Optimal feedback solutions to the disturbance attenuation regulator

James B. Rawlings and Davide Mannini

Department of Chemical Engineering

Systems Control and Optimization Laboratory
Universität Freiburg
Freiburg, Germany
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Introduction: two control paradigms

Objective Find feedback solution of disturbance attenuation regulator (DAR) using game theory and dynamic programming

Standard LQR:

$$\min_{u} V(x, u) \qquad x^{+} = Ax + Bu$$

Disturbance attenuation regulator (DAR):

$$\min_{u} \max_{w} V(x, u, w) \qquad x^{+} = Ax + Bu + Gw \qquad |w|^2 \leq \alpha$$

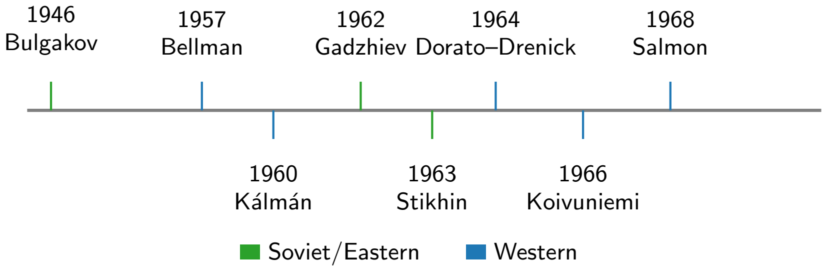
DAR: foundations & early East–West (1940s–1960s)

1957–1960: LQR foundations established

- **1957:** Bellman – dynamic programming solution
- **1960:** Kalman – first formulation and Riccati-based solution

1940s: Bulgakov laid the DAR foundations with his disturbance accumulation problem
1960s: The DAR race begins!

- Soviet researchers: Gadzhiev, Stikhin
- Western researchers: Dorato & Drenick influenced by Gadzhiev's research



Intense activity & a roadblock (Late 1960s–1970s)

- East: Ulanov (1971); Medanić & Andjelić (1971); Yakubovich (1975) (Ulanov, 1971; Medanić and Andjelić, 1971; Yakubovich, 1975)
- West: Rhodes–Luenberger (1969); Kimura (1970); Bertsekas–Rhodes (1973) (Rhodes and Luenberger, 1969; Kimura, 1970; Bertsekas and Rhodes, 1973)

Witsenhausen's early warning (1968) (Witsenhausen, 1968)

"The major feature of worst-case problems is this: a global minimum must be a local minimum, a global saddle-point must be a local saddle-point, but a global minimax need not be a local minimax."

(underemphasized for decades; modern *rediscovery* by Jin, Netrapalli, and Jordan (2020))

Consequence: progress stalled—global minmax pathology broke many "stationary-point" approaches

H_∞ turn and East–West threads (1980s–today)

1980s revival (mostly West)

- Zames (1981): proposed a frequency domain problem that happened to be the same as the 1960s–1970s time-domain problems (Zames, 1981)
- Glover–Doyle (1988), Başar (1989): mapped back to time/state space (Glover and Doyle, 1988; Basar, 1989)
- *But*: limited linkage back to the 60s/70s East/West corpus

Still missing

- A direct DP treatment for bounded disturbances with arbitrary initial conditions
- permitting *closed* disturbance sets
- and avoiding auxiliary problems

1990s and after

- Didinsky–Başar (1992): partial handling of nonzero initial conditions (Didinsky and Basar, 1992)
- Reviews: Dorato (1987) (Dorato, 1987); Khlebnikov–Polyak–Kuntsevich (2011) (Khlebnikov, Polyak, and Kuntsevich, 2011)

Big picture

LQR problem assumptions

Assumptions:

- (A, B) stabilizable and (A, Q) detectable
- $Q \succeq 0$, $R \succ 0$

One-stage LQR problem

$$\min_u V(x,u) = \ell(x,u) + V_f(x^+) \quad \text{s.t. } x^+ = Ax + Bu$$

with

$$\ell(x,u) = (1/2)(x'Qx + u'Ru) \qquad V_f(x) = (1/2)x'P_f x$$

Eliminate x^+ with the linear model

$$V(x,u) = (1/2)\big(\, |x|_Q^2 + |u|_R^2 + |Ax + Bu|_{P_f}^2 \,\big)$$

This is quadratic in u

$$V(x,u) = (1/2)\Big(u'(B'P_f B + R)u + 2u'B'P_f Ax + |x|^2_{Q+A'P_f A} \Big)$$

with $B'P_f B + R > 0$. Therefore solution to $\min_u V$ exists and is unique for all x

One-stage LQR problem solutions

$$u^*(x) = K\,x \qquad \text{(optimal control is linear state feedback)}$$
$$V^*(x) = (1/2)x'\Pi x \qquad \text{(optimal cost is quadratic in state)}$$

with

$$K = -(B'P_fB + R)^{-1}B'P_fA \qquad \text{(optimal feedback gain)}$$
$$\Pi = Q + A'P_fA - A'P_fB(B'P_fB + R)^{-1}B'P_fA \quad \text{(Riccati iteration)}$$

and the closed-loop system satisfies

$$x^+ = Ax + Bu = (A + BK)x$$

DAR problem assumptions

Assumptions:

- (A, B) stabilizable and (A, Q) detectable
- Range condition $\mathcal{R}(G) \subseteq \mathcal{R}(B)$
- $Q \succ 0,\ R \succ 0$

One-stage DAR problem

$$\min_u \max_w V(x,u,w) \quad \text{s.t. } x^+ = Ax + Bu + Gw \quad |w|^2 \le \alpha$$

Eliminate x^+ with the linear model

$$V(x,u,w) = (1/2)\big(|x|_Q^2 + |u|_R^2 + |Ax + Bu + Gw|_{P_f}^2 \big)$$

This is quadratic in (u, w)

$$V(x,u,w) = (1/2)\left(\begin{bmatrix} u \\ w \end{bmatrix}' \begin{bmatrix} B'P_fB + R & B'P_fG \\ (B'P_fG)' & G'P_fG \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} + 2 \begin{bmatrix} u \\ w \end{bmatrix}' \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A x + |x|_{Q+A'P_fA}^2 \right)$$

Proposition 1

Without loss of generality, $|w|^2 \le \alpha$ *can be replaced by* $|w|^2 = \alpha$

Lagrangian approach

Define the Lagrange multiplier $\lambda > 0$ and Lagrangian function

$$L(x, u, w, \lambda) = V(x, u, w) - (\lambda/2)(w'w - \alpha)$$

The constrained problem becomes

$$\min_u \max_w \min_\lambda L(x, u, w, \lambda)$$

With strong duality, we can switch to

$$\min_\lambda \min_u \max_w L(x, u, w, \lambda)$$

One-stage DAR problem

$$\min_{\lambda} \min_{u} \max_{w} L(x, u, w, \lambda)$$

$$L(x,u,w,\lambda) = (1/2)\Bigg(\begin{bmatrix} u \\ w \end{bmatrix}' \begin{bmatrix} B'P_f B + R & B'P_f G \\ (B'P_f G)' & G'P_f G - \lambda I \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} + \\ 2 \begin{bmatrix} u \\ w \end{bmatrix}' \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A\, x + |x|^2_{Q+A'P_f A} \Bigg) + (\lambda\alpha)/2$$

Solve $\min_u \max_w L(x, u, w, \lambda)$ and obtain saddle point $(u^*(\lambda), w^*(\lambda))$, which requires

$$B'P_f B + R \succeq 0 \qquad \text{convex in } u$$

$$G'P_f G - \lambda I \preceq 0 \qquad \text{concave in } w$$

One-stage DAR problem scalar minimization

Applying Theorem 9 (minmax fundamentals lecture), the remaining optimization is

$$\min_{\lambda} L(x, u^*(\lambda), w^*(\lambda)), \lambda) \quad \text{s.t. } G' P_f G - \lambda I \preceq 0$$

The optimal $\lambda^*(x, \alpha)$ solves

$$\min_{\lambda \geq |G' P_f G|} \left(\frac{1}{2} (\frac{x}{\sqrt{\alpha}})' \Pi(\lambda) (\frac{x}{\sqrt{\alpha}}) + \frac{\lambda}{2} \right)$$

with Riccati iteration

$$\Pi(\lambda) = Q + A' P_f A - A' P_f \begin{bmatrix} B & G \end{bmatrix} \begin{bmatrix} B' P_f B + R & B' P_f G \\ (B' P_f G)' & G' P_f G - \lambda I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A$$

One-stage DAR problem solutions

From Theorem 9

$$\begin{bmatrix} u^*(x) \\ z \end{bmatrix} = \begin{bmatrix} K(x,\alpha) \\ J(x,\alpha) \end{bmatrix} x$$

(optimal control is nonlinear)
(inner max dummy variable)

$$V^*(x) = \frac{1}{2}\left(\frac{x}{\sqrt{\alpha}}\right)' \Pi(\lambda^*(x,\alpha)) \left(\frac{x}{\sqrt{\alpha}}\right) + \frac{\lambda}{2}$$

(optimal cost is nonquadratic)

with optimal gains

$$K(x,\alpha) = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} B'P_fB + R & B'P_fG \\ (B'P_fG)' & G'P_fG - \lambda^*(x,\alpha)I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_fA$$

$$J(x,\alpha) = -\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} B'P_fB + R & B'P_fG \\ (B'P_fG)' & G'P_fG - \lambda^*(x,\alpha)I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_fA$$

and the closed-loop system satisfies

$$x^+ = Ax + Bu + Gw = (A + BK(x,\alpha))x + Gw$$

Summary

Objectives

LQR: $\min_{u} \; V(x,u)$

DAR: $\min_{u} \; \max_{w} \; V(x,u,w) \quad |w|^2 = \alpha$

$$\lambda^*(x,\alpha) = \arg \min_{\lambda \geq |G' P_f G|} \frac{1}{2}\Big(\frac{x}{\sqrt{\alpha}}\Big)' \Pi(\lambda) \Big(\frac{x}{\sqrt{\alpha}}\Big) + \frac{\lambda}{2}$$

Gains

LQR: $K = -(B' P_f B + R)^{-1} B' P_f A$

DAR: $K(x,\alpha) = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} B' P_f B + R & B' P_f G \\ G' P_f B & G' P_f G - \lambda^* I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A$

Riccati iteration

LQR: $\Pi = Q + A' P_f A - A' P_f B (B' P_f B + R)^{-1} B' P_f A$

DAR: $\Pi(x,\alpha) = Q + A' P_f A - A' P_f \begin{bmatrix} B & G \end{bmatrix} \begin{bmatrix} B' P_f B + R & B' P_f G \\ G' P_f B & G' P_f G - \lambda^* I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A$

An important number: $|x| / \sqrt{\alpha}$

The optimal DAR control $u^*(x, \alpha)$ depends on x and α

- Parameter α tells us the size of the disturbance
- Parameter x tells us the size of the initial state
- The ratio

$$\frac{|x|}{\sqrt{\alpha}}$$

  controls the scaling between the size of the disturbance and the size of the initial state

DAR solution for $x = 0$

The optimal $\lambda^*(x, \alpha)$ solves

$$\min_{\lambda \geq |G' P_f G|} \left(\frac{1}{2} (\frac{x}{\sqrt{\alpha}})' \Pi(\lambda) (\frac{x}{\sqrt{\alpha}}) + \frac{\lambda}{2} \right)$$

For $x = 0$, $\lambda^*(0, \alpha)$ solves

$$\min_{\lambda \geq |G' P_f G|} \frac{\lambda}{2}$$

Thus

$$\lambda^* = \left| G' P_f G \right| \qquad x = 0, \quad \text{all } \alpha > 0$$

Solution region $\mathcal{X}_L$

Definition 2 (Region $\mathcal{X}_L$)

For a given α, $\mathcal{X}_L$ is the region of initial states x where $\lambda^*(x) = |G'P_f G|$

Proposition 3

For a given α, the optimal control $u^*(x) = Kx$ is linear x in the region $\mathcal{X}_L$

$$K = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} B'P_f B + R & B'P_f G \\ G'P_f B & G'P_f G - |G'P_f G|\, I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A$$

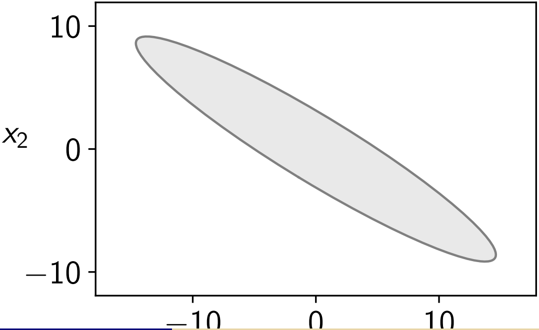
Note: $x = 0$ is always contained in $\mathcal{X}_L$

Solution region $\mathcal{X}_L$

The region $\mathcal{X}_L$ is an ellipsoid centered at the origin. Set $w'w \leq \alpha$

$$x' \, J' J \, x \leq \alpha$$

$$J = -\begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} B'P_f B + R & B'P_f G \\ G'P_f B & G'P_f G - |G'P_f G| \, I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A$$



Solution region $\mathcal{X}_{NL}$

Definition 4 (Region $\mathcal{X}_{NL}$)

For a given α, $\mathcal{X}_{NL}$ is the region of initial states x where $\lambda^*(x) > |G' P_f G|$

Proposition 5

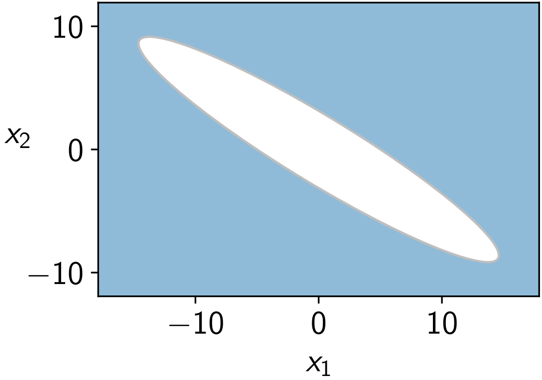
For a given α, the optimal control $u^*(x) = K(x)x$ is nonlinear x in the region $\mathcal{X}_{NL}$

$$K(x) = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} B' P_f B + R & B' P_f G \\ G' P_f B & G' P_f G - \lambda^*(x) I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A$$

Solution region $\mathcal{X}_{NL}$

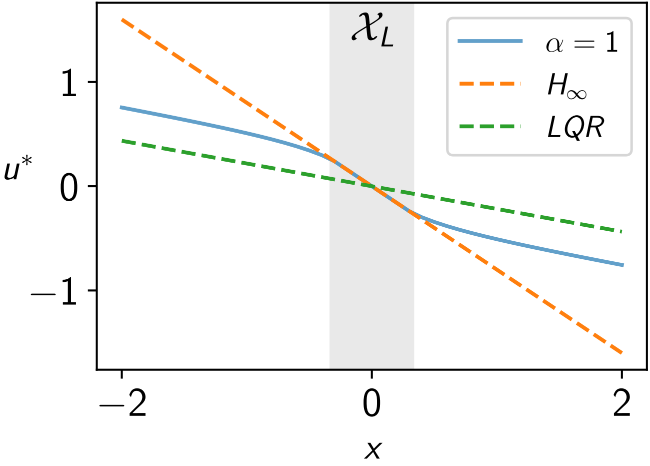
The region $\mathcal{X}_{NL}$ is the exterior of ellipsoid $\mathcal{X}_L$

$$x' \, J' J \, x > \alpha$$



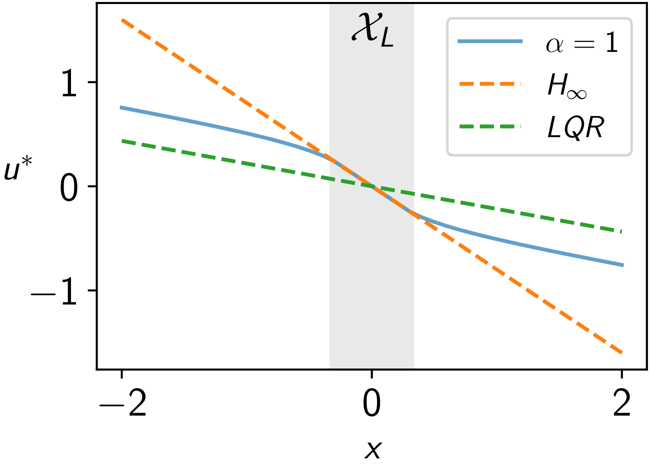
DAR optimal control

The DAR solution is a nonlinear control

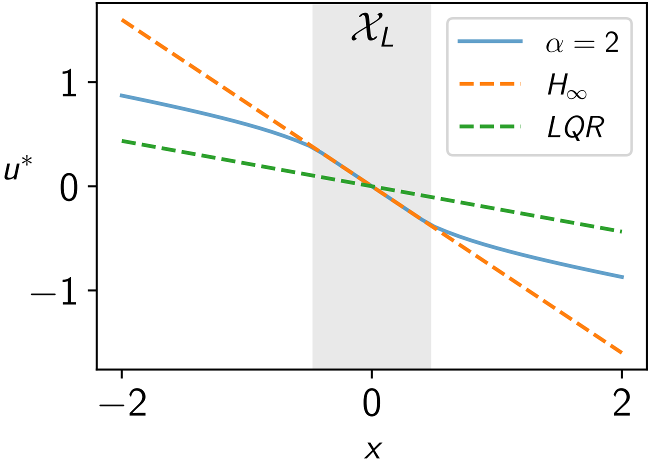


DAR optimal control: $\alpha = 1$

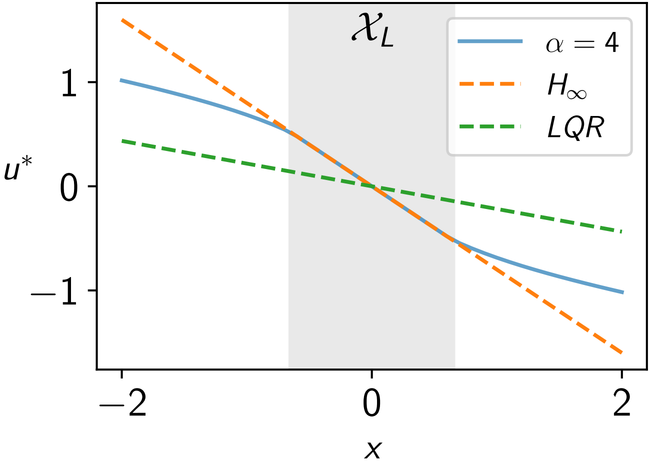
Let's increase α



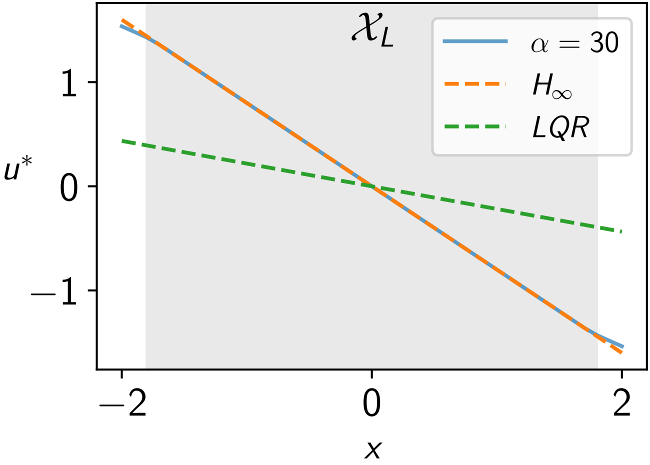
DAR optimal control: $\alpha = 2$



DAR optimal control: $\alpha = 4$

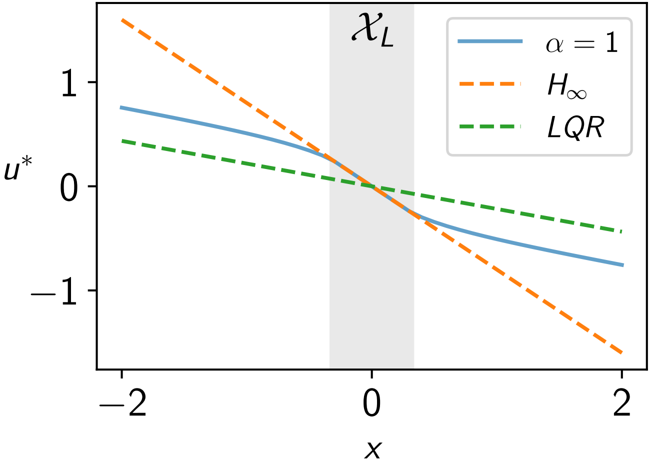


DAR optimal control: $\alpha = 30$

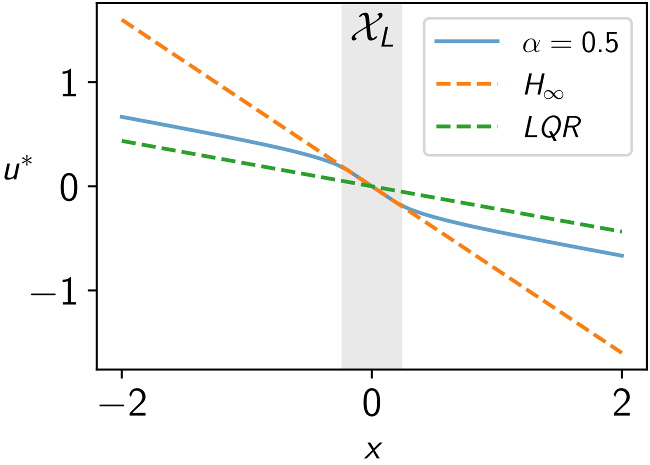


DAR optimal control: $\alpha = 1$

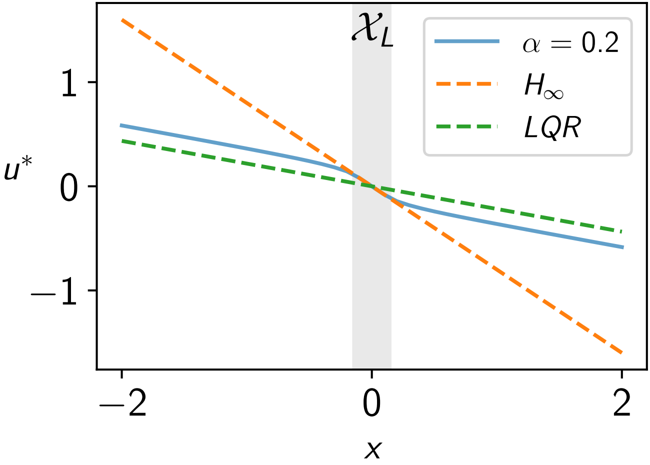
Let's decrease α



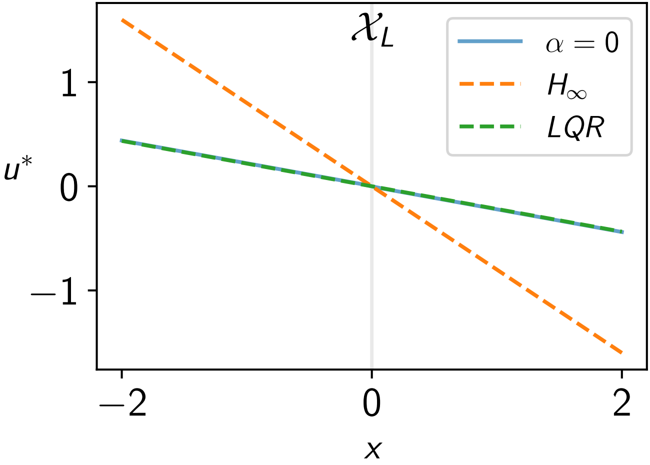
DAR optimal control: $\alpha = 0.5$



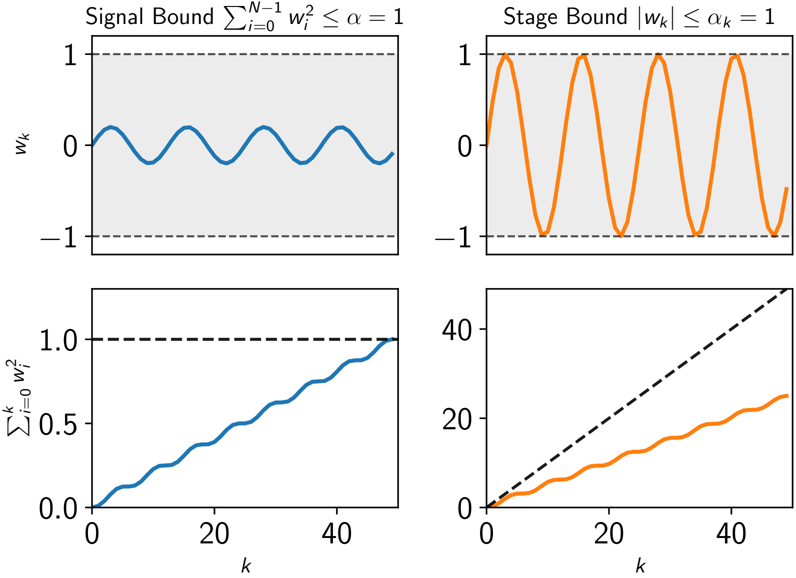
DAR optimal control: $\alpha = 0.2$



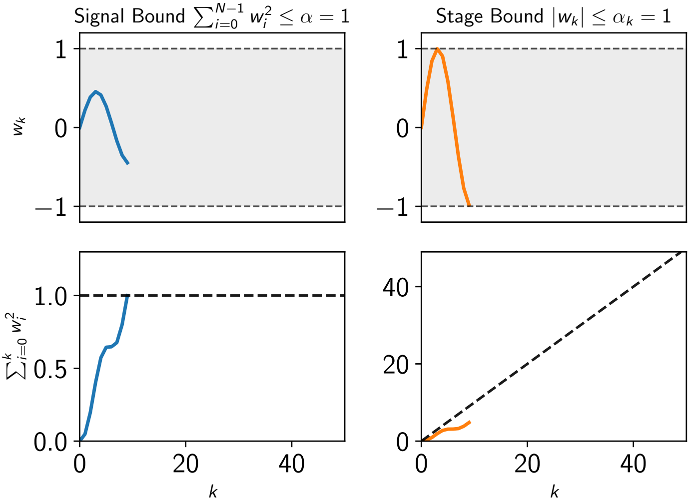
DAR optimal control: $\alpha = 0$



Two disturbance paradigms



Two disturbance paradigms horizon visualization



Finite horizon DAR

Given a horizon length N, the finite horizon DAR problem is

$$V^*(x_0) := \min_{u_0} \max_{w_0} \min_{u_1} \max_{w_1} \cdots \min_{u_{N-1}} \max_{w_{N-1}} V(x_0, \boldsymbol{u}, \boldsymbol{w}) \qquad \boldsymbol{w} \in \mathbb{W}$$

where $\mathbb{W}$ enforces either a signal bound or stage bound disturbance constraint

For $k = 0, 1, \dots, N-1$

$$\min_{\lambda \in [\lambda_1, \infty)} \frac{1}{2}\begin{pmatrix} \frac{x_0}{\sqrt{\alpha}} \end{pmatrix}' \Pi_0(\lambda) \begin{pmatrix} \frac{x_0}{\sqrt{\alpha}} \end{pmatrix} + \frac{\lambda}{2}$$

$$\lambda_k := \begin{cases} \min\limits_{\lambda \geq \lambda_{k+1}} \left\{ \lambda : \lambda = \left| G' \Pi_{k+1}(\lambda) G \right| \right\} & \text{if} \quad |G' \Pi_{k+1}(\lambda_{k+1}) G| > \lambda_{k+1} \\ \lambda_{k+1} & \text{if} \quad |G' \Pi_{k+1}(\lambda_{k+1}) G| \leq \lambda_{k+1} \end{cases}$$

$$\lambda_N := \left| G' P_f G \right| \qquad \Pi_N = P_f$$

$$\Pi_k(\lambda) = Q + A' \Pi_{k+1} A$$
$$- A' \Pi_{k+1} \begin{bmatrix} B & G \end{bmatrix} \begin{bmatrix} B' \Pi_{k+1} B + R & B' \Pi_{k+1} G \\ (B' \Pi_{k+1} G)' & G' \Pi_{k+1} G - \lambda I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} \Pi_{k+1} A$$

Finite horizon DAR: stage-bounded

For $k = 0, 1, \ldots, N-1$

$$\min_{\lambda_k} \; \frac{1}{2}\left(\frac{x_k}{\sqrt{\alpha}}\right)' \Pi_k(\lambda_k)\left(\frac{x_k}{\sqrt{\alpha}}\right) + \frac{\lambda_k}{2}$$

$$\lambda_k \geq \left|G'\Pi_{k+1}G\right| \qquad \Pi_N = P_f$$

$$\begin{aligned}\Pi_k(\lambda_k) &= Q + A'\Pi_{k+1}A \\ &\quad - A'\Pi_{k+1}\begin{bmatrix} B & G \end{bmatrix}\begin{bmatrix} B'\Pi_{k+1}B + R & B'\Pi_{k+1}G \\ (B'\Pi_{k+1}G)' & G'\Pi_{k+1}G - \lambda_k I \end{bmatrix}^{-1}\begin{bmatrix} B' \\ G' \end{bmatrix}\Pi_{k+1}A\end{aligned}$$

Computation: signal-bounded

Decision variables: $\lambda \in \mathbb{R}$

Minimize (final scalar solve):

$$\min_{\lambda \geq \lambda_1} \frac{1}{2}\left(\frac{x_0}{\sqrt{\alpha}}\right)' \Pi_0(\lambda)\left(\frac{x_0}{\sqrt{\alpha}}\right) + \frac{\lambda}{2}.$$

$$M_k(\lambda) = \begin{bmatrix} B'\Pi_{k+1}B + R & B'\Pi_{k+1}G \\ G'\Pi_{k+1}B & G'\Pi_{k+1}G - \lambda I \end{bmatrix} \qquad \phi_{k+1}(\lambda) := \left|G'\Pi_{k+1}(\lambda)G\right|$$

$$\lambda_k := \begin{cases} \min\limits_{\lambda \geq \lambda_{k+1}} \{\ \lambda :\ \lambda = \phi_{k+1}(\lambda)\ \} & \text{if } \ \phi_{k+1}(\lambda_{k+1}) > \lambda_{k+1} \\ \lambda_{k+1} & \text{otherwise} \end{cases}$$

Subject to: $\Pi_N = P_f \quad \lambda_N := |G'P_f G| \quad k = N-1, \ldots, 1$

$$\Pi_k(\lambda) = Q + A'\Pi_{k+1}A - A'\Pi_{k+1}\begin{bmatrix} B & G \end{bmatrix} M_k(\lambda)^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} \Pi_{k+1}A$$

Computation: stage-bounded

Decision variables: $\lambda_0, \ldots, \lambda_{N-1} \in \mathbb{R}$

Minimize:

$$\min_{\lambda_{0:N-1}} \ \frac{1}{2}\left(\frac{x_0}{\sqrt{\alpha}}\right)' \Pi_0(\lambda_{0:N-1})\left(\frac{x_0}{\sqrt{\alpha}}\right) + \frac{1}{2}\sum_{k=0}^{N-1}\lambda_k$$

$$M_k(\lambda_k) = \begin{bmatrix} B'\Pi_{k+1}B + R & B'\Pi_{k+1}G \\ G'\Pi_{k+1}B & G'\Pi_{k+1}G - \lambda_k I \end{bmatrix}$$

Subject to:

$$\Pi_N = P_f \qquad \lambda_k \ \geq \ \left|G'\Pi_{k+1}(\lambda_{k+1:N-1})G\right| \qquad k = N-1, \ldots, 0$$

$$\Pi_k(\lambda_{k:N-1}) = Q + A'\Pi_{k+1}A - A'\Pi_{k+1}\begin{bmatrix} B & G \end{bmatrix} M_k(\lambda_k)^{-1}\begin{bmatrix} B' \\ G' \end{bmatrix}\Pi_{k+1}A$$

Computation: signal-bounded (alternative)

Decision variables: $\lambda \in \mathbb{R}$, $\Pi_0, \ldots, \Pi_{N-1} \in \mathbb{R}^{n \times n}$
Minimize:

$$\frac{1}{2}\left(\frac{x_0}{\sqrt{\alpha}}\right)' \Pi_0 \left(\frac{x_0}{\sqrt{\alpha}}\right) + \frac{\lambda}{2}$$

Subject to:

$$\begin{aligned} \Pi_N &= P_f \\ \lambda I - G' \Pi_{k+1} G &\succeq 0 \\ \Pi_k &= Q + A'\Pi_{k+1}A - A'\Pi_{k+1}[B \;\; G] M_k^{-1} [B' \;\; G']' \Pi_{k+1} A \end{aligned}$$

where

$$M_k = \begin{bmatrix} B'\Pi_{k+1}B + R & B'\Pi_{k+1}G \\ G'\Pi_{k+1}B & G'\Pi_{k+1}G - \lambda I \end{bmatrix}$$

Decision variables: $\lambda_0, \ldots, \lambda_{N-1} \in \mathbb{R}$, $\Pi_0, \ldots, \Pi_{N-1} \in \mathbb{R}^{n \times n}$
Minimize:

$$\frac{1}{2}\left(\frac{x_0}{\sqrt{\alpha}}\right)' \Pi_0 \left(\frac{x_0}{\sqrt{\alpha}}\right) + \frac{1}{2}\sum_{k=0}^{N-1} \lambda_k$$

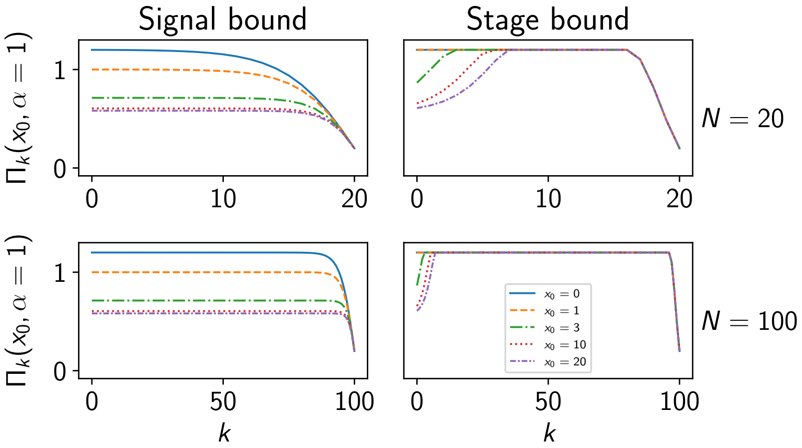
Subject to:

$$\begin{aligned} \Pi_N &= P_f \\ \lambda_k I - G'\Pi_{k+1} G &\succeq 0 \\ \Pi_k &= Q + A'\Pi_{k+1}A - A'\Pi_{k+1}[B \;\; G] M_k^{-1} [B' \;\; G']' \Pi_{k+1} A \end{aligned}$$

where

$$M_k = \begin{bmatrix} B'\Pi_{k+1}B + R & B'\Pi_{k+1}G \\ G'\Pi_{k+1}B & G'\Pi_{k+1}G - \lambda_k I \end{bmatrix}$$

Two disturbance paradigms: DAR optimal solutions



$\Pi_{H_\infty} = 1.2 \qquad \Pi_{LQR} = 0.55$

The steady-state problem

Define a steady-state signal bound problem in (Π, λ)

$$\min_{\Pi, \lambda \geq |G'\Pi G|} \left(\frac{1}{2} (\frac{x}{\sqrt{\alpha}})' \Pi (\frac{x}{\sqrt{\alpha}}) + \frac{\lambda}{2} \right)$$

subject to

$$\Pi = Q + A'\Pi A - A'\Pi \begin{bmatrix} B & G \end{bmatrix} \begin{bmatrix} B'\Pi B + R & B'\Pi G \\ (B'\Pi G)' & G'\Pi G - \lambda I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} \Pi A$$

Denote this steady-state solution $(\overline{\Pi}, \overline{\lambda})$

Signal bound disturbances: LMI formulation

The steady-state signal bound DAR problem can be solved efficiently as an LMI

Proposition 6 (Steady-state solution as an LMI)

$$\min_{\lambda,\chi,P,F} \frac{\chi}{2} + \frac{\lambda}{2}$$

subject to:

$$\begin{bmatrix} P & (AP - BF)' & 0 & (P\bar{Q}' - F'\bar{R}')' \\ AP - BF & P & G' & 0 \\ 0 & G & \lambda I & 0 \\ P\bar{Q}' - F'\bar{R}' & 0 & 0 & I \end{bmatrix} \succeq 0 \quad \begin{bmatrix} P & \frac{x}{\sqrt{\alpha}} \\ \frac{x'}{\sqrt{\alpha}} & \chi \end{bmatrix} \succeq 0$$

where

$$K = -FP^{-1} \quad \Pi = P^{-1}$$

This is a convex optimization with $O(n^3)$ complexity!

Signal bound disturbances: degenerate systems

Definition 7 (Degenerate systems)

A system is degenerate if the infinite horizon solution $(\overline{\lambda}, \overline{\Pi})$ obtained from the LMI satisfies for all x

$$\overline{\lambda} > \left| G'\overline{\Pi}G \right|$$

For x contained in the region $\mathcal{X}_L$, the finite horizon optimal solution is on the boundary

$$\lambda^* = \left| G' P_f G \right|$$

Proposition 8

For degenerate systems, the finite horizon DAR problem does **not** *converge as $N \to \infty$ to the steady-state solution obtained from the LMI for states x in the region $\mathcal{X}_L$*

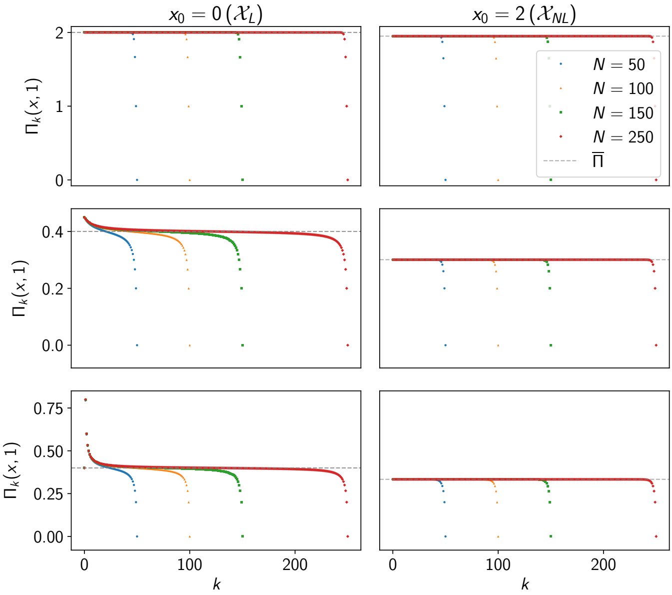
Degenerate systems: numerical examples

Consider three systems

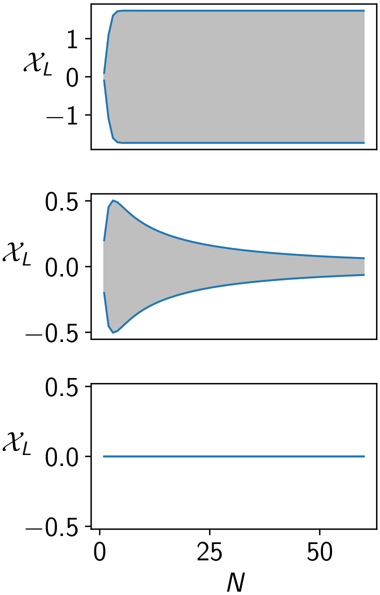
1. $A = 1.0 \quad B = 1 \quad G = 1 \quad R = 1 \quad Q = 1$
2. $A = 0.5 \quad B = 1 \quad G = 1 \quad R = 1 \quad Q = 0.2$
3. $A = 0.5 \quad B = 0 \quad G = 1 \quad R = 1 \quad Q = 0.2$

System 1 is non-degenerate, system 2 is degenerate, and system 3 violates Assumption 2, $\mathcal{R}(G) \subseteq \mathcal{R}(B)$

Degenerate systems: Riccati recursion



Degenerate systems: $\mathcal{X}_L$ regions



Signal bound disturbances: stability

The steady-state DAR solution with **signal bound disturbances** is stabilizing

Proposition 9 (Infinite horizon control is stabilizing)

Any feasible LMI solution yields:

1. *Stabilizing feedback:* $\rho(A + BK) < 1$
2. ℓ_2*-to-*ℓ_2 *gain bound:*

$$\|\boldsymbol{z}\|^2 \leq V(x_0) + (\lambda/2)\|\boldsymbol{w}\|^2$$

$$z := \frac{1}{\sqrt{2}} \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \qquad |z|^2 = \ell(x, u)$$

Connection to H_∞ control: For $x_0 = 0$, the DAR LMI and stability results reduce to the standard H_∞ LMI formulation

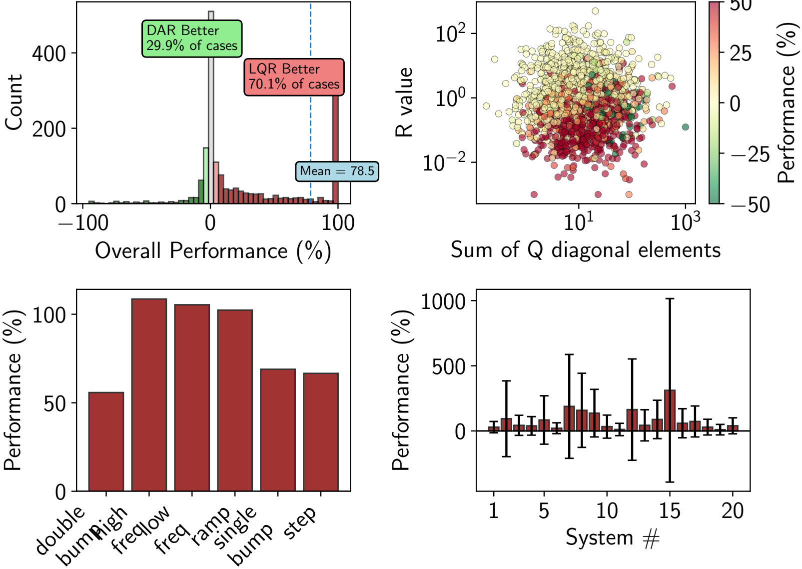
Monte Carlo validation: quarter-car suspension

Quarter-car suspension

$$
\begin{aligned}
m_b \ddot{x}_b &= -k_s(x_b - x_w) - b_s(\dot{x}_b - \dot{x}_w) + u \\
m_w \ddot{x}_w &= k_s(x_b - x_w) + b_s(\dot{x}_b - \dot{x}_w) - k_t(x_w - x_r) - u \\
\dot{x}_b &= v_b \\
\dot{x}_w &= v_w
\end{aligned}
$$

$$
x = \begin{bmatrix} x_b \\ v_b \\ x_w \\ v_w \end{bmatrix} = \begin{bmatrix} \text{body position} \\ \text{body velocity} \\ \text{wheel position} \\ \text{wheel velocity} \end{bmatrix}
$$

$u = F_a =$ actuator force $\qquad w = x_r =$ road disturbance


DAR Better
29.9% of cases
LQR Better
70.1% of cases
Mean = 78.5
Count
0
200
400
−100
0
100
Overall Performance (%)
R value
10^2
10^0
10^{-2}
10^1
10^3
Sum of Q diagonal elements
Performance (%)
50
25
0
−25
−50
Performance (%)
0
50
100
double bump
high freq
low freq
ramp
single bump
step
Performance (%)
1000
500
0
1
5
10
15
20
System #

Monte Carlo validation: CSTR

CSTR

$$\dot{C}_A = \frac{F}{V}(C_{Af} + \delta C_{Af} - C_A) - k_0 e^{-E/RT_r} C_A$$

$$\dot{T}_r = \frac{F}{V}(T_f + \delta T_f - T_r) + \frac{(-\Delta H)}{\rho C_p} k_0 e^{-E/RT_r} C_A - \frac{UA}{\rho C_p V}(T_r - T_j)$$

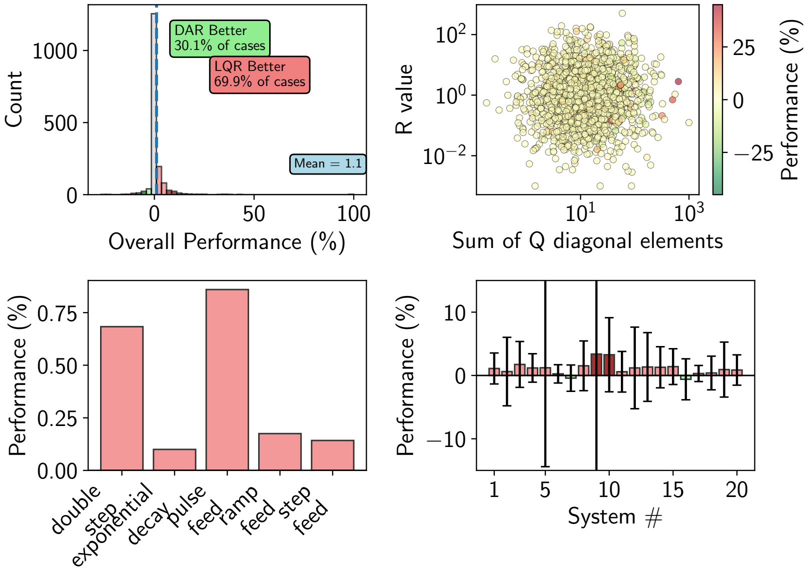
$$\dot{T}_j = \frac{q}{V_j}(T_{c,in} - T_j) + \frac{UA}{\rho_j C_{pj} V_j}(T_r - T_j)$$

$$\dot{q} = \frac{-q + u}{\tau_v}$$

$$x = \begin{bmatrix} C_A \\ T_r \\ T_j \\ q \end{bmatrix} = \begin{bmatrix} \text{concentration} \\ \text{reactor temp} \\ \text{jacket temp} \\ \text{actual coolant flow} \end{bmatrix}$$

$$u = q_{sp} = \text{coolant flow setpoint} \qquad w = \begin{bmatrix} \delta C_{Af} \\ \delta T_f \end{bmatrix} = \text{feed disturbances}$$

System linearized around steady-state operating point


DAR Better
30.1% of cases
LQR Better
69.9% of cases
Mean = 1.1
Count
Overall Performance (%)
R value
Sum of Q diagonal elements
Performance (%)
double step
exponential decay
pulse feed
ramp feed
step feed
System #

Summary

Objectives

LQR: $\min_u \; V(x,u)$

DAR: $\min_u \; \max_w \; V(x,u,w) \quad |w|^2 = \alpha$

$$\lambda^*(x,\alpha) = \arg \min_{\lambda \geq |G'P_f G|} \frac{1}{2}\left(\frac{x}{\sqrt{\alpha}}\right)' \Pi(\lambda) \left(\frac{x}{\sqrt{\alpha}}\right) + \frac{\lambda}{2}$$

Gains

LQR: $K = -(B'P_f B + R)^{-1} B'P_f A$

DAR: $K(x,\alpha) = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} B'P_f B + R & B'P_f G \\ G'P_f B & G'P_f G - \lambda^* I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A$

Riccati iteration

LQR: $\Pi = Q + A'P_f A - A'P_f B (B'P_f B + R)^{-1} B'P_f A$

DAR: $\Pi(x,\alpha) = Q + A'P_f A - A'P_f \begin{bmatrix} B & G \end{bmatrix} \begin{bmatrix} B'P_f B + R & B'P_f G \\ G'P_f B & G'P_f G - \lambda^* I \end{bmatrix}^{-1} \begin{bmatrix} B' \\ G' \end{bmatrix} P_f A$

For signal bound disturbances: DAR infinite horizon solution is stabilizing and obtained with an LMI

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