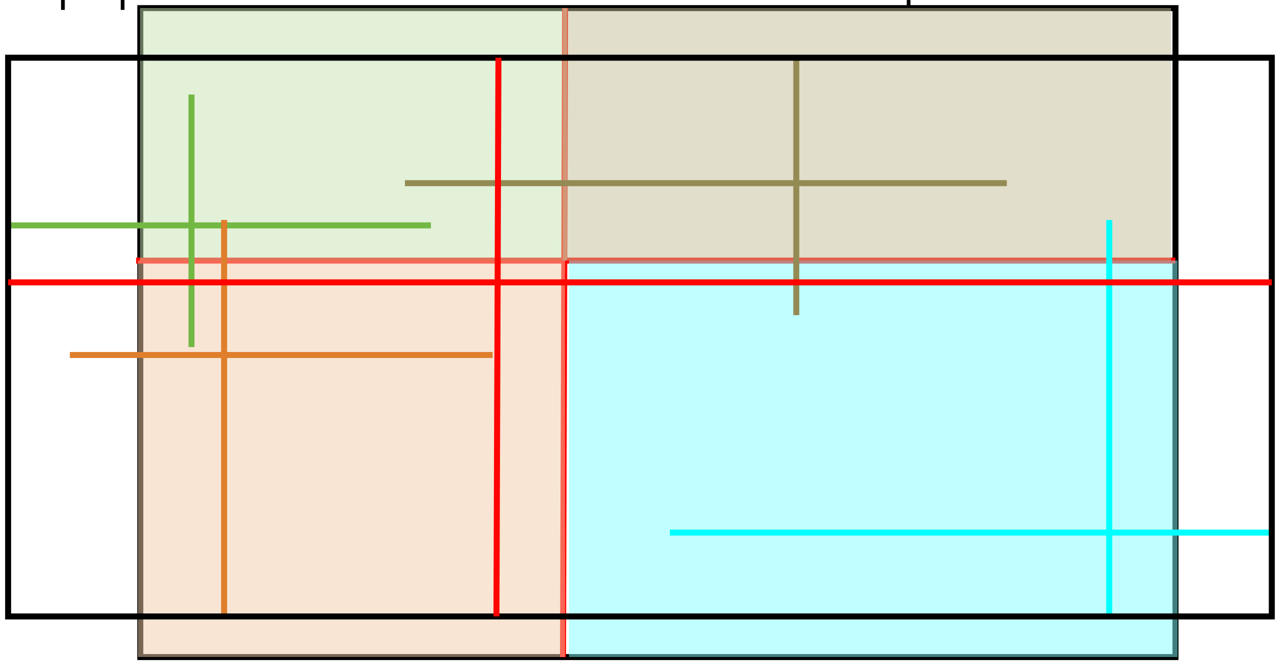
Dynamics preserving feedback policy

- Inspired by multi-stage MPC
- Reachable set after 1 time step
- Scenarios based on state after uncertainty realization
- Division of the reachable set -> Multiple subsets
- Positioning of partition a degree of freedom
- For each subset individual input
- Each subset is spanned by two points $[x_1^{s-}, x_1^{s+}]$



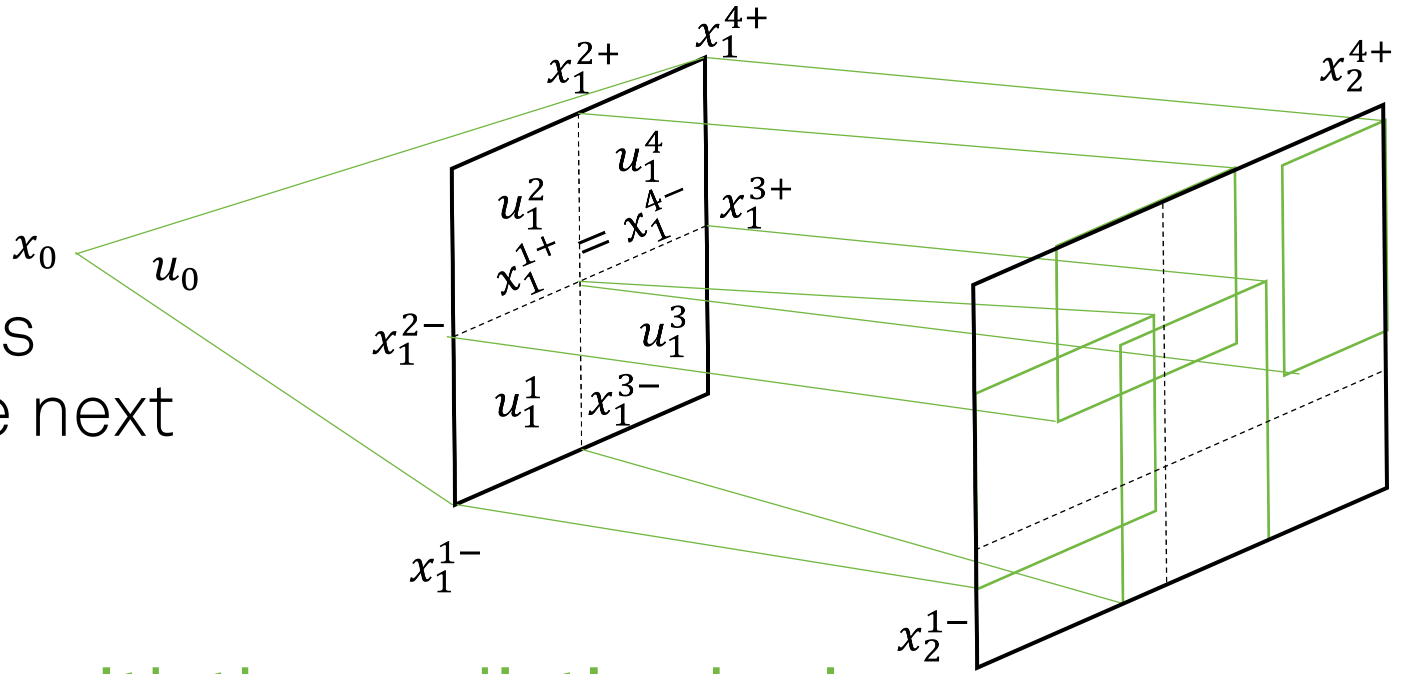
Multiple time steps

- Scenario tree possible: Dividing the propagation of each subset
- Exponential growth of scenarios with prediction horizon
- Or continue with open-loop prediction after few steps
- Idea:
  - Bound the propagation of all the subsets
  - Divide the bounding rectangle as before
  - Piecewise constant feedback policy



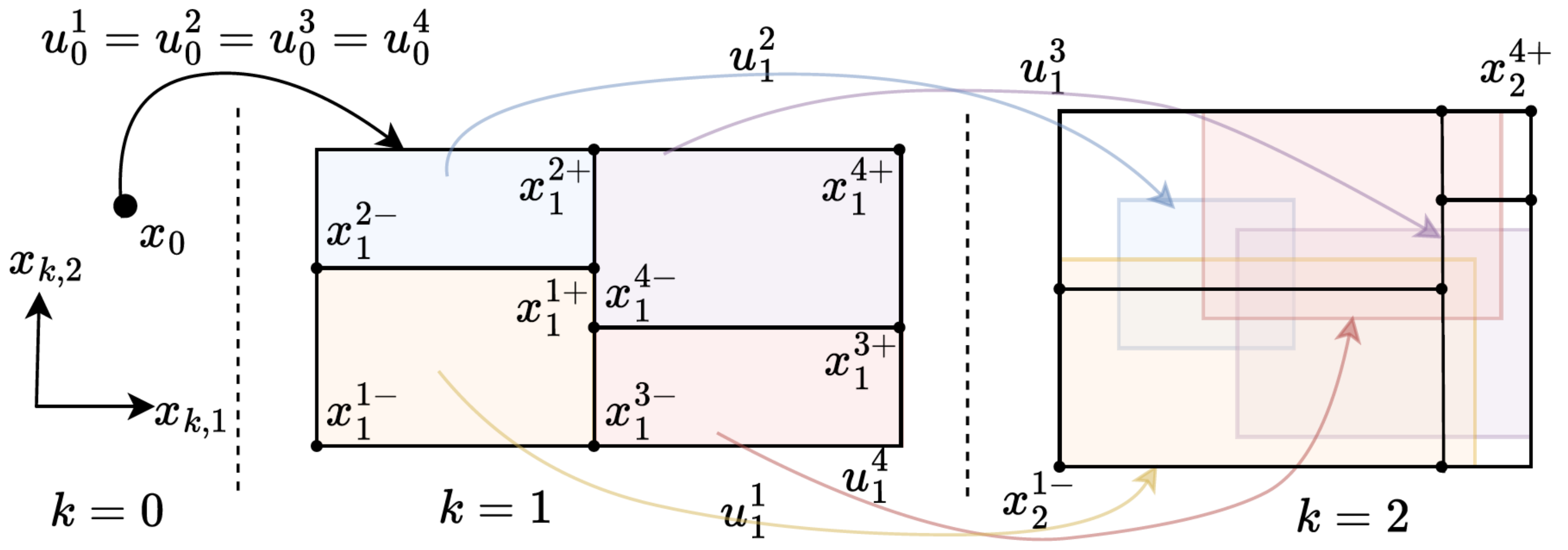
Dividing and bounding

- In each time step start with the reachble set
- Divide into subregions
- Propagate subregions
  - Maximum realization
  - Minimum realization
- Bound the propagations
  = Reachable set for the next
  time step



▶ No exponential growth with the prediction horizon

How are partitions implemented?



$$
\begin{array}{llll}
x_{k,2}^{1-} = x_{k,2}^{3-} & x_{k,2}^{4+} = x_{k,2}^{2+} & x_{k,1}^{1-} = x_{k,1}^{2-} & x_{k,2}^{1+} = x_{k,2}^{2-}, \\
x_{k,1}^{1+} = x_{k,1}^{2+} & x_{k,1}^{1+} = x_{k,1}^{3-} & x_{k,1}^{3-} = x_{k,1}^{4-} & x_{k,2}^{3+} = x_{k,2}^{4-}, \\
x_{k,1}^{3+} = x_{k,1}^{4+} & x_k^{s\pm} \geq x_k^{1-} & x_k^{4+} \geq x_k^{s\pm}, & \forall s \in \mathbb{S}
\end{array}
$$

- Partitions defined by linear inequality and equality constraints

Closed-loop robust monotone MPC approach

- μ_s number of subregions each time step
- Defines the bounding of the propagation
- $h(x_k^{[1:\mu_s]\pm})$ orders the division of subregions
- Recursive feasibility and constraint satisfaction proven for box constraints

$$\min_{\boldsymbol{x}_{[0:N]}^{s\pm}, \boldsymbol{u}_{[0:N-1]}^{s}, \forall s \in \mathbb{S}} \quad J(\boldsymbol{x}_{[0:N]}^{[1:\mu_s]\pm}, \boldsymbol{u}_{[0:N-1]}^{[1:\mu_s]})$$

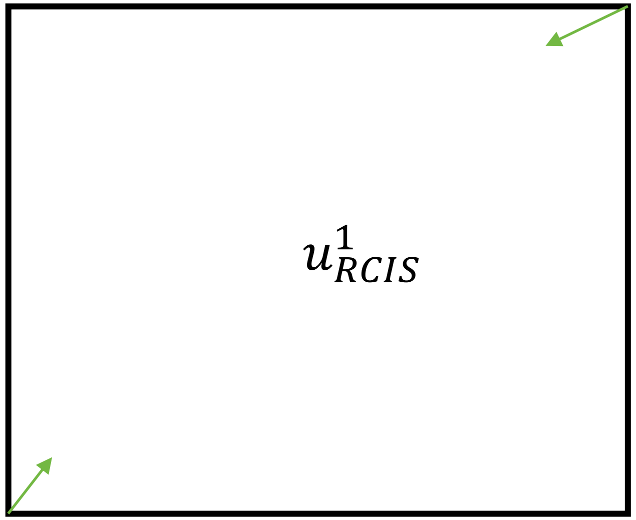
$$\begin{aligned}
\text{s.t}: \boldsymbol{x}_0^{s\pm} &= \boldsymbol{x}_0, \\
\boldsymbol{x}_{k+1}^{\mu_s+} &\geq f(\boldsymbol{x}_k^{s+}, \boldsymbol{u}_k^s, \boldsymbol{w}^+), \\
\boldsymbol{x}_{k+1}^{1-} &\leq f(\boldsymbol{x}_k^{s-}, \boldsymbol{u}_k^s, \boldsymbol{w}^-), \\
\left[\boldsymbol{x}_k^{1-}, \boldsymbol{x}_k^{\mu_s+}\right] &\subseteq \mathbb{X}, \\
\boldsymbol{u}_k^s &\in \mathbb{U}, \\
\left[\boldsymbol{x}_N^{1-}, \boldsymbol{x}_N^{\mu_s+}\right] &\subseteq \mathbb{X}_f, \\
\boldsymbol{u}_0^1 &= \boldsymbol{u}_0^s, \\
\boldsymbol{h}(\boldsymbol{x}_k^{[1:\mu_s]\pm}) &\leq \boldsymbol{0},
\end{aligned}$$

M. Heinlein, S. Subramanian, M. Molnar, and S. Lucia, "Robust MPC approaches for monotone systems*," in 2022 IEEE 61st Conf. Decis. and Control (CDC), Dec. 2022, pp. 2354–2360. doi: 10.1109/CDC51059.2022.9992502.

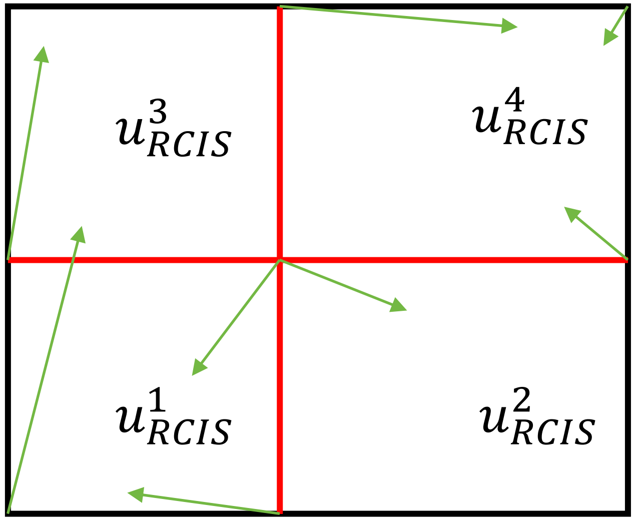
Robust control invariant sets

- Recursive feasibility needs robust control invariant set (RCIS)
- A set, in which **the method** will find an input to remain in the set regardless of the uncertainty

Open-loop



Closed-loop



Calculation of robust control invariant set

- This can be formulated as an optimization problem
- $V\left(x_i^{\mu_s +}, x_i^{1-}\right)$ as measure of the RCIS size (e.g. volume)
- $h(x^{[1:\mu_s]\pm})$ orders subregions
- RCIS property is enforced
- Can be added as a constraint in the MPC problem for more flexibility

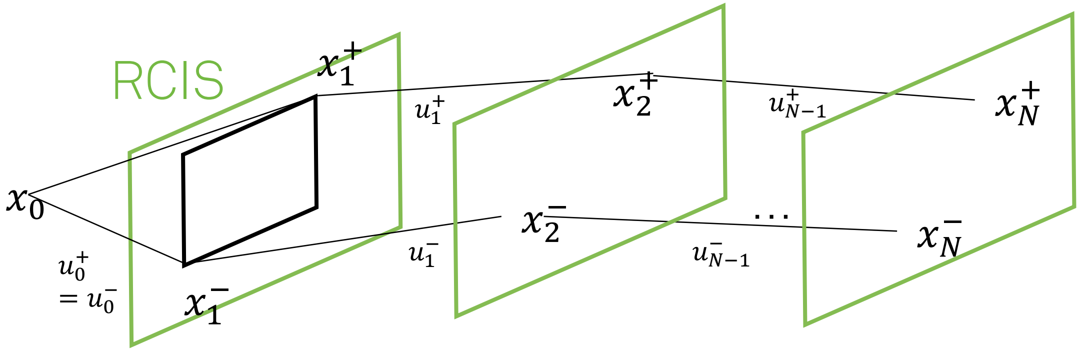
$$\max_{\boldsymbol{x}^{s+}, \boldsymbol{x}^{s-}, \boldsymbol{u}^s,\ \forall s \in \mathbb{S}} \quad V(\boldsymbol{x}_i^{\mu_s^+}, \boldsymbol{x}_i^{1-})$$

$$\begin{aligned} \text{s.t} : \ & \boldsymbol{x}^{s\pm} \in \mathbb{X},\ \forall s \in \mathbb{S}, \\ & \boldsymbol{u}^s \in \mathbb{U},\ \forall s \in \mathbb{S}, \\ & \boldsymbol{h}(\boldsymbol{x}_k^{[1:\mu_s]\pm}) \leq \boldsymbol{0}, \\ & \boldsymbol{x}^{\mu_s +} \geq f(\boldsymbol{x}^{s\pm}, \boldsymbol{u}^s, \boldsymbol{w}^\pm) \geq \boldsymbol{x}^{1-},\ \forall s \in \mathbb{S}. \end{aligned}$$

Robust control invariant set as safety filter

- For monotone systems RCI sets easy to calculate
- When in RCI set, use monotonicity to check if input safe
- Can be implemented as constraint
- Can be combined with nominal or approximate MPC
- Use RCI input as fallback

J. Adamek, M. Heinlein, L. Lüken, and S. Lucia, “Deterministic Safety Guarantees for Learning-Based Control of Monotone Nonlinear Systems Under Uncertainty,” *IEEE Control Systems Letters*, vol. 8, pp. 1030–1035, 2024, doi: 10.1109/LCSYS.2024.3407635.



$$\min_{\boldsymbol{x}^+_{[0:N]},\boldsymbol{x}^-_{[0:N]},\boldsymbol{u}^+_{[0:N-1]},\boldsymbol{u}^-_{[0:N-1]}} \sum_{k=0}^{N-1} \left(\ell(\boldsymbol{x}_k^+,\boldsymbol{u}_k^+) + \ell(\boldsymbol{x}_k^-,\boldsymbol{u}_k^-)\right) + V_f(\boldsymbol{x}_N^+) + V_f(\boldsymbol{x}_N^-) \tag{1a}$$

$$\text{s.t}: \quad \boldsymbol{x}_0^\pm = \boldsymbol{x}_0, \tag{1b}$$

$$\boldsymbol{x}_{k+1}^\pm = f(\boldsymbol{x}_k^\pm, \boldsymbol{u}_k^\pm, \boldsymbol{w}^\pm),\ \forall k \in \{0,\dots,N-1\} \tag{1c}$$

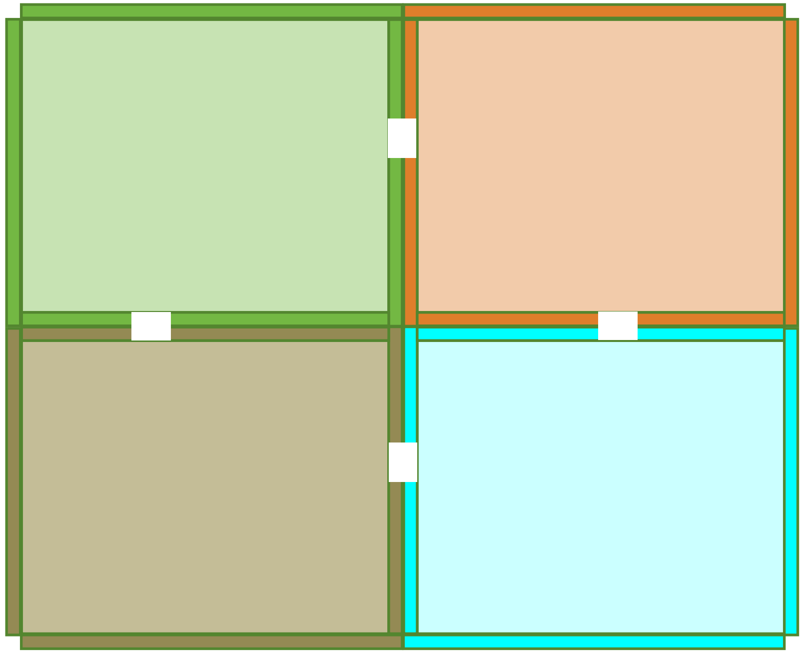
$$\boldsymbol{x}_k^\pm \in \mathbb{X}_{\text{RCIS}},\ k \in \{1,\dots,N\} \tag{1d}$$

$$\boldsymbol{u}_k^\pm \in \mathbb{U},\ \forall k \in \{0,\dots,N-1\}, \tag{1e}$$

$$\boldsymbol{u}_0^+ = \boldsymbol{u}_0^-. \tag{1f}$$

Case study

- Temperature control in a building
- 4 rooms in rectangle
- Each room 3 states: interior wall, exterior wall, interior temperature
- 2 inputs: heating and cooling
- Additional exchange between adjacent rooms



Building model

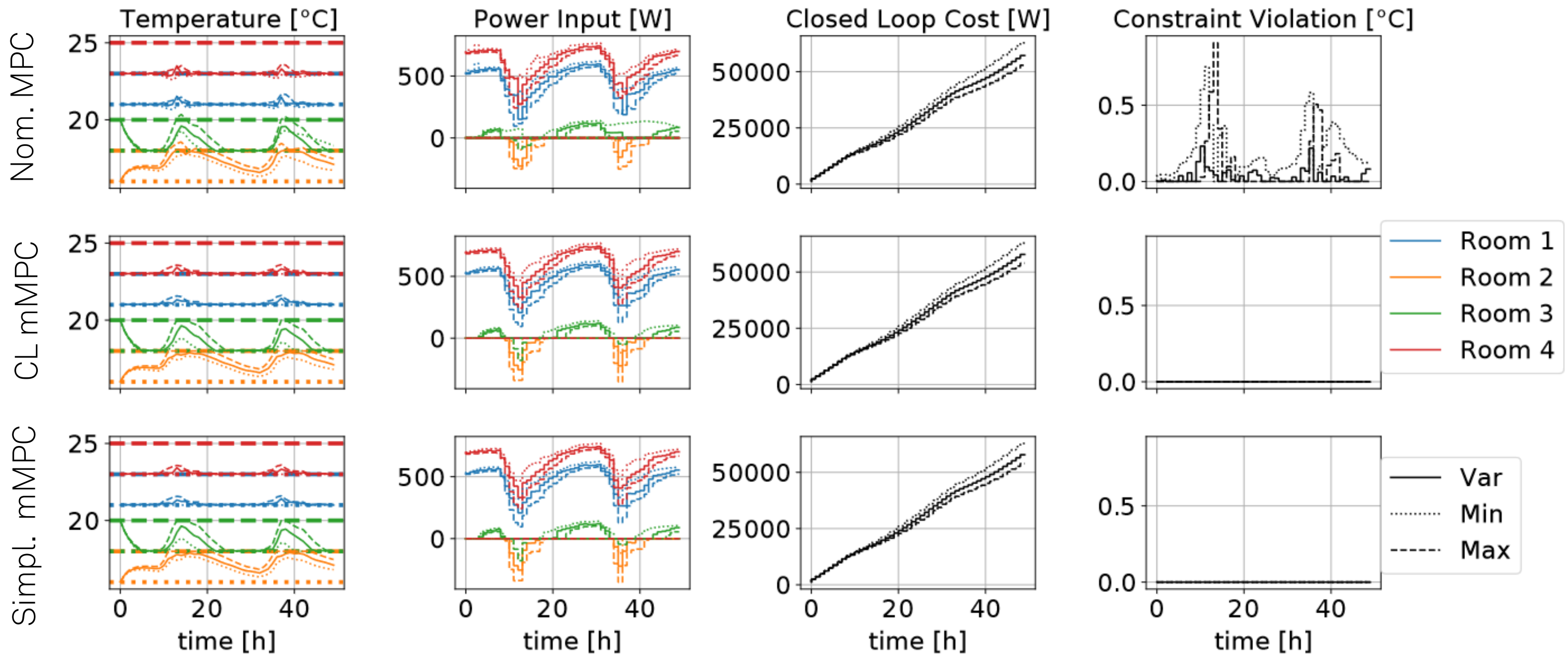
- Linear model
- External influences with uncertainty:
  - External temperature $\pm 1\ °C$
  - Solar radiation $\pm 25\ \%$
  - Internal gains in each room uncertain during work hours $\pm 30\ \%$

$$\min_{\boldsymbol{u}_h, \boldsymbol{u}_c} \sum_{k=0}^{12-1} (u_{h,k} + u_{c,k})$$

$$\text{s.t.}: \begin{bmatrix} T_{\mathrm{r},i}^{+} \\ T_{\mathrm{w,\,int},i}^{+} \\ T_{\mathrm{w,\,ext},i}^{+} \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} T_{\mathrm{r},i} \\ T_{\mathrm{w,\,int},i} \\ T_{\mathrm{w,\,ext},i} \end{bmatrix} + \boldsymbol{B} \begin{bmatrix} u_{h,i} \\ u_{c,i} \end{bmatrix} + \boldsymbol{E} \begin{bmatrix} T_{\mathrm{ext}} + \Delta T_{\mathrm{ext}} \\ s_r(1+\Delta s_r) \\ g_{\mathrm{int},i}(1+\Delta g_{\mathrm{int},i}) \end{bmatrix} + \sum_j \delta_{i,j} \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} (T_{\mathrm{r,j}} - T_{\mathrm{r,i}}), \forall i \in \{1, \ldots, 4\},$$

$$21°C \leq T_{\mathrm{r},1} \leq 23°C,$$
$$18°C \leq T_{\mathrm{r},2} \leq 20°C,$$
$$16°C \leq T_{\mathrm{r},3} \leq 18°C,$$
$$23°C \leq T_{\mathrm{r},4} \leq 25°C,$$
$$0 \leq \boldsymbol{u}_c \leq 1000\mathrm{W},$$
$$0 \leq \boldsymbol{u}_h \leq 1000\mathrm{W}.$$

Comparison of approaches



- Comparison of
  - Nominal MPC
  - Closed-loop MPC for monotone systems
  - Simplified approach
- 16 subregions for closed-loop approach
- Full approach ~16 s per solution
- Simpl. approach ~42 ms per solution

Simplifying general difference inequality

$$S\left[\mathbb{X}_{k+1}(\boldsymbol{u}_k)\right](c) \geq \max_{\boldsymbol{x}\in\mathbb{X}_k,\boldsymbol{w}\in\mathbb{W}} \boldsymbol{c}^\intercal f_\mathrm{d}(\boldsymbol{x},\boldsymbol{u}_k,\boldsymbol{w}) \quad \forall \boldsymbol{c} \in \mathbb{R}^n$$

- Monotonicity gives tight solution to the inner maximization problem

- Partitioning based feedback strategy leaves dynamics unaltered
- Increase in complexity with the number of subregions

- Interval sets
- Computationally cheap

- What to do with non-monotone systems?

Beyond monotone systems

State transformations and mixed monotonicity

Monotonicity through state transformation

- Monotonicity can be shown by signs of Jacobian
  - All elements positive for discrete system
  - Off-diagonal positive for continuous system
- Consider the biochemical system
- Under a linear transformation, the system is monotone

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{G}\dot{\boldsymbol{x}} = \boldsymbol{G} f_c(\boldsymbol{G}^{-1}\tilde{\boldsymbol{x}}, \boldsymbol{u}, \boldsymbol{w})$$

- Can be checked with graph consistency

$$\frac{\partial L}{\partial t} = -v_1$$
$$\frac{\partial R}{\partial t} = -v_1$$
$$\frac{\partial C}{\partial t} = v_1$$
$$v_1 = k_1 L R - k_2 C$$

$$\text{sign}\left(\frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right) = \begin{bmatrix} - & - & + \\ - & - & + \\ + & + & - \end{bmatrix}$$

$$\frac{\partial (L+C)}{\partial t} = 0$$
$$\frac{\partial (R+C)}{\partial t} = 0$$
$$\frac{\partial C}{\partial t} = v_1$$
$$v_1 = k_1((L+C)-C)((R+C)-C) - k_2 C$$

$$\boldsymbol{G} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

C. Kallies, M. Schliemann, R. Findeisen, S. Lucia, and E. Bullinger, "Monotonicity of Kinetic Proofreading," *IFAC-PapersOnLine*, vol. 49, pp. 306–311, Dec. 2016, doi: 10.1016/j.ifacol.2016.12.144.

Finding solutions to difference inequalities

- Can we solve the inner optimization problem in advance?

$$\boldsymbol{x}_{k+1,i}^{\min} \leq \min_{\xi \in [\boldsymbol{x}_k^{\min}, \boldsymbol{x}_k^{\max}], \boldsymbol{w} \in \mathbb{W}} \boldsymbol{f}_{\mathrm{d},i}(\xi, \boldsymbol{u}_k, \boldsymbol{w}) \qquad \boldsymbol{x}_{k+1,i}^{\max} \geq \max_{\xi \in [\boldsymbol{x}_k^{\min}, \boldsymbol{x}_k^{\max}], \boldsymbol{w} \in \mathbb{W}} \boldsymbol{f}_{\mathrm{d},i}(\xi, \boldsymbol{u}_k, \boldsymbol{w})$$

- Assume continuity and $\boldsymbol{w} \in \mathbb{W} = [\boldsymbol{w}^{\min}, \boldsymbol{w}^{\max}]$
- Mixed monotonicity: There exists a decomposition function

$$d_i(\boldsymbol{x}_k^{\max}, \boldsymbol{w}^{\max}, \boldsymbol{u}_k, \boldsymbol{x}_k^{\min}, \boldsymbol{w}^{\min}) \geq \max_{\xi \in [\boldsymbol{x}_k^{\min}, \boldsymbol{x}_k^{\max}], \rho \in \mathbb{W}} \boldsymbol{f}_{\mathrm{d},i}(\xi, \boldsymbol{u}_k, \rho) \qquad d_i(\boldsymbol{x}_k^{\min}, \boldsymbol{w}^{\min}, \boldsymbol{u}_k, \boldsymbol{x}_k^{\max}, \boldsymbol{w}^{\max}) \leq \min_{\xi \in [\boldsymbol{x}_k^{\min}, \boldsymbol{x}_k^{\max}], \rho \in \mathbb{W}} \boldsymbol{f}_{\mathrm{d},i}(\xi, \boldsymbol{u}_k, \rho)$$

- Decomposes systems into increasing and decreasing components

S. Coogan, "Mixed Monotonicity for Reachability and Safety in Dynamical Systems," in *59th IEEE Conference on Decision and Control*, Jeju, Korea (South): IEEE, Dec. 2020, pp. 5074–5085. doi: 10.1109/CDC42340.2020.9304391.

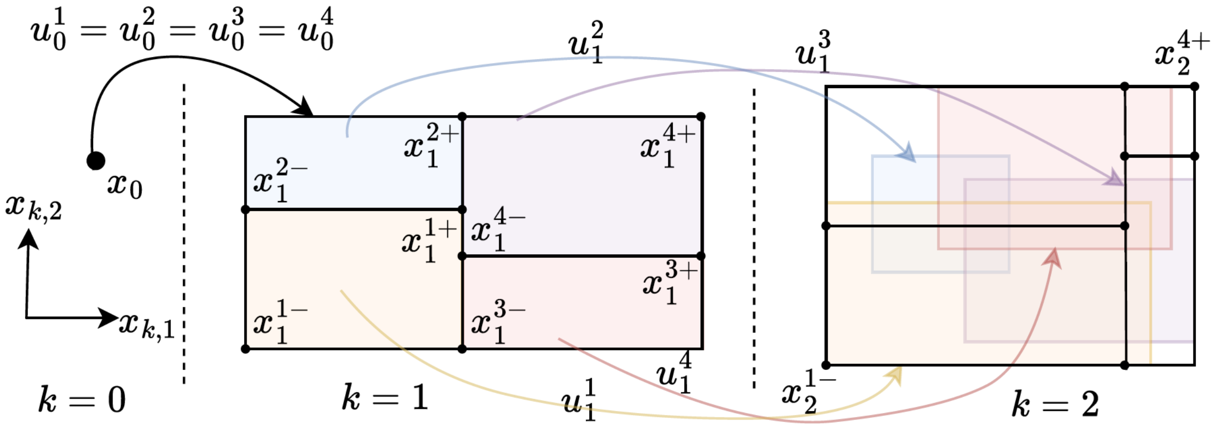
Mixed monotone robust MPC

- Similar to monotone MPC
- Instead of system function decomposition function
- Consideration of feedback with partition policy
- Recursive feasibility can be shown
- Terminal set also possible by adding constraint

$$\min_{\boldsymbol{x}^{s\pm}_{[0:N]},\boldsymbol{u}^{s}_{[0:N-1]},\forall s\in\mathbb{S}} \quad J(\boldsymbol{x}^{[1:\mu_s]\pm}_{[0:N]}, \boldsymbol{u}^{[1:\mu_s]}_{[0:N-1]})$$

$$\begin{aligned}
\text{s.t} : \boldsymbol{x}^{s\pm}_0 &= \boldsymbol{x}_0,\\
\boldsymbol{x}^{\mu_s+}_{k+1} &\geq \boldsymbol{d}(\boldsymbol{x}^{s+}_k, \boldsymbol{w}^{\max}, \boldsymbol{u}^s_k, \boldsymbol{x}^{s-}_k, \boldsymbol{w}^{\min}),\\
\boldsymbol{x}^{1-}_{k+1} &\leq \boldsymbol{d}(\boldsymbol{x}^{s-}_k, \boldsymbol{w}^{\min}, \boldsymbol{u}^s_k, \boldsymbol{x}^{s+}_k, \boldsymbol{w}^{\max}),\\
&[\boldsymbol{x}^{1-}_k, \boldsymbol{x}^{\mu_s+}_k] \subseteq \mathbb{X},\\
&\boldsymbol{u}^s_k \in \mathbb{U},\\
&[\boldsymbol{x}^{1-}_N, \boldsymbol{x}^{\mu_s+}_N] \subseteq \mathbb{X}_f,\\
&\boldsymbol{u}^1_0 = \boldsymbol{u}^s_0,\\
&\boldsymbol{h}(\boldsymbol{x}^{[1:\mu_s]\pm}_k) \leq \boldsymbol{0},
\end{aligned}$$

M. Heinlein, S. Subramanian, and S. Lucia, "Robust Model Predictive Control Exploiting Monotonicity Properties," *IEEE Transactions on Automatic Control*, vol. 70, no. 9, pp. 6260–6267, Sep. 2025, doi: 10.1109/TAC.2025.3558137.



Availability of decomposition functions

- Mostly rephrases the problem of difference inequalities
- For linear systems $\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{E}\boldsymbol{w}_k$

$$d(\hat{\boldsymbol{x}}, \hat{\boldsymbol{w}}, \boldsymbol{u}, \check{\boldsymbol{x}}, \check{\boldsymbol{w}}) = \boldsymbol{A}^+\hat{\boldsymbol{x}} + \boldsymbol{A}^-\check{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{E}^+\hat{\boldsymbol{w}} + \boldsymbol{E}^-\check{\boldsymbol{w}}$$

$$\boldsymbol{A}^+ = \max\{\boldsymbol{A}, \boldsymbol{0}\}$$
$$\boldsymbol{A}^- = \min\{\boldsymbol{A}, \boldsymbol{0}\}$$

- Combining this with linear state transformations gives "Low complexity tube-based MPC" [1]
- Works similarly for bounded Jacobians
- Interval arithmetics can also be used [2]
- Introduces over-approximation errors

[1] B. Kouvaritakis and M. Cannon, Model Predictive Control. in Advanced Textbooks in Control and Signal Processing. Cham: Springer International Publishing, 2016, doi: 10.1007/978-3-319-24853-0.

[2] T. Alamo, D. Limon, E. F. Camacho, and J. M. Bravo, "Robust MPC of constrained nonlinear systems based on interval arithmetic," *IEEE Proc. Control Therory and Appl.*, vol. 152, no. 3, pp. 325–332, May 2005, doi: 10.1049/ip-cta:20040480.

How to handle constraints

- For interval sets, box constraints are trivial
- For linear constraints, use Farkas Lemma

Lemma 1 (Farkas Lemma). *Let* $\mathbb{X}_i \doteq \{\boldsymbol{x} : \boldsymbol{F}_i \boldsymbol{x} \leq b_i\}, i = 1, 2,$ *be non-empty subsets of* $\mathbb{R}^{n_x}$. *Then* $\mathbb{X}_1 \subseteq \mathbb{X}_2$ *if and only if there exists a nonnegative matrix* $\boldsymbol{H} \geq \boldsymbol{0}$ *satisfying*

$$\boldsymbol{H}\boldsymbol{F}_1 = \boldsymbol{F}_2,$$
$$\boldsymbol{H} b_1 \leq b_2.$$

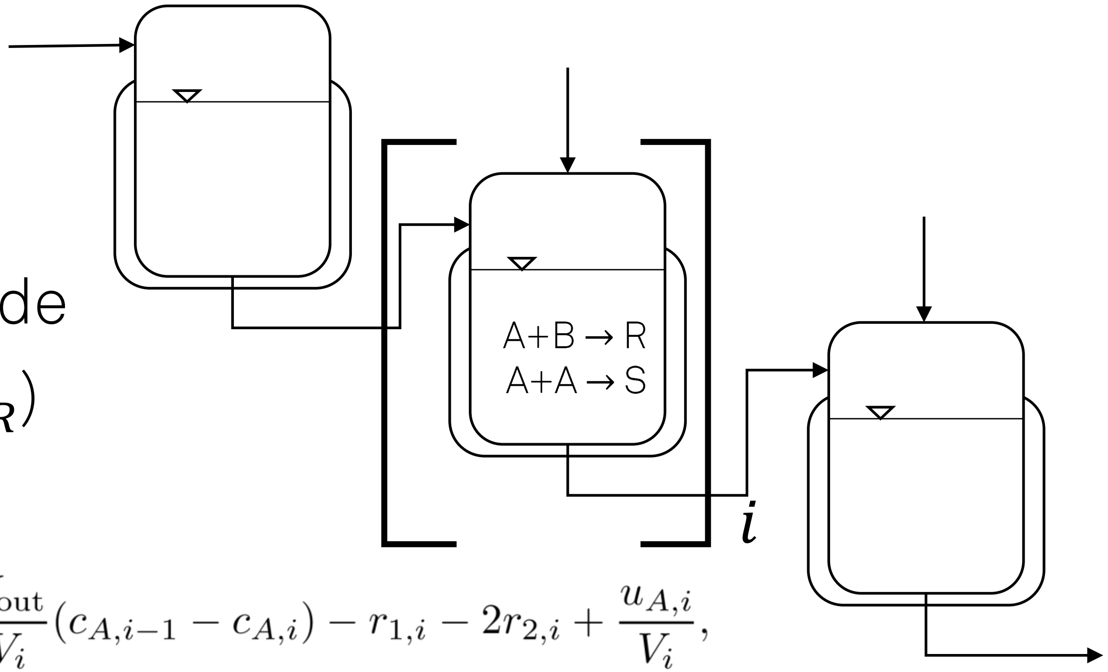
- Precompute $\boldsymbol{H}$ to avoid nonlinearity $\boldsymbol{h}_i^* = \arg \min_{\boldsymbol{h} \in \mathbb{R}^{n_{\boldsymbol{F}_1}}} \boldsymbol{1}^T \boldsymbol{h}$ subject to $\boldsymbol{h}^T \boldsymbol{F}_1$ and $\boldsymbol{h} \geq \boldsymbol{0}$
- For nonlinear constraint $\boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k) \leq 0$, decomposition needed

$$\boldsymbol{e}_{\mathbb{W}}(\boldsymbol{x}_k^+, \boldsymbol{x}_k^-, \boldsymbol{u}_k) \geq \max_{\xi \in \mathbb{X}_k^{\mathrm{Int}},\ \boldsymbol{w} \in \mathbb{W}} \boldsymbol{g}(\xi, \boldsymbol{u}_k, \boldsymbol{w})$$

Case Study

Nonlinear mixed monotone CSTR Cascade

Case Study



- Nonlinear & non-monotone CSTR-Cascade
- Scalable in states & uncertainties ($k; \Delta H_R$)
- Minimize $\left|\left|c_{R,max} - c_R\right|\right|_{Q_R}^2 - ||c_S||_{Q_S}^2$
- Reachable sets via decomposition function & mixed monotonicity
- Partition policy for feedback

$$r_{1,i} = k_{1,i} \exp - \frac{E_{A1,i}}{R_{\text{gas}}(T_{R,i} + 273.15)} c_{A,i} c_{B,i}$$

$$r_{2,i} = k_{2,i} \exp - \frac{E_{A2,i}}{R_{\text{gas}}(T_{R,i} + 273.15)} c_{A,i}^2$$

$$\dot{c}_{A,i} = \frac{\dot{V}_{\text{out}}}{V_i}(c_{A,i-1} - c_{A,i}) - r_{1,i} - 2r_{2,i} + \frac{u_{A,i}}{V_i},$$

$$\dot{c}_{B,i} = \frac{\dot{V}_{\text{out}}}{V_i}(c_{B,i-1} - c_{B,i}) - r_{1,i} + \frac{u_{B,i}}{V_i},$$

$$\dot{c}_{R,i} = \frac{\dot{V}_{\text{out}}}{V_i}(c_{R,i-1} - c_{R,i}) + r_{1,i},$$

$$\dot{c}_{S,i} = \frac{\dot{V}_{\text{out}}}{V_i}(c_{S,i-1} - c_{S,i}) + r_{2,i},$$

$$\dot{T}_{R,i} = \frac{\dot{V}_{\text{out}}}{V_i}(T_{R,i-1} - T_{R,i}) - r_{1,i}\frac{\Delta H_{R1,i}}{\rho c_p} - r_{2,i}\frac{\Delta H_{R2,i}}{\rho c_p} + \frac{kA}{\rho c_p V_i}(T_{J,i} - T_{R,i}).$$

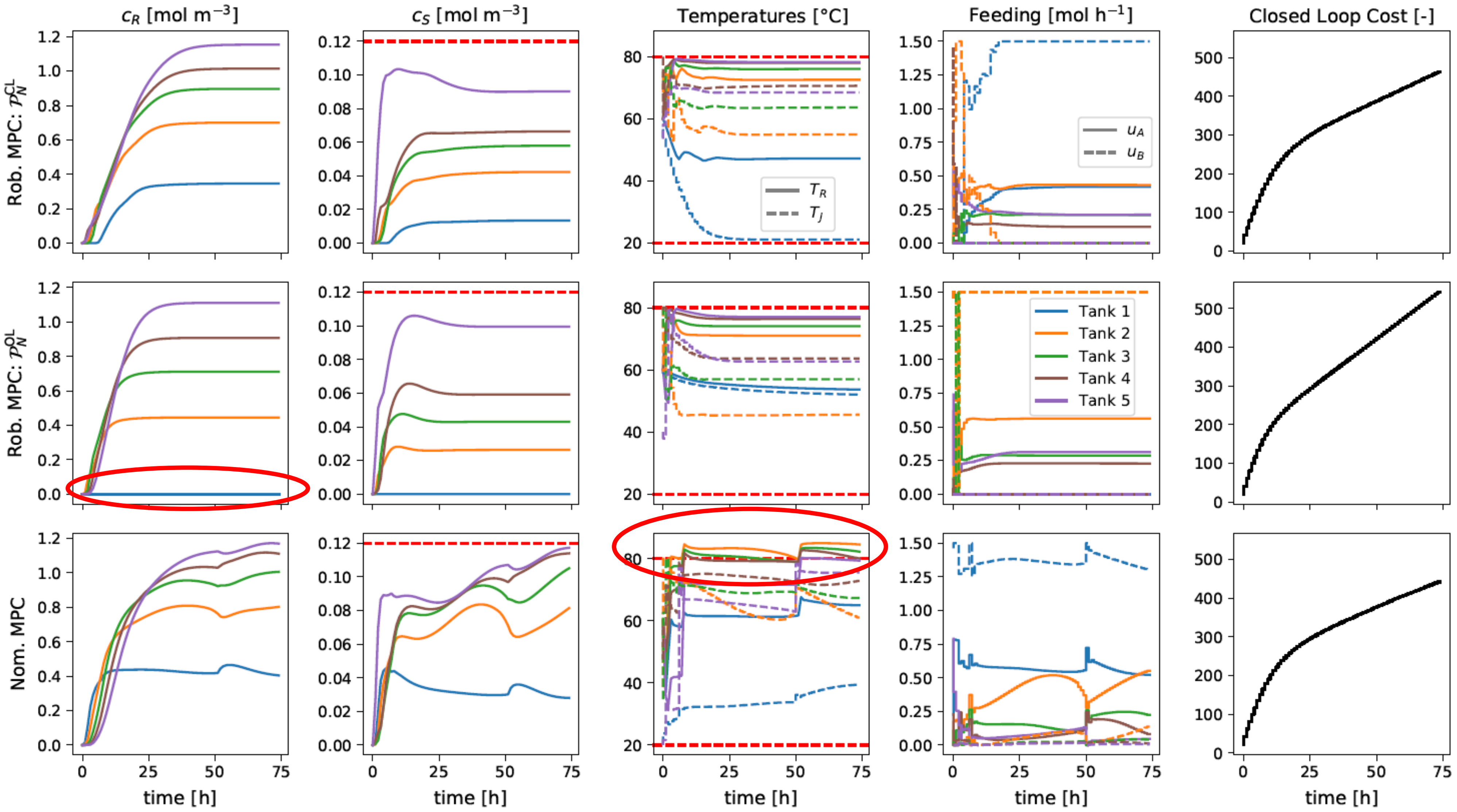
Comparison of closed loop and open loop

n_R	n_x	n_p	μ_s for $\mathcal{P}_N^{\mathrm{CL}_d}$	Closed-loop cost $\mathcal{P}_N^{\mathrm{OL}_d}$	Closed-loop cost $\mathcal{P}_N^{\mathrm{CL}_d}$	Comp. time $[s]$ $\mathcal{P}_N^{\mathrm{OL}_d}$	Comp. time $[s]$ $\mathcal{P}_N^{\mathrm{CL}_d}$
1	5	4	32	149%	129%	1.70	36.40
3	15	12	16	161%	114%	16.35	111.45
5	25	20	4	142%	122%	42.76	72.38

- Values averaged over 50 closed-loop simulations
- Closed loop cost (CLC) compared to MPC with correct parameters
- For 5 reactors, partitioning three times in $c_{S,1}$
- Rigorous robustness guarantees

M. Heinlein, S. Subramanian, and S. Lucia, “Robust Model Predictive Control Exploiting Monotonicity Properties,” *IEEE Transactions on Automatic Control*, vol. 70, no. 9, pp. 6260–6267, Sep. 2025, doi: 10.1109/TAC.2025.3558137.

Simulation for 5 reactors



General difference inequality – non-monotone

$$S\left[\mathbb{X}_{k+1}(\boldsymbol{u}_k)\right](c) \geq \max_{\boldsymbol{x}\in\mathbb{X}_k,\boldsymbol{w}\in\mathbb{W}} \boldsymbol{c}^\intercal f_\mathrm{d}(\boldsymbol{x},\boldsymbol{u}_k,\boldsymbol{w}) \quad \forall \boldsymbol{c} \in \mathbb{R}^n$$

- Solution to max over-approximated
  - Mixed monotonicity
  - Interval arithmetics
- Introduces overapproximation error for non-monotone systems

- Partitioning based feedback strategy leaves dynamics unaltered
- Increase in complexity with the number of subregions

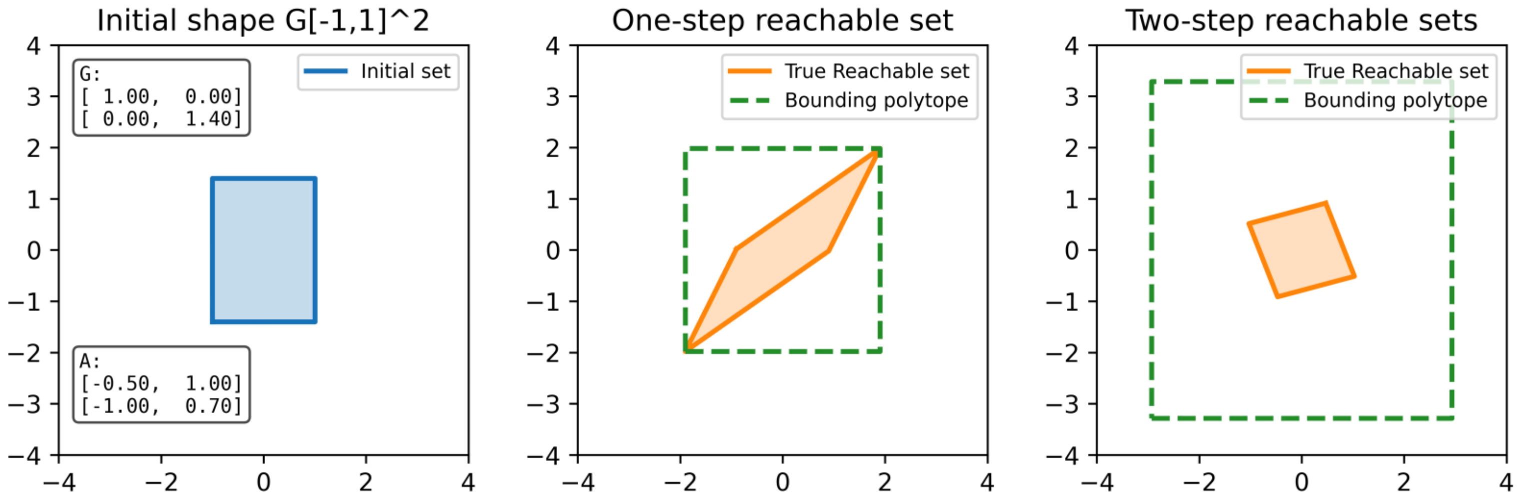
- Interval sets
- Computationally cheap
- Cannot capture rotation through dynamics

More than rectangles

Exploring more complex shapes to avoid the wrapping effect

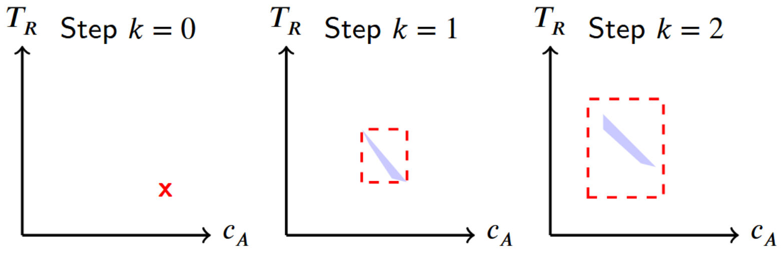
The wrapping effect

- Interval sets capture monotone dynamics well
- They cannot represent rotation (non-monotonicity)
- This leads to conservativeness due to over-approximation
- The error accumulates over multiple time steps
- Termed "wrapping effect"
- Best orientation depends on dynamics



Edgy reachable sets in the linear case

- Rotated sets may capture the dynamics better
- Zonotopes $\mathbb{X}^{\text{zono}} = \{\boldsymbol{x} \in \mathbb{R}^{n_x} | \alpha^- \leq \boldsymbol{G}\boldsymbol{x} \leq \alpha^+\}$
  - $\boldsymbol{G} \in \mathbb{R}^{n_g \times n_x}$ with $n_g \geq n_x$
  - Linear transformed intervals
  - Generalized by polytopes $\mathbb{X}^{\text{poly}} = \{\boldsymbol{x} \in \mathbb{R}^{n_x} | \boldsymbol{V}\boldsymbol{x} \leq \alpha\}$ $\boldsymbol{V}_{\text{zono as poly}} = \begin{bmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{G} \end{bmatrix}, \quad \alpha_{\text{zono as poly}} = \begin{bmatrix} \alpha^+ \\ -\alpha^- \end{bmatrix}$
  - If $n_g = n_x$ and linear systems-> “Low complexity tube-base MPC”



- Polytopes propagated for linear systems with Farkas Lemma -> “General complexity tube-based MPC”

$$\boldsymbol{H}\boldsymbol{F}_1 = \boldsymbol{F}_2, \quad \boldsymbol{H}b_1 \leq b_2. \qquad \boldsymbol{H} \geq \boldsymbol{0}$$

$$\boldsymbol{F}_1 = \boldsymbol{V}, \qquad \text{from } \boldsymbol{V}\boldsymbol{x}_k \leq \underbrace{\alpha_k}_{b_1}$$

$$\boldsymbol{F}_2 = \boldsymbol{V}\boldsymbol{A}, \qquad \text{from } \boldsymbol{V} \underbrace{\boldsymbol{x}_{k+1}}_{\boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k} \leq \alpha_{k+1} \Leftrightarrow \boldsymbol{V}\boldsymbol{A}\boldsymbol{x}_k \leq \underbrace{\alpha_{k+1} - \boldsymbol{B}\boldsymbol{u}_k}_{b_2}$$

B. Kouvaritakis and M. Cannon, Model Predictive Control. in Advanced Textbooks in Control and Signal Processing. Cham: Springer International Publishing, 2016, doi: 10.1007/978-3-319-24853-0.

Solving zonotopic difference inequality

- Difference inclusion

$$\alpha_{k+1,i}^{-} \leq \min_{\alpha_k^- \leq \boldsymbol{G}\boldsymbol{x} \leq \alpha_k^+, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{G} f_i(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w}),$$

$$\alpha_{k+1,i}^{+} \geq \max_{\alpha_k^- \leq \boldsymbol{G}\boldsymbol{x} \leq \alpha_k^+, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{G} f_i(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w})$$

- For bounded Jacobian or with a lot of analytical effort Decomposition:

$$\boldsymbol{d}_{\mathbb{W},i}^{+}(\alpha_k^+, \alpha_k^-, \boldsymbol{u}_k) \geq \max_{\alpha_k^- \leq \boldsymbol{G}\boldsymbol{x} \leq \alpha_k^+, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{G} f_i(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w}),$$

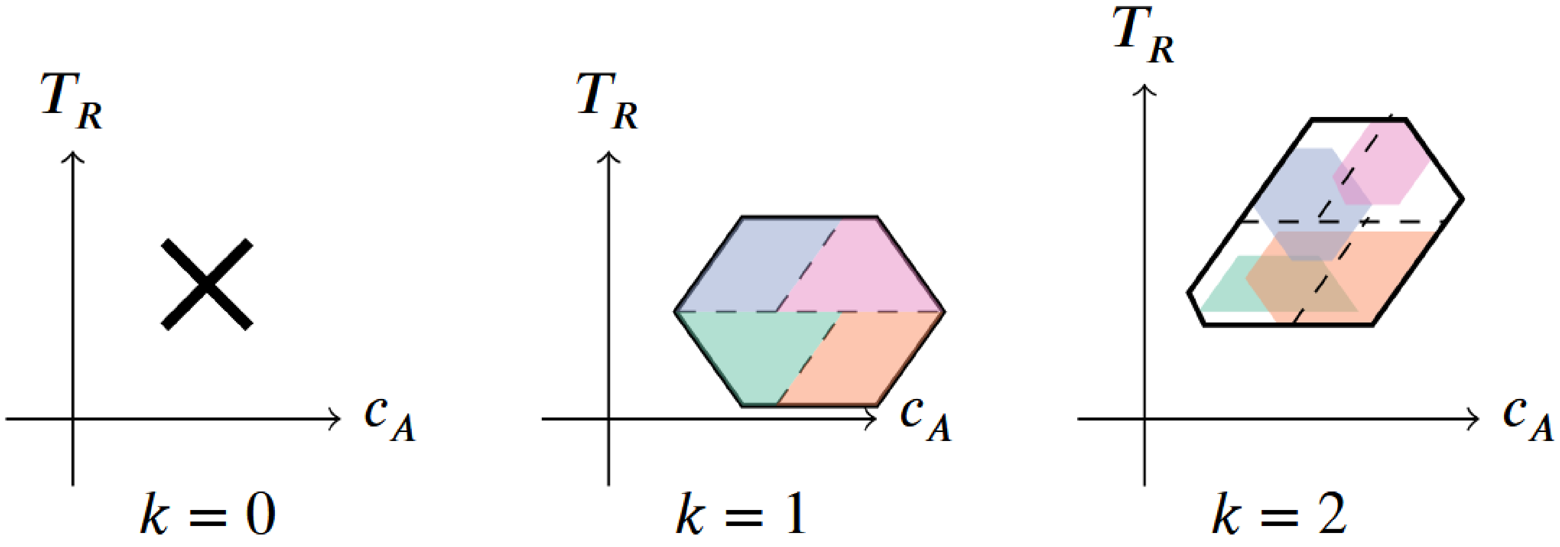
$$\boldsymbol{d}_{\mathbb{W},i}^{-}(\alpha_k^+, \alpha_k^-, \boldsymbol{u}_k) \leq \min_{\alpha_k^- \leq \boldsymbol{G}\boldsymbol{x} \leq \alpha_k^+, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{G} f_i(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w})$$

- Assume also decomposition for constraints

$$\boldsymbol{e}_{\mathbb{W}}(\alpha_k^+, \alpha_k^-, \boldsymbol{u}_k) \geq \max_{\alpha_k^- \leq \boldsymbol{G}\boldsymbol{x} \leq \alpha_k^+,\ \boldsymbol{w} \in \mathbb{W}} \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w})$$

Feedback strategy for zonotopic sets

- Optimizing over feedback policies complicates decomposition
- Use same partitioning strategy



- For zonotope both sides of partition again zonotope with same $\boldsymbol{G}$

Robust MPC with zonotopes

- Similar to previous approaches
- $\alpha^{s\pm}_{[0:N]}$ for subregions, $\alpha^{\max\pm}_{[0:N]}$ for bound
- Introducing auxiliary variable to ensure non-emptiness
- Can also be used in cost
- Terminal set RCI, if replaced by

$$\alpha^{\max-}_{N-1} \le \alpha^{\max-}_{N}, \quad \alpha^{\max+}_{N-1} \ge \alpha^{\max+}_{N}$$

$$\min_{\boldsymbol{x}^{s\pm}_{[0:N]}, \alpha^{s\pm}_{[0:N]}, \alpha^{\max\pm}_{[0:N]}, \boldsymbol{u}^{s}_{[0:N-1]}, \forall s \in \mathbb{S}} J(\boldsymbol{x}^{[1:\mu_s]\pm}_{[0:N]}, \boldsymbol{u}^{[1:\mu_s]}_{[0:N-1]})$$

$$\begin{aligned}
\text{s.t}: \ & \alpha^{s-}_0 = \boldsymbol{G}\boldsymbol{x}_0 = \alpha^{s+}_0, \\
& \alpha^{\max+}_{k+1} \ge \boldsymbol{d}^{+}_{\mathbb{W}}(\alpha^{s+}_k, \alpha^{s-}_k, \boldsymbol{u}^{s}_k), \\
& \alpha^{\max-}_{k+1} \le \boldsymbol{d}^{-}_{\mathbb{W}}(\alpha^{s+}_k, \alpha^{s-}_k, \boldsymbol{u}^{s}_k), \\
& \boldsymbol{e}_{\mathbb{W}}(\alpha^{s+}_k, \alpha^{s-}_k, \boldsymbol{u}^{s}_k) \le 0, \\
& \alpha^{-}_{\text{term}} \le \alpha^{\max-}_{N}, \quad \alpha^{+}_{\text{term}} \ge \alpha^{\max+}_{N} \\
& \boldsymbol{u}^1_0 = \boldsymbol{u}^s_0, \quad \alpha^1_0 = \alpha^s_0 \\
& \alpha^{s-}_k \le \boldsymbol{G}\boldsymbol{x}^s_k \le \alpha^{s+}_k, \\
& \boldsymbol{h}(\alpha^{[1:\mu_s]\pm}_k, \alpha^{\max\pm}_k) \le \boldsymbol{0},
\end{aligned}$$

▶ But how to get these decompositions reliably?

Learning reachable set propagations

Surrogate modeling of decomposition functions

Surrogate decomposition function

- For notational convenience, assume general polytope

$$\mathbb{X}^{\text{poly}} = \{\boldsymbol{x} \in \mathbb{R}^{n_x} | \boldsymbol{V}\boldsymbol{x} \leq \alpha\}$$

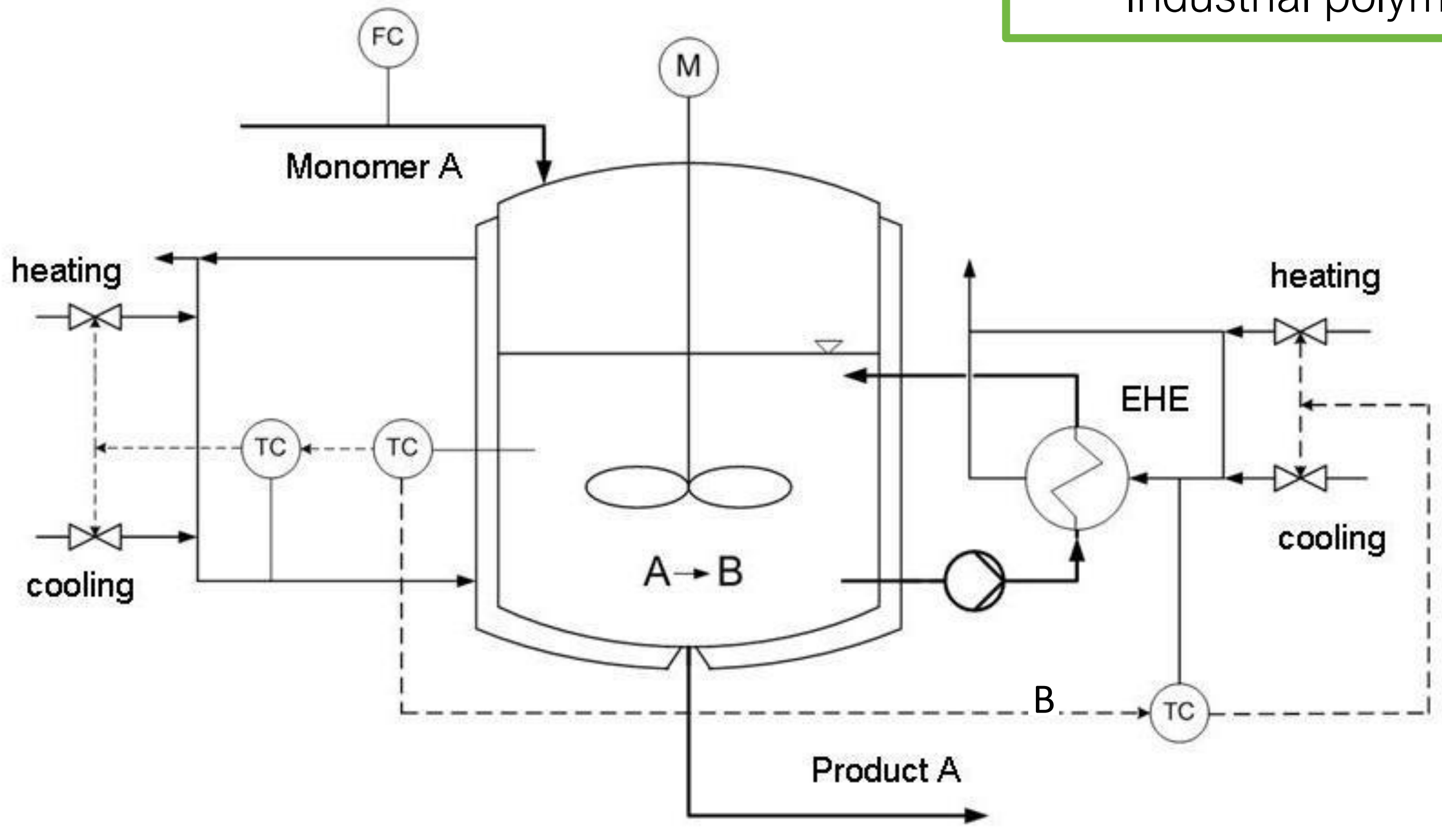
- We seek surrogate model for constraints and propagation

$$\mathcal{N}(\alpha_k, \boldsymbol{u}_k) \geq \begin{bmatrix} \max_{\boldsymbol{V}\boldsymbol{x} \leq \alpha_k, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{V}\boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w}) \\ \max_{\boldsymbol{V}\boldsymbol{x} \leq \alpha_k, \boldsymbol{w} \in \mathbb{W}} \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w}) \end{bmatrix}$$

- ToDos
  - Choice of $\boldsymbol{V}$ / $\boldsymbol{G}$ (zonotope to be used in RMPC)
  - Data generation
  - Training
  - Implementation in RMPC

Case study

Industrial polymerization reactor



An industrial batch polymerization reactor

9 differential states
3 control inputs
9 uncertain parameters ($\pm 10\%$)
1 nonlinear constraint

$$\dot{m}_{\mathrm{W}} = \dot{m}_{\mathrm{F}}\omega_{\mathrm{W,F}},$$

$$\dot{m}_{\mathrm{A}} = \dot{m}_{\mathrm{F}}\omega_{\mathrm{A,F}} - k_{R1}m_{\mathrm{A,R}} - k_{R2}m_{\mathrm{AWT}}\frac{m_{\mathrm{A}}}{m_{\mathrm{ges}}},$$

$$\dot{m}_{\mathrm{P}} = k_{R1}m_{\mathrm{A,R}} + p_1 k_{R2}m_{\mathrm{AWT}}\frac{m_{\mathrm{A}}}{m_{\mathrm{ges}}},$$

$$\dot{m}_{\mathrm{A}}^{\mathrm{acc}} = \dot{m}_{\mathrm{F}},$$

$$\dot{T}_{\mathrm{R}} = \frac{1}{c_{p,\mathrm{R}}m_{\mathrm{ges}}}\left[\dot{m}_{\mathrm{F}}c_{p,\mathrm{F}}(T_{\mathrm{F}} - T_{\mathrm{R}})\Delta H_R k_{R1}m_{\mathrm{A,R}} - k_K A(T_{\mathrm{R}} - T_{\mathrm{S}}) - \dot{m}_{\mathrm{AWT}}c_{p,\mathrm{R}}(T_{\mathrm{R}} - T_{\mathrm{EK}})\right],$$

$$\dot{T}_{\mathrm{S}} = \frac{1}{c_{p,\mathrm{S}}m_{\mathrm{S}}}\left[k_K A(T_{\mathrm{R}} - T_{\mathrm{S}}) - k_K A(T_{\mathrm{S}} - T_{\mathrm{M}})\right],$$

$$\dot{T}_{\mathrm{M}} = \frac{1}{c_{p,\mathrm{W}}m_{\mathrm{M,KW}}}\left[\dot{m}_{\mathrm{M,KW}}c_{p,\mathrm{W}}(T_{\mathrm{M}}^{\mathrm{IN}} - T_{\mathrm{M}}) + k_K A(T_{\mathrm{S}} - T_{\mathrm{M}})\right],$$

$$\dot{T}_{\mathrm{EK}} = \frac{1}{c_{p,\mathrm{R}}m_{\mathrm{AWT}}}\left[\dot{m}_{\mathrm{AWT}}c_{p,\mathrm{W}}(T_{\mathrm{R}} - T_{\mathrm{EK}}) - \alpha(T_{\mathrm{EK}} - T_{\mathrm{AWT}}) + k_{R2}m_{\mathrm{A}}m_{\mathrm{AWT}}\frac{\Delta H_R}{m_{\mathrm{ges}}}\right],$$

$$\dot{T}_{\mathrm{AWT}} = \left[\dot{m}_{\mathrm{AWT,KW}}c_{p,\mathrm{W}}(T_{\mathrm{AWT}}^{\mathrm{IN}} - T_{\mathrm{AWT}}) - \alpha(T_{\mathrm{AWT}} - T_{\mathrm{EK}})\right](c_{p,\mathrm{W}}m_{\mathrm{AWT,KW}}).$$

$$U = \frac{m_{\mathrm{P}}}{m_{\mathrm{A}} + m_{\mathrm{P}}},$$

$$m_{\mathrm{ges}} = m_{\mathrm{W}} + m_{\mathrm{A}} + m_{\mathrm{P}},$$

$$k_{R1} = k_0 \exp\left(-\frac{E_a}{R(T_{\mathrm{R}} + 273.15)}\right)(k_{U1}(1-U) + k_{U2}U),$$

$$k_{R2} = k_0 \exp\left(-\frac{E_a}{R(T_{\mathrm{EK}} + 273.15)}\right)(k_{U1}(1-U) + k_{U2}U),$$

$$k_K = \frac{m_{\mathrm{W}}k_{\mathrm{WS}} + m_{\mathrm{A}}k_{\mathrm{AS}} + m_{\mathrm{P}}k_{\mathrm{PS}}}{m_{\mathrm{ges}}},$$

$$m_{\mathrm{A,R}} = m_{\mathrm{A}} - \frac{m_{\mathrm{A}}m_{\mathrm{AWT}}}{m_{\mathrm{ges}}},$$

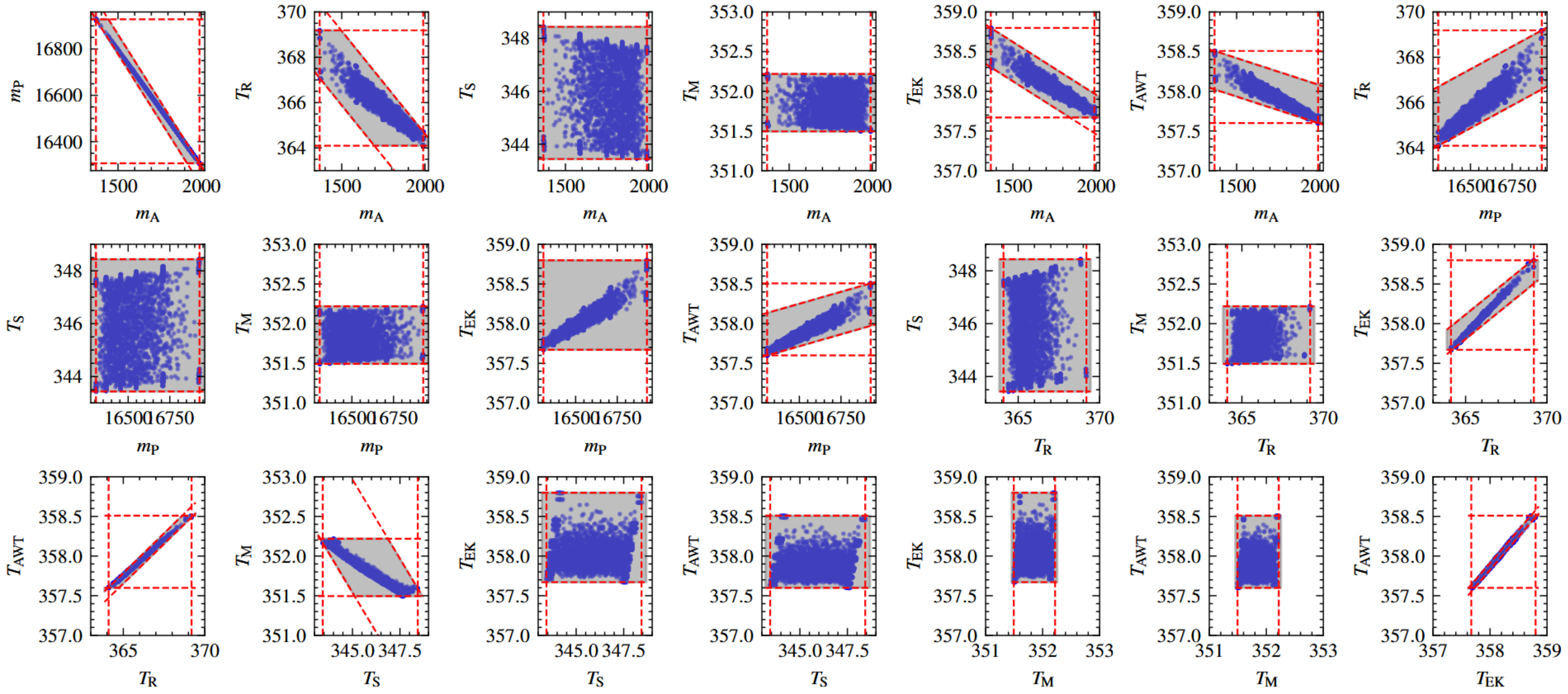
$$T_{\mathrm{adiabat}} = \frac{\Delta H_R}{m_{\mathrm{ges}}c_{p,\mathrm{R}}}m_{\mathrm{A}} + T_{\mathrm{R}}.$$

- Tight temperature constraints
- Maximization of product in batch

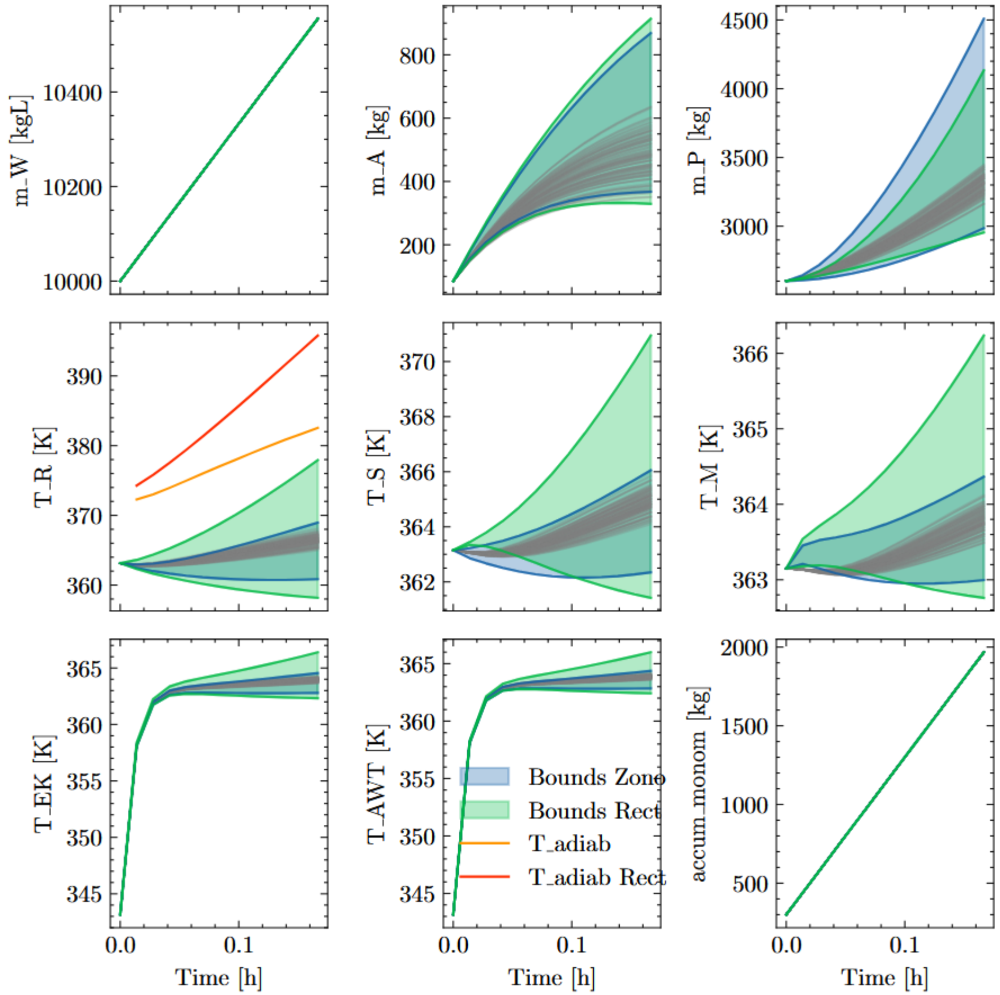
S. Lucia, J. A. E. Andersson, H. Brandt, M. Diehl, und S. Engell, „Handling uncertainty in economic nonlinear model predictive control: A comparative case study“, *J. Process Control*, Bd. 24, Nr. 8, S. 1247–1259, Aug. 2014, doi: 10.1016/j.jprocont.2014.05.008.

Choice of the orientation of the zonotope

- Heuristic approach because of nonlinearity
- Sampling reachable sets from multiple initial states
- Halfplanes from 2D projections
- Focus on $\boldsymbol{T_R}$ and $\boldsymbol{m_A}$
- Keep axis aligned halfspaces for easy implementation of box constraints: $\boldsymbol{G} \in \mathbb{R}^{19\times 9}$ / $\boldsymbol{V} \in \mathbb{R}^{38\times 9}$

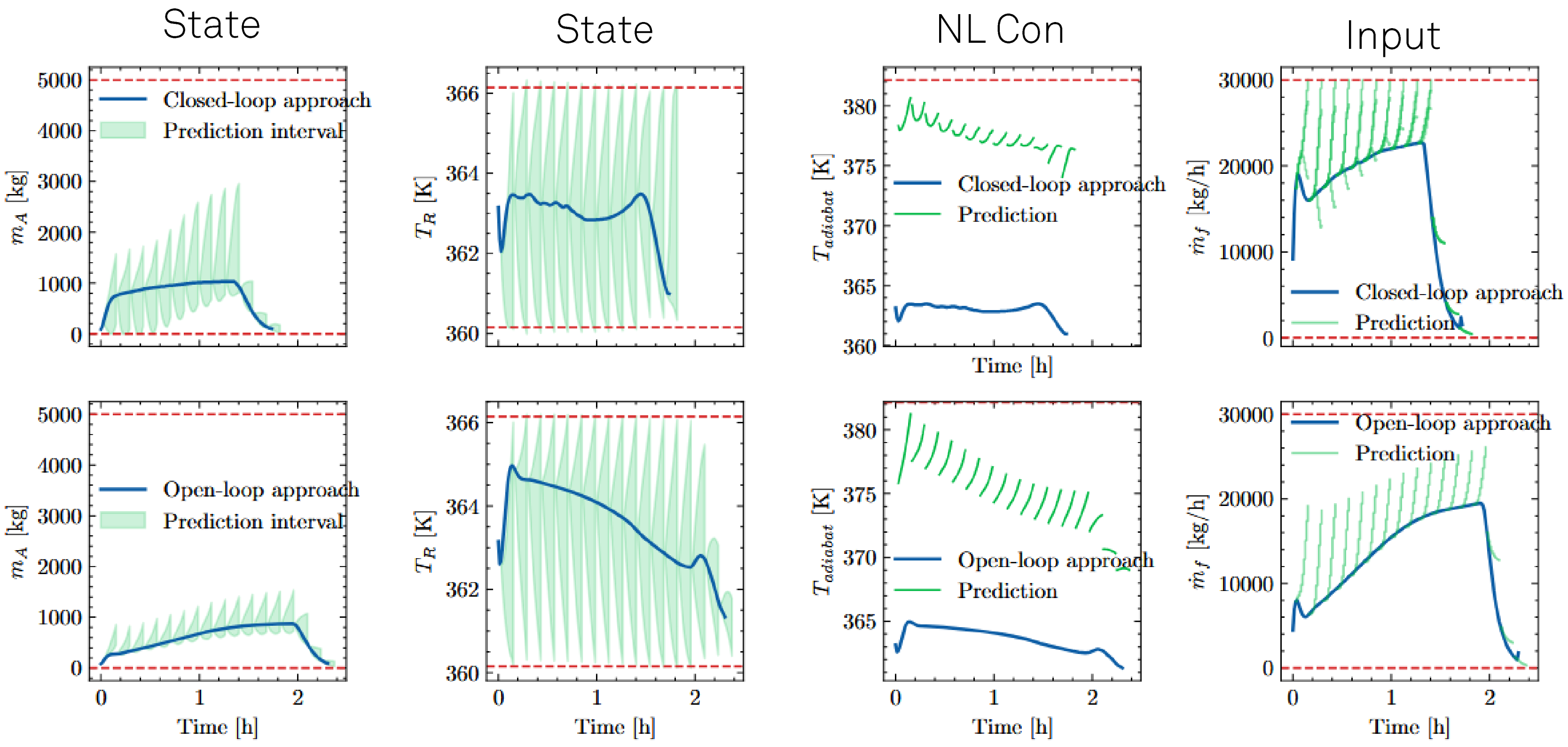


Prediction capability – Zonotope vs Interval sets



- Neural networks with 1 layer and 120 neurons
- Open-loop predictions for constant input
- Zonotopes tighter bounds
- Small violations

Implementation on closed loop



Metrics	$\mu_s = 1$	$\mu_s = 2$	$\mu_s = 3$
Cost [-]	77.52	50.00	48.50
Comp. Time [s]	2.85	9.62	22.09
Batch time [h]	2.43	1.89	1.85
Constraint Viol. [K]	0	0	0

- Top: 2 partitions in T_R
- Bottom: no partitions
- Recourse enables more aggressive feeding strategy
- Smaller batch times

General difference inequality – Zonotopes

$$S\left[\mathbb{X}_{k+1}(\boldsymbol{u}_k)\right](c) \geq \max_{\boldsymbol{x}\in\mathbb{X}_k, \boldsymbol{w}\in\mathbb{W}} \boldsymbol{c}^\intercal f_\mathrm{d}(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w}) \quad \forall \boldsymbol{c} \in \mathbb{R}^n$$

- Solution to max learned with neural network
- Complicates solution
- No need for analytical decomposition function

- Partitioning based feedback strategy leaves dynamics unaltered
- Increase in complexity with the number of subregions

- Zonotopic sets
- Less prone to wrapping effect
- Increased complexity
  - Number of parameters
  - Handling of constraints

▶ That's it, folks!

Conclusion – Main takeaways

- Reachable sets are monotone, but not boring
- Propagation of reachable sets describable by generalized difference/differential inequalities $S\left[\mathbb{X}_{k+1}(\boldsymbol{u}_k)\right](c) \geq \max_{\boldsymbol{x}\in\mathbb{X}_k, \boldsymbol{w}\in\mathbb{W}} \boldsymbol{c}^\intercal f_\mathrm{d}(\boldsymbol{x}, \boldsymbol{u}_k, \boldsymbol{w})$
- Tractability by convex parameterizations
  - Intervals: Cheap, but conservative for non-monotone systems
  - Zonotopes/Polytopes: More flexible, but more complex
  - Ellipsoids: Also great option, but not covered here
- Overapproximation of inner maximization problem
  - Analytical (decomposition function, monotonicity)
  - Data-based
- Recourse by partitioning leaves dynamics unaltered