Robust Dynamic Optimization

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Summer School on Robust Model Predictive Control with CasADi, University of Freiburg
September 15-19, 2025

slides jointly developed with **Moritz Diehl**, Katrin Baumgärtner, Titus Quah, Jim Rawling

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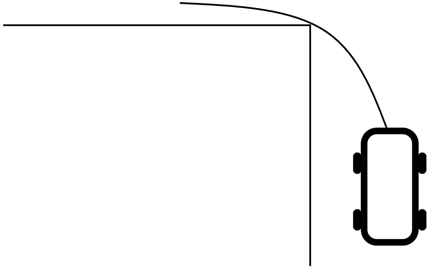
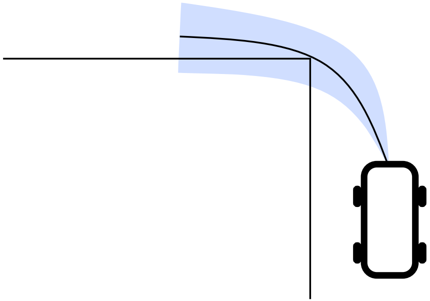
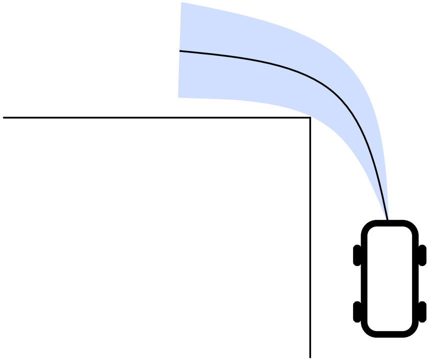


The predicted trajectory cuts the corner tightly, in *nominal MPC*.

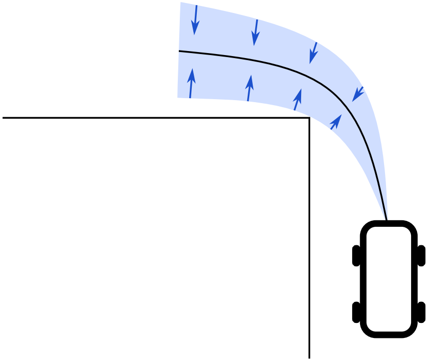


Predicting an uncertainty set (“tube”), we see that the car would often crash.

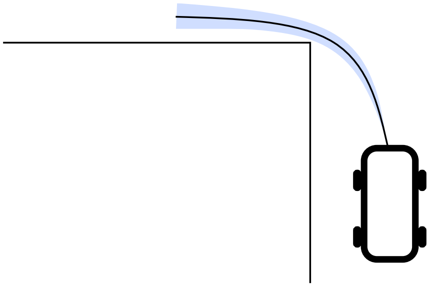


Due to uncertainty, the center of the tube needs to keep a distance ("backoff") from the corner.

This corresponds to *open-loop robust MPC*.



DBut: we know that in the future we will apply feedback.



Considering future feedback allows for a more realistic, less conservative prediction.

This corresponds to *closed-loop robust MPC*.

Three challenges of robust dynamic optimization

When formulating and solving the robust dynamic optimization problems, one needs to address three major challenges:

- **Challenge 1: Robust constraint satisfaction.** How can the state uncertainty be approximated and propagated over the prediction horizon in order to guarantee robust constraint satisfaction?
- **Challenge 2: Feedback predictions.** How can feedback control policies be approximated and incorporated into the robust MPC optimization problem in order to reduce its conservatism?
- **Challenge 3: Dual control.** How can we reduce uncertainty by systematically and purposefully collecting information? (explore-exploit-tradeoff)

In this lecture, we address Challenges 1 and 2.
Challenge 3 will be addressed on Friday.

Uncertain optimal control problem statement

Uncertain optimal control problem in discrete time

$$
\begin{aligned}
\min_{x,\,u} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_{\mathrm{f}}(x_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0, \\
& x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, \ldots, N-1, \\
& 0 \geq h(x_k, u_k), \quad k = 0, \ldots, N-1, \\
& 0 \geq r(x_N).
\end{aligned}
$$

- The future disturbance trajectory $w = (w_0, \ldots, w_{N-1})$ is unknown, such that the above OCP is insufficiently specified.
- Otherwise, we could simply solve a standard OCP.
- Instead, we robustify the OCP against all possible $w \in \mathbb{W}$ for a given set $\mathbb{W} \subset \mathbb{R}^{n_w}$.
- ... facing the three challenges of robust dynamic optimization.

Three perspectives

We consider three perspectives in order to address the challenges. They are not mutually exclusive and sometimes go hand-in-hand or yield the same answers.

- **Perspective 1: Robust optimization.** Bring OCP into high-level standard form and use results from the *Robust Optimization* or *Minmax MPC* lecture.
- **Perspective 2: OCP with set-valued trajectories.** Explicitly predict and compute sets of values that the state trajectory may take.
- **Perspective 3: Robust dynamic programming.** Describe solution via DP recursion / Bellman operator. Especially important as a conceptual tool.

Perspective 1: Robust Optimization

Eliminate state trajectory – as in single shooting – via a recursion started at $\tilde{x}_0(u,w) := \bar{x}_0$ and looping through the state transitions $\boxed{\tilde{x}_{k+1}(u,w) := f(\tilde{x}_k(u,w), u_k, w_k)}$ for $k = 0, \ldots, N-1$:

Open-loop min-max robust OCP (as in single shooting)

$$\begin{aligned} \min_{u} \ \max_{w \in \mathbb{W}} \ & \sum_{k=0}^{N-1} \ell(\tilde{x}_k(u,w), u_k) + V_\mathrm{f}(\tilde{x}_N(u,w)) \\ \text{s.t.} \ \max_{w \in \mathbb{W}} \ & h(\tilde{x}_k(u,w), u_k) \leq 0, \quad k = 0, \ldots, N-1 \\ \max_{w \in \mathbb{W}} \ & \quad r(\tilde{x}_N(u,w)) \leq 0 \end{aligned}$$

Identify the cost with $F_0(u,w)$ and the constraints componentwise with $F_i(u,w)$:

$$\boxed{\min_{u} \max_{w \in \mathbb{W}} F_0(u,w) \quad \text{s.t.} \quad \max_{w \in \mathbb{W}} F_i(u,w) \leq 0, \quad i = 1, \ldots, n_F}$$

Thus, all methods from the *Robust Optimization* lecture apply. We will look at their specific instantiation later.

Perspective 2: OCP with set-valued trajectories

Set-based robust OCP

$$\begin{aligned}
\min_{\mathbb{X},\,\pi(\cdot)} \quad & \sum_{k=0}^{N-1} \mathcal{L}(\mathbb{X}_k, \pi_k(\cdot)) + \mathcal{L}_\mathrm{f}(\mathbb{X}_N) \\
\text{s.t.} \quad & \mathbb{X}_0 = \{\bar{x}_0\}, \\
& \mathbb{X}_{k+1} = \mathcal{F}(\mathbb{X}_k, \pi_k(\cdot)), && k = 0, \dots, N-1, \\
& 0 \geq h(x_k, \pi_k(\cdot)), \quad \forall x_k \in \mathbb{X}_k, && k = 0, \dots, N-1, \\
& 0 \geq r(x_N), \quad \forall x_N \in \mathbb{X}_N.
\end{aligned}$$

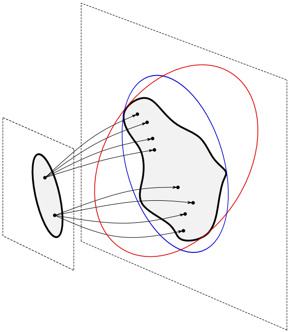
- Set dynamics: $\boxed{\mathcal{F}(\mathbb{X}_k, \pi_k(\cdot)) = \{ f(x_k, \pi_k(x_k), w_k) \mid x_k \in \mathbb{X}_k, w_k \in \bar{\mathbb{W}} \}}$
- Feedback policy: $\boxed{u_k = \pi_k(x_k)}$
  - Careful: Assumes state is exactly observed!
- Assign costs $\mathcal{L}(\mathbb{X}_k, \pi_k(\cdot))$ to set $\mathbb{X}_k$ based on $\ell(x_k, u_k)$, e.g., worst-case or average.
- Covers both tube and scenario-tree approaches.

Perspective 2: OCP with set-valued trajectories

Set-based robust OCP

$$
\begin{aligned}
\min_{\mathbb{X},\,\pi(\cdot)} \quad & \sum_{k=0}^{N-1} \mathcal{L}(\mathbb{X}_k, u_k) + \mathcal{L}_\mathrm{f}(\mathbb{X}_N) \\
\text{s.t.} \quad & \mathbb{X}_0 = \{\bar{x}_0\}, \\
& \mathbb{X}_{k+1} = \mathcal{F}(\mathbb{X}_k, \pi_k(\cdot)), && k = 0, \dots, N-1, \\
& 0 \geq h(x_k, \pi_k(\cdot)), \quad \forall x_k \in \mathbb{X}_k, && k = 0, \dots, N-1, \\
& 0 \geq r(x_N), \quad \forall x_N \in \mathbb{X}_N.
\end{aligned}
$$

- Optimization over policy functions $\pi_k(\cdot)$ makes this an infinite dimensional problem
  - Parametrize feedback law to gain finite dimensional problem.
  - Constant $\pi_k(x_k) \equiv \bar{u}_k$ yields open loop robust OCP.
- Parametrize state sets $\mathbb{X}_k$, e.g., by basic shapes such as ellipsoids or polyhedra.
  - Shape typically not preserved by nonlinear dynamics. Require overapproximation instead: $\mathbb{X}_{k+1} \supseteq \mathcal{F}(\mathbb{X}_k, \pi_k(\cdot))$.

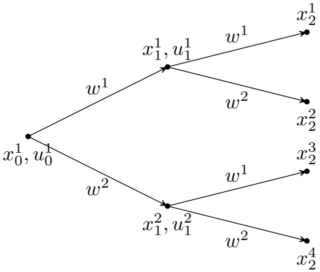


The nonlinear transformation of an ellipsoid is in general not ellipsoidal.

Special case: scenario-tree OCP for finite disturbances

Also known as multistage robust OCP

- In each stage: m disturbance values $\{w^1, \dots, w^m\}$
- Exact state set parameterization $\mathbb{X}_k = \{x_k^1, \dots, x_k^{m^k}\}$
- One control u_k^i for each state x_k^i parametrizes feedback
- "Epigraph slack control" v_k^i collects worst-case objective



Exact scenario-tree OCP

$$\begin{aligned}
\min_{x,u,v} \quad & \ell(x_0^1, u_0^1) + v_0^1 \\
\text{s.t.} \quad & x_0^1 = \bar{x}_0, \\
& x_{k+1}^i = f(x_k^{\lceil i/m^k \rceil}, u_k^{\lceil i/m^k \rceil}, w^{\,i]_1^m}), && k = 0, \dots, N-1, \\
& v_k^{\lceil i/m^k \rceil} \geq \ell(x_{k+1}^i, u_{k+1}^i) + v_{k+1}^i, && i = 1, \dots, m^{k+1}, \\
& 0 \geq h(x_k^{\lceil i/m^k \rceil}, u_k^{\lceil i/m^k \rceil}), && i = 1, \dots, m^k, \\
& 0 \geq r(x_N^j),\ v_N^j \geq V_\mathrm{f}(x_N^j), && j = 1, \dots, m^N.
\end{aligned}$$

$\lceil \cdot \rceil$: ceiling function, $i]_1^m$ wraps i to $\{1, \dots, m\}$.

Given discrete disturbances, scenario trees exactly solve Challenge 1 (constraints) and Challenge 2 (feedback) for **nonlinear** systems! However, we need to deal with exponential scenario growth.

Prelude of Perspective 3: Extended Cost Values

Assign infinite cost to infeasible points, using the extended reals $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$

Constrained OCP

$$\begin{aligned} \min_{x,u} \; & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_\mathrm{f}(s_N) \\ \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & x_{k+1} = f(x_k, u_k, w_k), \\ & 0 \geq h(x_k, u_k), \; k = 0, \dots, N-1, \\ & 0 \geq r(x_N). \end{aligned}$$

Equivalent unconstrained formulation

$$\begin{aligned} \min_{x,u} \; & \sum_{k=0}^{N-1} \bar{\ell}(x_k, u_k) + \bar{V}_\mathrm{f}(x_N) \\ \text{s.t.} \quad & x_0 = \bar{x}_0, \\ & x_{k+1} = f(x_k, u_k, w_k), \; k = 0, \dots, N-1, \end{aligned}$$

with $\bar{\ell}(x,u) = \left\{ \begin{array}{cc} \ell(x,u) & \text{if } h(x,u) \leq 0 \\ \infty & \text{else} \end{array} \right\}$

and $\bar{V}_\mathrm{f}(x) = \left\{ \begin{array}{cc} V_\mathrm{f}(x) & \text{if } r(x) \leq 0 \\ \infty & \text{else} \end{array} \right\}.$

Prelude of Perspective 3: Extended Cost Values

Assign infinite cost to infeasible points, using the extended reals $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$

Equivalent unconstrained formulation

$$\begin{aligned}\min_{x,u} \quad & \sum_{k=0}^{N-1} \bar{\ell}(x_k, u_k) + \bar{V}_\mathrm{f}(x_N) \\ \text{s.t.} \quad & x_0 = \bar{x}_0, \\ & x_{k+1} = f(x_k, u_k, w_k), \; k = 0, \dots, N-1,\end{aligned}$$

with $\bar{\ell}(x,u) = \begin{Bmatrix} \ell(x,u) & \text{if } h(x,u) \leq 0 \\ \infty & \text{else} \end{Bmatrix}$

and $\bar{V}_\mathrm{f}(x) = \begin{Bmatrix} V_\mathrm{f}(x) & \text{if } r(x) \leq 0 \\ \infty & \text{else} \end{Bmatrix}.$

Prelude of Perspective 3: Extended Cost Values

Assign infinite cost to infeasible points, using the extended reals $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$

Equivalent unconstrained formulation

$$\begin{aligned}
\min_{x,u} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_\mathrm{f}(x_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0 \\
& x_{k+1} = f(x_k, u_k, w_k), \; k = 0, \dots, N-1,
\end{aligned}$$

with $\quad \ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \bar{\mathbb{R}}$

and $\quad V_\mathrm{f} : \mathbb{R}^{n_x} \to \bar{\mathbb{R}}$.

Perspective 3: Robust Dynamic Programming (robust DP)

Assume uncertainty is restricted to set $w_k \in \bar{\mathbb{W}}$ in each time step.

Robust DP Recursion

Starting with the terminal cost, iterate backwards using the robust Bellman equation

$$
\begin{aligned}
J_N(x_N) =& V_\mathrm{f}(x_N), \\
J_k(x_k) =& \min_{u_k} \max_{w_k \in \bar{\mathbb{W}}} \ \ell(x_k, u_k) + J_{k+1}(f(x_k, u_k, w_k)), \quad k = N-1, \dots, 0.
\end{aligned}
$$

The corresponding optimal policy is

$$
\pi_k^\star(x_k) = \arg\min_{u_k} \max_{w_k \in \bar{\mathbb{W}}} \ \ell(x_k, u_k) + J_{k+1}(f(x_k, u_k, w_k)).
$$

- Robust DP exactly characterizes the solution of the closed-loop robust OCP without needing to explicitly consider policy parametrizations nor sets in state space.
- Perfectly addresses Challenges 1 to 3, at least conceptually.
- Intractable in this general form, but important conceptual tool, e.g., for proofs.

Monotonicity of Robust Dynamic Programming

The "cost-to-go" J_k is often also called "value function" V_k.

The *robust dynamic programming operator* T mapping between value functions is defined by

$$\boxed{T[J](x) := \min_u \max_{w \in \bar{\mathbb{W}}} \ \ell(x,u) + J(f(x,u,w)).}$$

Write DP recursion compactly as $J_k = T[J_{k+1}]$.

We write $J \geq J'$ if $J(x) \geq J'(x)$ for all $x \in \mathbb{R}^{n_x}$.

One can prove that

$$J \geq J' \quad \Rightarrow \quad T[J] \geq T[J'].$$

This is called "monotonicity" of dynamic programming. It holds also for deterministic or stochastic dynamic programming. It can e.g. be used in existence proofs for solutions of the stationary Bellman equation, or in stability proofs for MPC ($J_N \geq J_{N-1} \Rightarrow J_1 \geq J_0$).

Convex Robust Dynamic Programming

Certain RDP operators T preserve convexity of the value function $J : \mathbb{R}^{n_x} \to \bar{\mathbb{R}}$:

Theorem

If

- system is affine $\boxed{f(x,u,w) = A(w)x + B(w)u + c(w)}$ and
- stage cost $\ell(x,u)$ convex in (x,u),

then the **robust DP operator** T **preserves convexity** of J, i.e.,

$$\boxed{J \text{ convex} \quad \Rightarrow \quad T[J] \text{ convex.}}$$

Note: no assumptions on disturbance set $\bar{\mathbb{W}}$ or on how w enters cost and dynamics.

M. Diehl. Formulation of closed loop min-max MPC as a quadratically constrained quadratic program. *IEEE Transactions on Automatic Control*, 52(2):339–343, 2007

Proof of Convexity Preservation

The function

$$\ell(x, u) + J(\ A(w)x + B(w)u + c(w)\ )$$

is convex in (x, u) for any fixed w, as concatenation of an affine function inside a convex one.

Because the maximum over convex functions (indexed by w) preserves convexity, the function

$$Q(x, u) := \max_{w \in \bar{\mathbb{W}}} \ \ell(x, u) + J(\ A(w)x + B(w)u + c(w)\ )$$

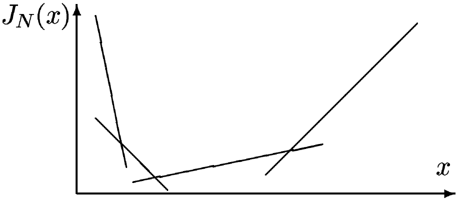
is also convex in (x, u).

Finally, the minimization of a convex function over one of its arguments preserves convexity, i.e., the resulting value function $T[J]$ defined by

$$T[J](x) = \min_{u} Q(x, u)$$

is convex. □

Why is convexity of the value function important?



- Value function $J(x)$ can be represented (or approximated) as the maximum of affine functions with vectors $a_i \in \mathbb{R}^{1+n_x}$ with indices i in some (finite or infinite) set S

$$J(x) = \max_{i \in S} \; a_i^\top \begin{bmatrix} 1 \\ x \end{bmatrix}$$

- Computation of feedback law $\arg\min_u Q(x,u)$ is convex
- Convexity of value function allows us to conclude, in case of polytopic uncertainty, that worst case is assumed on boundary of the polytope, making scenario-tree formulation possible [D.: Formulation of Closed-Loop Min-Max MPC as a QCQP. IEEE TAC 2007]

Overview

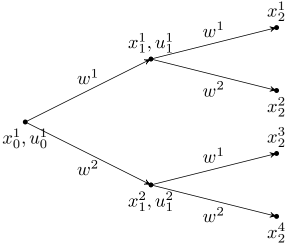
Scenario Tree for Polytopic Systems with Convex Costs and Constraints

Extension of scenario-tree formulation to infinite polytopic disturbance sets, using convexity of RDP cost-to-go

Assume:

- Polytopic uncertainty $\bar{\mathbb{W}} = \mathrm{conv}\{w^1, \ldots, w^m\} \subset \mathbb{R}^{n_w}$,
- Affine dynamics $x_{k+1} = A(w_k)x_k + B(w_k)u_k + c(w_k)$,
- Affine dependence of $A(w), B(w), c(w)$ on $w \in \mathbb{R}^{n_w}$,
- convexity of functions ℓ, h, V_f, r.

Then worst-case is taken in vertices of $\bar{\mathbb{W}}$ and scenario-tree suffices.



Exact Convex Scenario Tree for Polytopic Systems [D., IEEE TAC 2007]

$$
\begin{aligned}
\min_{x,u,v} \quad & \ell(x_0^1, u_0^1) + v_0^1 \\
\text{s.t.} \quad & x_0^1 = \bar{x}_0, \\
& x_{k+1}^i = A\big(w^{i]_1^m}\big)x_k^{\lceil i/m^k \rceil} + B\big(w^{i]_1^m}\big)u_k^{\lceil i/m^k \rceil} + c\big(w^{i]_1^m}\big), \\
& v_k^{\lceil i/m^k \rceil} \geq \ell(x_{k+1}^i, u_{k+1}^i) + v_{k+1}^i, \qquad k = 0, \ldots, N-1, \\
& 0 \geq h(x_k^{\lceil i/m^k \rceil}, u_k^{\lceil i/m^k \rceil}), \qquad i = 1, \ldots, m^{k+1}, \\
& 0 \geq r(x_N^j),\ v_N^j \geq V_\mathrm{f}(x_N^j), \qquad j = 1, \ldots, m^N.
\end{aligned}
$$

Exactly solves Challenge 1 and (implicitly) Challenge 2! However: exponential scenario growth.

(Proof via Perspective 3 and convexity of value function.)
$\lceil \cdot \rceil$: ceiling function, $i]_1^m$ wraps i to $\{1, \ldots, m\}$.

Dual norm formulations for uncertainty affine systems

Back to Perspective 1:
Regard disturbance trajectories $w = (w_0, \ldots, w_{N-1}) \in \mathbb{R}^{Nn_{\bar{w}}}$ in norm ball $\mathbb{W} = \{w \in \mathbb{R}^{n_w} \mid \|w\| \leq 1\}$ for any norm $\|\cdot\|$, with $n_w = Nn_{\bar{w}}$.[1]

Define "single shooting" state trajectory $\tilde{x}_k(u, w)$ at time k as function of (u, w) trajectories, where $u = (\bar{u}_0, \ldots, \bar{u}_{N-1}) \in \mathbb{R}^{n_u}$, and $n_u = Nn_{\bar{u}}$.

For simplicity, omit terminal constraint and uncertainty in objective.

Open-loop robust OCP

$$\begin{aligned} \min_{u} \; & F_0(u) \\ \text{s.t.} \; & \max_{w \in \mathbb{W}} \underbrace{h_j(\tilde{x}_k(u, w), \bar{u}_k)}_{=: F_{k,j}(u,w)} \leq 0, \quad k = 0, \ldots, N-1, \; j = 1, \ldots, n_h. \end{aligned}$$

If functions $F_{k,j}(u, w)$ are affine in uncertainty w, the dual norm formulation is applicable (cf. *Robust Optimization* lecture).

[1] A mixed ℓ_∞-ℓ_p-norm covers the case of independent, stage-wise p-norm bounded uncertainties, $\mathbb{W} = \bar{\mathbb{W}} \times \ldots \times \bar{\mathbb{W}}$ with ℓ_p-norm balls $\bar{\mathbb{W}} = \{\bar{w} \in \mathbb{R}^{n_{\bar{w}}} \mid \|\bar{w}\|_p \leq 1\}$.

Dual norm formulations for uncertainty affine systems

For constraints affine in the uncertainty trajectory we obtain

$$\max_{w \in \mathbb{W}} F_{k,j}(u,w) \quad = \quad h_j(\tilde{x}_k(u,0), \bar{u}_k) \quad + \quad \|\nabla_w \tilde{x}_k(u,0) \nabla_x h_j(\tilde{x}_k(u,0), \bar{u}_k)\|_*.$$

For uncertainty affine systems $\boxed{x_{k+1} = a(u_k) + A(u_k)x_k + \Gamma(u_k)w_k}$

the derivative of state x_k w.r.t. disturbance w_m is given by

$$\boxed{G_{k,m}(u) := \frac{\partial \tilde{x}_k}{\partial w_m}(u,w) = A(u_{k-1}) \cdots A(u_{m+1})\Gamma(u_m)}$$

so that we obtain

$$\max_{w \in \mathbb{W}} F_{k,j}(u,w) \quad = \quad h_j(\tilde{x}_k(u,0), \bar{u}_k) \quad + \quad \left\| \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} \underbrace{\nabla_x h_j(\tilde{x}_k(u,0), \bar{u}_k)}_{=: g_{k,j}(u)} \right\|_* .$$

Dual norm formulations for uncertainty affine systems

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$$\max_{w \in \mathbb{W}} F_{k,j}(u,w) \quad = \quad h_j(\tilde{x}_k(u,0), \bar{u}_k) \quad + \quad \|\nabla_w \tilde{x}_k(u,0) \nabla_x h_j(\tilde{x}_k(u,0), \bar{u}_k)\|_*.$$

For uncertainty affine systems $\boxed{x_{k+1} = a(u_k) + A(u_k)x_k + \Gamma(u_k)w_k}$

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In detail, this looks different for different norms...

Infinity Norm – Exact Dual Norm Formulation

Dual of infinity norm is ℓ_1-norm.

$$\left\| \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} g_{k,j}(u) \right\|_1 = \sum_{m=0}^{k-1} \|G_{k,m}(u)^\top g_{k,j}(u)\|_1$$

This formulation is very expensive, because one needs to compute all matrices $G_{k,m}(u)$ for $k = 1, \ldots, N-1$ and $m = 0, \ldots, k-1$, corresponding to $O(N^2 n_x n_{\bar{w}})$ extra variables.

Infinity Norm – Exact Dual Norm Formulation

Dual of infinity norm is ℓ_1-norm.

Exact open-loop robust OCP for ℓ_∞-norm bounded disturbances

$$\begin{aligned} \min_{u} \quad & F_0(u) \\ \text{s.t.} \quad & h_j(\tilde{x}_k(u,w), \bar{u}_k) + \sum_{m=0}^{k-1} \|G_{k,m}(u)^\top g_{k,j}(u)\|_1 \leq 0, \\ & k = 0, \dots, N-1, \ j = 1, \dots, n_h. \end{aligned}$$

This formulation is very expensive, because one needs to compute all matrices $G_{k,m}(u)$ for $k = 1, \dots, N-1$ and $m = 0, \dots, k-1$, corresponding to $O(N^2 n_x n_{\bar{w}})$ extra variables.

Euclidean Norm – Exact Formulation

Euclidean ℓ_2-norm is self-dual, so its dual is also the ℓ_2-norm.

$$\left\| \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} g_{k,j}(u) \right\|_2^2 = g_{k,j}(u)^\top \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix}^\top \begin{bmatrix} G_{k,0}(u)^\top \\ \vdots \\ G_{k,k-1}(u)^\top \\ 0 \\ \vdots \end{bmatrix} g_{k,j}(u)$$

$$= g_{k,j}(u)^\top \left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top \right) g_{k,j}(u)$$

Euclidean Norm – Exact Formulation

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Exact open-loop robust OCP for ℓ_2-norm bounded disturbances

$$
\begin{aligned}
\min_{u} \quad & F_0(u) \\
\text{s.t.} \quad & h_j(\tilde{x}_k(u,w), \bar{u}_k) + \sqrt{g_{k,j}(u)^\top \left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top \right) g_{k,j}(u)} \quad \leq \quad 0, \\
& \qquad k = 0, \dots, N-1, \; j = 1, \dots, n_h.
\end{aligned}
$$

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Exact open-loop robust OCP for ℓ_2-norm bounded disturbances

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\min_{u} \quad & F_0(u) \\
\text{s.t.} \quad & h_j(\tilde{x}_k(u,w),\bar{u}_k) + \sqrt{g_{k,j}(u)^\top \left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top\right) g_{k,j}(u)} \leq 0, \\
& \qquad k = 0,\dots,N-1,\ j = 1,\dots,n_h.
\end{aligned}
$$

The computations can be much more efficient if one computes the matrix sums differently:

$$
\underbrace{\sum_{m=0}^{k} G_{k+1,m}(u) G_{k+1,m}(u)^\top}_{=P_{k+1}(u)} = A(u_k) \underbrace{\left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top\right)}_{=P_k(u)} A(u_k)^\top + \underbrace{G_{k+1,k}(u) G_{k+1,k}(u)^\top}_{=\Gamma(u_k)\Gamma(u_k)^\top}
$$

Euclidean Norm – Exact Formulation

Euclidean ℓ_2-norm is self-dual, so its dual is also the ℓ_2-norm.

Exact open-loop robust OCP for ℓ_2-norm bounded disturbances

$$
\begin{aligned}
\min_{u} \quad & F_0(u) \\
\text{s.t.} \quad & h_j(\tilde{x}_k(u,w), \bar{u}_k) + \sqrt{g_{k,j}(u)^\top \left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top \right) g_{k,j}(u)} \quad \leq \quad 0, \\
& \qquad k = 0, \dots, N-1, \; j = 1, \dots, n_h.
\end{aligned}
$$

The computations can be much more efficient if one computes the matrix sums differently:

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\underbrace{\sum_{m=0}^{k} G_{k+1,m}(u) G_{k+1,m}(u)^\top}_{=P_{k+1}(u)} = A(u_k) \underbrace{\left(\sum_{m=0}^{k-1} G_{k,m}(u) G_{k,m}(u)^\top \right)}_{=P_k(u)} A(u_k)^\top + \underbrace{G_{k+1,k}(u) G_{k+1,k}(u)^\top}_{=\Gamma(u_k)\Gamma(u_k)^\top}
$$

Start at $P_0(u) := 0 \in \mathbb{R}^{n_x \times n_x}$, compute $\boxed{P_{k+1}(u) := A(u_k) P_k(u) A(u_k)^\top + \Gamma(u_k)\Gamma(u_k)^\top}$

Euclidean Norm – Exact formulation with smart sensitivity computation

Lift formulation, resulting in a sparse NLP in only $\mathcal{O}(N)$ variables: $u = (u_0, \dots, u_{N-1})$, $x = (x_0, \dots, x_N)$, $P = (P_0, \dots, P_N)$, with $P_k \in \mathbb{R}^{n_x \times n_x}$, $P = P^\top$.

Exact open-loop robust OCP for ℓ_2-norm bounded disturbances (lifted)

$$
\begin{aligned}
\min_{u,x,P} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_\mathrm{f}(x_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0, \quad P_0 = 0, \\
& x_{k+1} = f(x_k, u_k, 0) \\
& P_{k+1} = A(x_k)\; P_k\; A(x_k)^\top \; + \; \Gamma(u_k)\Gamma(u_k)^\top \\
& \quad 0 \geq h_j(x_k, u_k) + \sqrt{\nabla_x h_j(x_k, u_k)^\top P_k \nabla_x h_j(x_k, u_k)}, \\
& \qquad\qquad k = 0, \dots, N-1, \; j = 1, \dots, n_h.
\end{aligned}
$$

Euclidean Norm – Exact formulation with smart sensitivity computation

Lift formulation, resulting in a sparse NLP in only $\mathcal{O}(N)$ variables: $u = (u_0, \ldots, u_{N-1})$, $x = (x_0, \ldots, x_N)$, $P = (P_0, \ldots, P_N)$, with $P_k \in \mathbb{R}^{n_x \times n_x}$, $P = P^\top$.

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\begin{aligned}
\min_{u,x,P} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_\mathrm{f}(x_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0, \quad P_0 = 0, \\
& x_{k+1} = f(x_k, u_k, 0) \\
& P_{k+1} = A(x_k)\ P_k\ A(x_k)^\top \ + \ \Gamma(u_k)\Gamma(u_k)^\top \\
& \quad 0 \geq h_j(x_k, u_k) + \sqrt{\nabla_x h_j(x_k, u_k)^\top P_k \nabla_x h_j(x_k, u_k)}, \\
& \qquad\qquad k = 0, \ldots, N-1,\ j = 1, \ldots, n_h.
\end{aligned}
$$

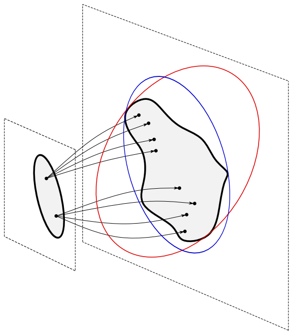
- **Exact** for $f(x, u, w) = a(u_k) + A(u_k)x_k + \Gamma(u_k)w_k$ and $h(x_k, u_k)$ affine in x_k.
- Or: use as **linearization-based approximation** for any nonlinear system $x_+ = f(x, u, w)$.

Tube-based robust OCP

OCP with set-valued trajectory (Perspective 2)

$$
\begin{aligned}
\min_{\mathbb{X},\, \pi} \quad & \sum_{k=0}^{N-1} \mathcal{L}(\mathbb{X}_k, \pi_k) + \mathcal{L}_N(\mathbb{X}_N) \\
\text{s.t.} \quad & \mathbb{X}_0 = \{\bar{x}_0\}, \\
& \mathbb{X}_{k+1} = \mathcal{F}(\mathbb{X}_k, \pi_k), \qquad k = 0, \dots, N-1, \\
& 0 \geq h(x_k, \pi_k(x_k)), \forall x_k \in \mathbb{X}_k, \quad k = 0, \dots, N-1, \\
& 0 \geq r(x_N), \qquad \forall x_N \in \mathbb{X}_N.
\end{aligned}
$$

- Tube-based OCP: parametrize $\mathbb{X}_k$ as continuous, compact, and connected set, e.g.,:
  - **ellipsoids**,
  - various flavors of polyhedra.
- We also need some (simple) parametrization of the policy π.
- Nonlinearity in general leads to non-parametrizable sets $\to$ overapproximate.



Ellipsoidal tubes – dynamics

Consider the linear time-varying system, for $k = 0, \dots, N-1$,

$$x_0 = \bar{\bar{x}}_0, \quad x_{k+1} = A_k x_k + B_k u_k + \Gamma_k w_k, \quad \text{with} \quad w = (w_0, \dots, w_{N-1}) \in \mathbb{W} = \{w \mid \|w\|_2 \leq 1\}.$$

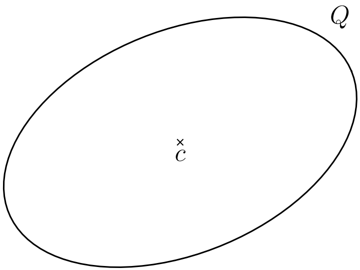
What is the sequence of sets $\mathbb{X}_k$, so that $x_k \in \mathbb{X}_k$ for all disturbance realizations ("tube")?

- **Variant 1, open-loop control trajectory:**
  $\pi_k(x_k) \equiv \bar{u}_k$.
  This results in ellipsoidal state uncertainty sets $\mathbb{X}_k = \mathcal{E}(\bar{x}_k, P_k)$, with

$$\bar{x}_0 = \bar{\bar{x}}_0, \quad \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k,$$
$$P_0 = 0, \quad P_{k+1} = A_k P_k A_k^\top + \Gamma_k \Gamma_k^\top.$$

- **Variant 2, with additional linear feedback:**
  $\pi_k(x_k) = \bar{u}_k + K_k(x_k - \bar{x}_k)$.
  Only the ellipsoid dynamics are modified:

$$P_{k+1} = (A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + \Gamma_k \Gamma_k^\top.$$



Ellipsoids can be defined via center c and shape matrix ("variance") $Q \succ 0$.
$\mathcal{E}(c, Q) := \{x \mid (x-c)^\top Q^{-1}(x-c) \leq 1\}$

Ellipsoidal tubes – constraints

Given ellipsoidal uncertainty set $\mathbb{X}_k = \mathcal{E}(\bar{x}_k, P_k)$, how to treat constraints?

$$\boxed{b + a^\top x_k \leq 0 \quad \forall x_k \in \mathcal{E}(\bar{x}_k, P_k)}$$

Reformulate as

$$b + \max_{x_k \in \mathcal{E}(\bar{x}_k, P_k)} a^\top x_k \leq 0.$$

For affine constraints we can compute the maximum analytically (e.g. via dual norm) as

$$\max_{x_k \in \mathcal{E}(\bar{x}_k, P_k)} a^\top x_k = a^\top \bar{x}_k + \sqrt{a^\top P_k a},$$

resulting in

$$\boxed{b + c^\top \bar{x}_k + \sqrt{a^\top P_k a} \leq 0.}$$

Ellipsoidal tubes – resulting OCP

Ellipsoidal tube OCP for linear systems with linear state feedback (ℓ_2-norm bounded dist.)

$$
\begin{aligned}
\min_{\bar{x}, \bar{u}, P, K} \quad & \sum_{k=0}^{N-1} \ell(\bar{x}_k, \bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \quad P_0 = 0, \\
& \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k, && k = 0, \dots, N-1, \\
& P_{k+1} = (A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + \Gamma_k \Gamma_k^\top, && k = 0, \dots, N-1, \\
& 0 \geq b_i + a_i^\top \bar{x}_k + \sqrt{a_i^\top P_k a_i}, && i = 1, \dots, n_c, \\
& 0 \geq \tilde{b}_j + \tilde{a}_j^\top \bar{u}_k + \sqrt{\tilde{a}_j^\top K_k P_k K_k^\top \tilde{a}_j}, && j = 1, \dots, n_{\tilde{c}}.
\end{aligned}
$$

- For $K = 0$: Same OCP as from dual norm derivation.
- Nonconvex due to optimization over state feedback gains K_k ($\mathcal{O}(N)$ variables).
- If K_k fix, then also P_k fix, resulting in standard OCP with backoff.
- Side remark: Stochastic i.i.d. noise $w_k \sim \mathcal{N}(0, I)$, and chance constraints lead to numerically identical OCP, with stochastic tube $x_k \sim \mathcal{N}(\bar{x}_k, P_k)$.

Affine Disturbance Feedback Parameterization

- Optimization over state feedback matrices K is nonconvex and can be challenging to solve.
- Alternative 2: Disturbance feedback instead of state feedback.

$$\boxed{u_k = \bar{u}_k + \sum_{m=0}^{k-1} M_{k,m} w_m}$$

  - Ties in neatly with Perspective 1 and dual norm formulation.
  - Affinely adjustable robust counterpart[1] for causal structure of OCP.
  - Leads to convex problem for linear systems: $\boxed{x_{k+1} = A_k x_k + B_k u_k + \Gamma_k w_k.}$
  - For linear systems: equivalent to linear state feedback on all past states.[2]
  - Same principle behind system level synthesis (SLS).[3]
  - Many feedback gains $\rightarrow$ large-dimensional, expensive optimization problems ($O(N^2)$ variables).

---

[1] A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004

[2] P. J. Goulart, E. C. Kerrigan, and J. M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42:523–533, 2006

[3] J. Anderson, J. C. Doyle, S. H. Low, and N. Matni. System level synthesis. *Annual Reviews in Control*, 47, 2019

Affine disturbance feedback formulation for ℓ_2-norm

Robust OCP (ℓ_2-norm bounded dist.) with aff. dist. feedback ($\mathcal{O}(N^2)$ variables)

$$\begin{aligned}
\min_{\bar{x}, \bar{u}, G, M} \quad & \sum_{k=0}^{N-1} \ell_k(\bar{x}_k, \bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \\
& \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k, && k = 0, \dots, N-1, \\
& G_{k+1,k} = \Gamma_k, && k = 0, \dots, N-1, \\
& G_{k+1,n} = A_k G_{k,n} + B_k M_{k,n} && n = 0, \dots, k-1, \\
& 0 \geq b_i + a_i^\top \bar{x}_k + \sqrt{a_i^\top \left(\sum_{m=0}^{k-1} G_{k,m} G_{k,m}^\top \right) a_i}, && i = 1, \dots, n_c, \\
& 0 \geq \tilde{b}_j + \tilde{a}_j^\top \bar{u}_k + \sqrt{\tilde{a}_j^\top \left(\sum_{m=0}^{k-1} M_{k,m} M_{k,m}^\top \right) \tilde{a}_j}, && j = 1, \dots, n_{\tilde{c}}.
\end{aligned}$$

For convex quadratic objectives, this is a convex second order cone program (SOCP).

Affine disturbance feedback formulation for ℓ_∞-ℓ_2-norm

Robust OCP (ℓ_∞-ℓ_2-norm bounded dist.) with aff. dist. feedback ($\mathcal{O}(N^2)$ variables)

$$
\begin{aligned}
\min_{\bar{x}, \bar{u}, G, M} \quad & \sum_{k=0}^{N-1} \ell_k(\bar{x}_k, \bar{u}_k) + V_\mathrm{f}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \\
& \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k, && k = 0, \dots, N-1, \\
& G_{k+1,k} = \Gamma_k, && k = 0, \dots, N-1, \\
& G_{k+1,n} = A_k G_{k,n} + B_k M_{k,n} && n = 0, \dots, k-1, \\
& 0 \geq b_i + a_i^\top \bar{x}_k + \sum_{m=0}^{k-1} \| G_{k,m}^\top a_i \|_2, && i = 1, \dots, n_c, \\
& 0 \geq \tilde{b}_j + \tilde{a}_j^\top \bar{u}_k + \sum_{m=0}^{k-1} \| M_{k,m}^\top a_i \|_2, && j = 1, \dots, n_{\tilde{c}}.
\end{aligned}
$$

For convex quadratic objectives, this is a convex second order cone program (SOCP).

Mini summary of the last few slides

- Use dual norm to derive robust counterpart for norm-bounded disturbances (Perspective 1)
  - Typically, we need to compute many derivatives ($\mathcal{O}(N^2)$ variables in lifted form)
  - Special case ℓ_2-norm: Compute derivatives in a smart way to reduce to $\mathcal{O}(N)$ variables!
- Derived ellipsoidal tube formulation for ℓ_2-norm bounded noise (Perspective 2)
  - Without feedback: corresponds to OCP derived via dual norm in Perspective 1.
- Feedback policy parametrizations:

	Aff. state FB	Aff. disturbance FB
Convex for linear systems	No	Yes
Parameters	$\mathcal{O}(N)$	$\mathcal{O}(N^2)$
Ties in neatly with	Perspective 2 (Tube)	Perspective 1 (RO)
	& ℓ_2-norm	& any norm

- All of them: Exact (tight) constraint robustification (Challenge 1), but suboptimal feedback (Challenge 2).

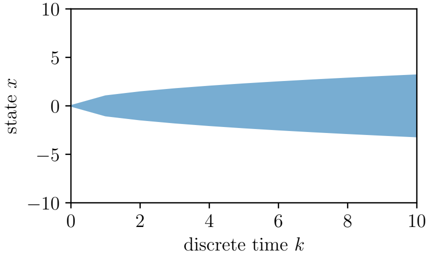
A closer look at the assumptions on w – Case 1

$$x_0 = 0, \quad x_{k+1} = x_k + w_k, \quad k = 0, \dots, N-1, \quad w = (w_0, \dots, w_{N-1}) \in \mathbb{W}$$

- Case 1: Full trajectory is ℓ_2-norm-bounded:

$$\mathbb{W} = \{w \in \mathbb{R}^{N n_w} \mid \|w\|_2 \leq 1\}$$

  - Encodes dependence across time: w_k cannot take an extreme value for all k.
  - Similar effect as i.i.d. assumption in stochastic context.



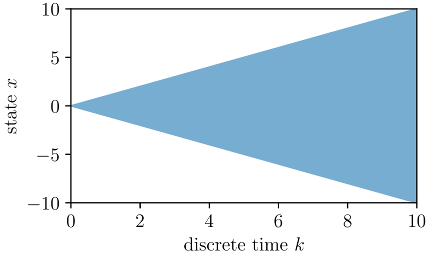
A closer look at the assumptions on w – Case 2

$$\boxed{x_0 = 0, \quad x_{k+1} = x_k + w_k, \quad k = 0, \dots, N-1, \quad w = (w_0, \dots, w_{N-1}) \in \mathbb{W}}$$

- Case 2: Each w_k is ℓ_2-norm-bounded independently:

$$\mathbb{W} = \bar{\mathbb{W}} \times \cdots \times \bar{\mathbb{W}},$$
$$\text{with } \bar{\mathbb{W}} = \{\bar{w} \in \mathbb{R}^{n_{\bar{w}}} \mid \bar{w}^\top \bar{w} \le 1\}$$

  - Encodes independence across time: w_k can take an extreme value for all k.
  - Corresponds to mixed ℓ_∞-ℓ_2-norm bound on full trajectory.



Extending ellipsoidal tubes to independent stage noise?

$$\boxed{x_0 = \bar{\bar{x}}_0, \quad x_{k+1} = A_k x_k + B_k u_k + \Gamma_k w_k.}$$

So far, we assumed $w = (w_0, \ldots, w_N) \in \mathbb{W} = \{w \in \mathbb{R}^{N n_w} \mid \|w\|_2 \leq 1\}$. This contains the assumption that the noise is dependent across time.
Alternative assumption: noise is norm-bounded independently at each time

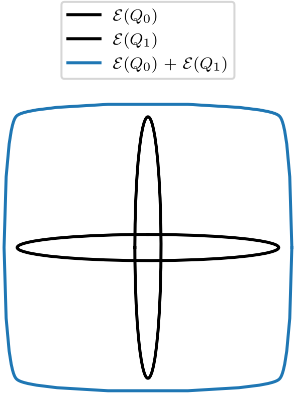
$$\mathbb{W} = \underbrace{\bar{\mathbb{W}} \times \cdots \times \bar{\mathbb{W}}}_{N-\text{times}} \quad \text{with} \quad \bar{\mathbb{W}} = \{w \in \mathbb{R}^{n_{\bar{w}}} \mid w^\top w \leq 1\}.$$

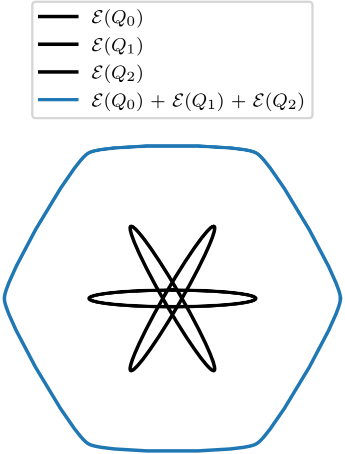
Can in principle be addressed using the affine case with mixed ℓ_∞-ℓ_2-norm, combined with any feedback parameterization – but this is expensive. Can we use ellipsoidal tubes instead?
Assume we have $\mathbb{X}_k = \mathcal{E}(\bar{x}_k, P_k)$. Then

$$\begin{aligned} \mathbb{X}_{k+1} &= A_k \mathbb{X}_k + B_k u_k + \Gamma_k \bar{\mathbb{W}} \\ &= \mathcal{E}(A_k \bar{x}_k + B_k u_k, A_k P_k A_k^\top) + \mathcal{E}(0, \Gamma_k \Gamma_k^\top) \end{aligned}$$

Problem: The sum of two ellipsoids is not an ellipsoid.

Sum of ellipsoids (Minkowski sum)





The sum of ellipsoids is not ellipsoidal.

Overapproximating sum of ellipsoids by ellipsoid

- Aim: find Q such that $\mathcal{E}(Q) \supseteq \mathcal{E}(Q_1) + \mathcal{E}(Q_2)$
- More general: Find Q such that $\mathcal{E}(Q) \supseteq \sum_{k=1}^{N} \mathcal{E}(Q_k)$
- Construct family of outer approximations parametrized by $\alpha \in \mathbb{R}^N_{++}$

$$Q(\alpha) = \sum_{k=1}^{N} \frac{1}{\alpha_k} Q_k \quad \Rightarrow \quad \mathcal{E}(Q(\alpha)) \supseteq \sum_{k=1}^{N} \mathcal{E}(Q_k) \quad \forall \alpha \in \mathbb{R}^N_{++} \quad \text{with} \quad \sum_{k=1}^{N} \alpha_k = 1$$

- Denote set of feasible α by $\mathcal{A}^N$ (basically a simplex)
- Parametrized outer approximation is tight:

$$\bigcap_{\alpha \in \mathcal{A}^N} \mathcal{E}(Q(\alpha)) = \sum_{k=1}^{N} \mathcal{E}(Q_k)$$

A. Kurzhanski and P. Valyi. *Ellipsoidal Calculus for Estimation and Control*. Birkhäuser Boston, 1997
B. Houska. *Robust Optimization of Dynamic Systems*. PhD thesis, KU Leuven, 2011. (ISBN: 978-94-6018-394-2)

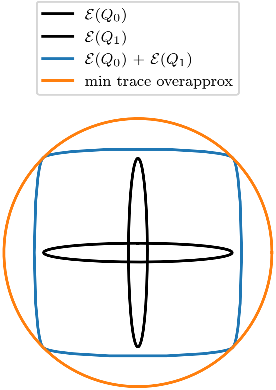
Overapproximating sum of ellipsoids by ellipsoid (cont.)

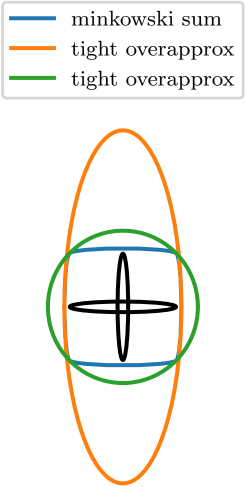
- In general: Choose α according to some criterion
  - e.g., such that $\mathcal{E}(Q(\alpha))$ has minimal size, e.g., $\min_{\alpha \in \mathcal{A}^N} \operatorname{tr}(Q(\alpha))$
  - or $\mathcal{E}(Q(\alpha))$ tight in a given direction $g \in \mathbb{R}^n$ (approximation touches true sum)

$$\min_{\alpha \in \mathcal{A}^N} \left(\max_{x \in \mathbb{R}^n} \; g^\top x \quad \text{s.t.} \quad x \in \mathcal{E}(Q(\alpha)) \right) = \min_{\alpha \in \mathcal{A}^N} \sqrt{g^\top Q(\alpha) g} \;\hat{=}\; \min_{\alpha \in \mathcal{A}^N} \operatorname{tr}(g g^\top Q(\alpha))$$

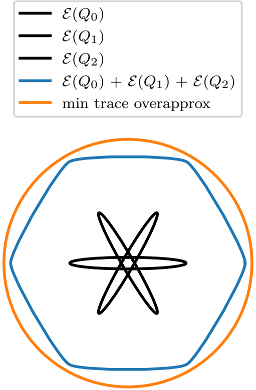
- Special case $N = 2$
  - $Q(\alpha) = \frac{1}{\alpha_1} Q_1 + \frac{1}{\alpha_2} Q_2$ with $\alpha_1 + \alpha_2 = 1$
  - Reparametrize: $\alpha_2 = 1 - \alpha_1$, $\beta = \frac{1}{1-\alpha_1} > 0$
  - $\tilde{Q}(\beta) = (1 + \frac{1}{\beta}) Q_1 + (1 + \beta) Q_2$

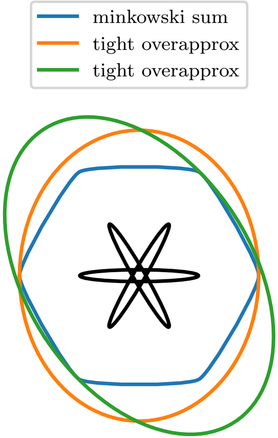
Overapproximations of sum of two ellipsoids





Overapproximations of sum of three ellipsoids





Uncertain linear dynamical system

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k w_k,$$

- Reachable set

$$\begin{aligned} x_k \in \mathcal{E}(\bar{x}_k, P_k),\ w_k \in \bar{\mathbb{W}} \\ \Rightarrow x_{k+1} \in \tilde{\mathbb{X}}_{k+1} = \mathcal{E}(A_k \bar{x}_k + B_k u_k, A_k P_k A_k^\top) + \mathcal{E}(\Gamma_k \Gamma_k^\top) \end{aligned}$$

  - $\tilde{\mathbb{X}}_{k+1}$ not ellipsoidal
  - Overapproximate by ellipsoid

- Overapproximation of reachable set

$$\begin{aligned} P_{k+1} &= (1+\beta_k) A_k P_k A_k^\top + (1 + \tfrac{1}{\beta_k}) \Gamma_k \Gamma_k^\top \\ &\Rightarrow \tilde{\mathbb{X}}_{k+1} \subseteq \mathcal{E}(P_{k+1}) \\ &\Rightarrow x_{k+1} \in \mathcal{E}(P_{k+1}) \end{aligned}$$

Overapproximating tubes for stagewise ellipsoidal uncertainty

Ellipsoidal tube OCP for linear systems with linear state feedback ($\ell_\infty - \ell_2$-norm bounded dist.)

$$
\begin{aligned}
\min_{\bar{x},\bar{u},\beta,P,K} \quad & \sum_{k=0}^{N-1} \ell(\bar{x}_k,\bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \quad P_0 = 0, \quad \beta \geq 0, \\
& \bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k, \qquad k = 0,\dots,N-1, \\
& P_{k+1} = (1+\beta_k)(A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + (1 + (1/\beta_k))\Gamma_k \Gamma_k^\top, \\
& 0 \geq b_i + a_i^\top \bar{x}_k + \sqrt{a_i^\top P_k a_i}, \qquad i = 1,\dots,n_c, \\
& 0 \geq \tilde{b}_j + \tilde{a}_j^\top \bar{u}_k + \sqrt{\tilde{a}_j^\top K_k P_k K_k^\top \tilde{a}_j}, \quad j = 1,\dots,n_{\tilde{c}}.
\end{aligned}
$$

- Conservative (non-tight) constraint satisfaction (Challenge 1) but suboptimal feedback. Nonconvex. $\mathcal{O}(N)$ variables.
- Not the same as – and cheaper than – dual norm formulation for ℓ_∞-ℓ_2-norm.
- Three types of “controls” with two different tasks
  - Nominal $\bar{u} = (\bar{u}_0,\dots,\bar{u}_{N-1})$ influence $\bar{x}_k$.
  - Gains $K = (K_0,\dots,K_{N-1})$ and “Minkowski-multipliers” $\beta = (\beta_0,\dots,\beta_{N-1})$ influence P_k.

Overview

Ellipsoidal tube approximation for robust nonlinear MPC

- We switch to a nonlinear system

$$x_0 = \bar{\bar{x}}_0, \qquad x_{k+1} = f_k(x_k, u_k, w_k), \qquad k = 0, \dots, N-1.$$

- $w = (w_0, \dots, w_{N-1})$ is drawn from ℓ_2-ball with radius σ, i.e., $w \in \mathcal{E}(0, I)$
- Similar approach with ellipsoids as before, but we will only have "approximate robustness" based on linearization at nominal trajectory

$$\bar{x}_0 = \bar{\bar{x}}_0, \qquad \bar{x}_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0)$$

$$A_k = \frac{\partial f_k}{\partial x_k}(\bar{x}_k, \bar{u}_k, 0), \quad B_k = \frac{\partial f_k}{\partial u_k}(\bar{x}_k, \bar{u}_k, 0), \quad \Gamma_k = \frac{\partial f_k}{\partial w_k}(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \dots, N-1.$$

- In principle it is possible to bound the linearization error.

Z. Nagy and R. Braatz. Open-loop and closed-loop robust optimal control of batch processes using distributional and worst-case analysis. *Journal of Process Control*, 14:411–422, 2004

M. Diehl, H. Bock, and E. Kostina. An approximation technique for robust nonlinear optimization. *Mathematical Programming*, 107:213–230, 2006

B. Houska. *Robust Optimization of Dynamic Systems*. PhD thesis, KU Leuven, 2011. (ISBN: 978-94-6018-394-2)

Feedback to reduce the uncertainty

- Plan with linear feedback law to reduce uncertainty

$$u_k = \kappa_k(x_k) = \bar{u}_k + K_k(x_k - \bar{x}_k), \quad k = 0, \dots, N-1, \quad K_0 = 0.$$

- Propagate ellipsoids according to linearized dynamics

$$P_0 = 0, \quad P_{k+1} = \underbrace{(A_k + B_k K_k) P_k (A_k + B_k K_k)^\top + \Gamma_k \Gamma_k^\top}_{=: \psi(\bar{x}_k, \bar{u}_k, P_k, K_k)}$$

- Left out here, but could also generalize to ℓ_∞-ℓ_2-norms by including Minkowski-multipliers, β_k, or to affine disturbance feedback

Nonlinear ellipsoidal-tube OCP with linear state feedback

Nonlinear ellipsoidal-tube OCP with linear state feedback ("approximately robust")

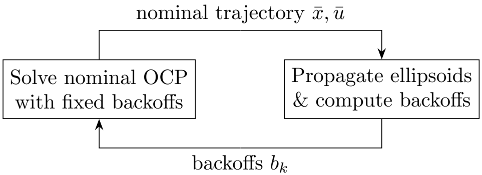
$$
\begin{aligned}
\min_{\bar{x}, \bar{u}, P, K} \quad & \sum_{k=0}^{N-1} \ell_k(\bar{x}_k, \bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \quad P_0 = 0, \\
& \bar{x}_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \dots, N-1, \\
& P_{k+1} = \psi_k(\bar{x}_k, \bar{u}_k, P_k, K_k), \\
& 0 \geq h_k(\bar{x}_k, \bar{u}_k) + b_k(\bar{x}_k, \bar{u}_k, P_k, K_k), \\
& 0 \geq h_N(\bar{x}_N) + b_N(\bar{x}_N, P_N).
\end{aligned}
$$

$$
\begin{aligned}
b_k^i(\bar{x}_k, \bar{u}_k, P_k, K_k) &= \sqrt{\nabla h_k^i(\bar{x}_k, \bar{u}_k)^\top \begin{bmatrix} I & K_k^\top \end{bmatrix}^\top P_k \begin{bmatrix} I & K_k^\top \end{bmatrix} \nabla h_k^i(\bar{x}_k, \bar{u}_k)}, \\
b_N^i(\bar{x}_N, P_N) &= \sqrt{\nabla h_N^i(\bar{x}_N)^\top P_N \nabla h_N^i(\bar{x}_N)},
\end{aligned}
$$

Zero-order robust optimization (zoRO) algorithm

- Consider nonlinear ellipsoidal-tube OCP with fixed feedback gains (e.g. $K = 0$)
  - Augmented state: $x_k \in \mathbb{R}^{n_x}$, $P_k \in \mathbb{R}^{n_x^2}$
  - Controls: $u_k \in \mathbb{R}^{n_u}$
  - Complexity per iteration of standard OCP algorithm: $\mathcal{O}(N n_x^6)$ (Riccati recursion)
- Zero-order robust optimization (zoRO)
  - Alternate between nominal OCP with fixed backoff and uncertainty propagation.
    $\rightarrow$ Complexity per iteration: $\mathcal{O}(N n_x^3)$
  - Neglects some sensitivities, such that it converges to feasible but suboptimal point.
  - Efficient implementation in acados

zoRO



A. Zanelli, J. Frey, F. Messerer, and M. Diehl. Zero-order robust nonlinear model predictive control with ellipsoidal uncertainty sets. *Proceedings of the IFAC Conference on Nonlinear Model Predictive Control (NMPC)*, 2021

J. Frey, Y. Gao, F. Messerer, A. Lahr, M. N. Zeilinger, and M. Diehl. Efficient zero-order robust optimization for real-time model predictive control with acados. In *Proceedings of the European Control Conference (ECC)*, 2024

zoRO iterations

- Iterate between (A) nominal problem with fixed backoffs, and (B) matrix propagation

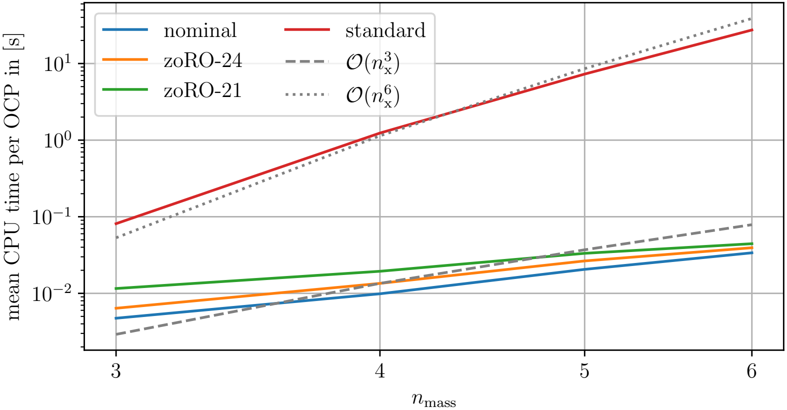
(A) Nominal problem with backoffs – standard NMPC problem

$$\begin{aligned}
\min_{\bar{x},\,\bar{u}} \quad & \sum_{k=0}^{N-1} \ell_k(\bar{x}_k, \bar{u}_k) + V_{\mathrm{f}}(\bar{x}_N) \\
\text{s.t.} \quad & \bar{x}_0 = \bar{\bar{x}}_0, \quad x_{k+1} = f_k(\bar{x}_k, \bar{u}_k, 0), \qquad k = 0, \dots, N-1, \\
& 0 \geq h_k(\bar{x}_k, \bar{u}_k) + b_k, \quad 0 \geq h_N(\bar{x}_N) + b_N.
\end{aligned}$$

(B) Matrix propagation to compute backoffs

$$\begin{aligned}
& P_0 := 0, \quad P_{k+1} := \psi_k(\bar{x}_k, \bar{u}_k, P_k, K_k), \\
& b_k^i := \sqrt{\nabla h_k^i(\bar{x}_k, \bar{u}_k)^\top \begin{bmatrix} I & K_k^\top \end{bmatrix}^\top P_k \begin{bmatrix} I & K_k^\top \end{bmatrix} \nabla h_k^i(\bar{x}_k, \bar{u}_k)}, \qquad k = 0, \dots, N-1 \\
& b_N^i := \sqrt{\nabla h_N^i(\bar{x}_N)^\top P_N \nabla h_N^i(\bar{x}_N)}.
\end{aligned}$$

Computation times zoRO

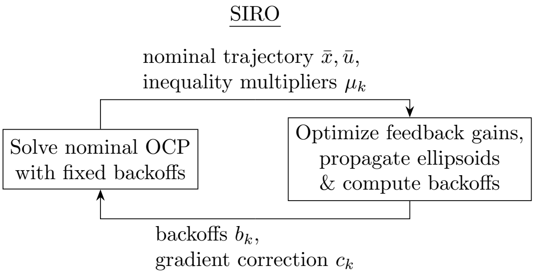


Code available at `www.github.com/FreyJo/zoro-NMPC-2021`,

J. Frey, Y. Gao, F. Messerer, A. Lahr, M. N. Zeilinger, and M. Diehl. Efficient zero-order robust optimization for real-time model predictive control with acados. In *Proceedings of the European Control Conference (ECC)*, 2024

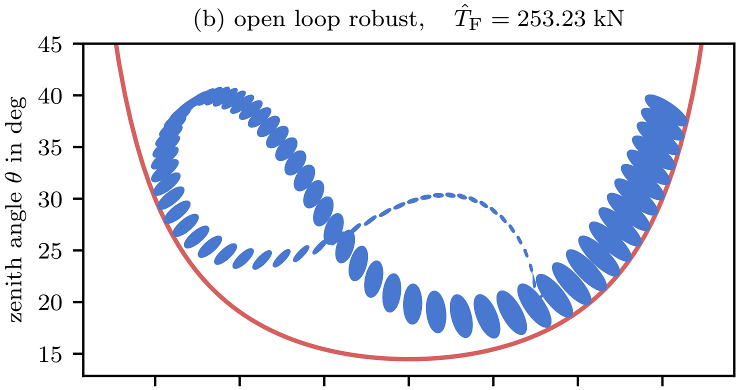
Feedback optimization with Sequential Inexact Robust Optimization (SIRO)

- Consider nonlinear ellipsoidal-tube OCP with feedback gain optimization
  - Augmented state: $x_k \in \mathbb{R}^{n_x}$, $P_k \in \mathbb{R}^{n_x^2}$
  - Augmented controls: $u_k \in \mathbb{R}^{n_u}$, $K_k \in \mathbb{R}^{n_x n_u}$
  - Complexity per iteration, standard algorithm: $\mathcal{O}(N n_x^6)$
  - Additionally highly nonconvex and challenging to solve
- Sequential inexact robust optimization (SIRO)
  - Split into nominal variables $\bar{x}, \bar{u}$ and uncertainty variables P, K.
  - Alternate between:
    - Nominal OCP for the nominal variables
    - Riccati recursion and uncertainty propagation for the uncertainty variables
  - Complexity per iteration: $\mathcal{O}(N n_x^3)$



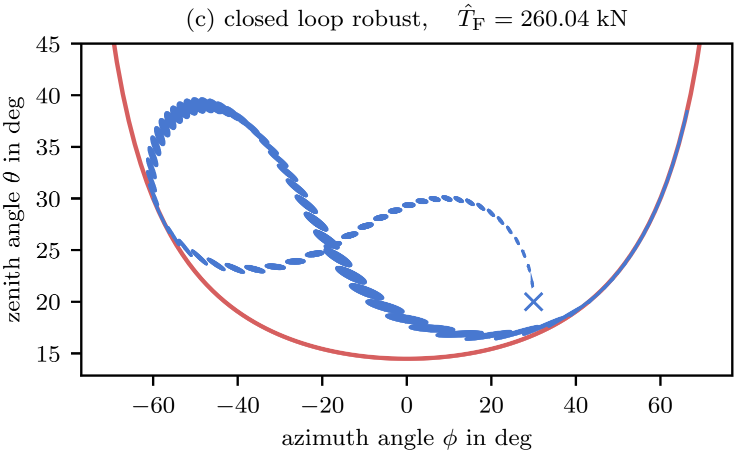
F. Messerer and M. Diehl. An efficient algorithm for tube-based robust nonlinear optimal control with optimal linear feedback. In *Proceedings of the IEEE Conference on Decision and Control (CDC)*, 2021

Open-loop robust trajectory



Code available at `www.github.com/fmesserer/plain-siro`.

Closed-loop robust trajectory with optimal linear feedback



Code available at `www.github.com/fmesserer/plain-siro`.

Related algorithm: fast-SLS

- Consider: linear dynamics, ℓ_∞-ℓ_2-bounded disturbance, disturbance feedback, convex quadratic cost.
  - Results in convex SOCP with $\mathcal{O}(N^2)$ decision variables
  - Standard sparsity algorithm, complexity per iteration: $\mathcal{O}(N^4)$
- Algorithm fast-SLS[1] employs SIRO-like alternation of nominal OCP and N Riccati recursions
  - Complexity per iteration: $\mathcal{O}(N^2)$
  - Reduction to $\mathcal{O}(N)$ via straightforward parallelization
- For QP resulting from ℓ_∞-ℓ_2 or ℓ_∞-ℓ_1 bounded noise, reformulation resulting in different sparsity pattern[2] yields $\mathcal{O}(N^3)$

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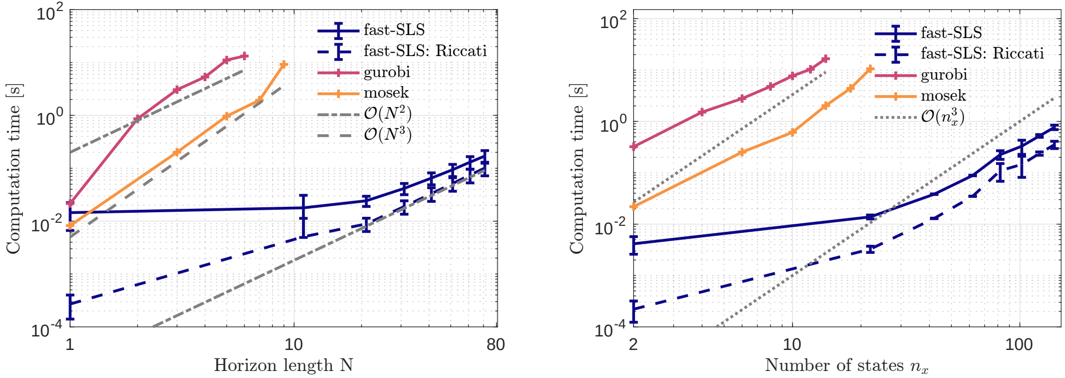
[1] A. P. Leeman, J. Kohler, F. Messerer, A. Lahr, M. Diehl, and M. N. Zeilinger. Fast system level synthesis: Robust model predictive control using riccati recursions. *IFAC-PapersOnLine*, 58(18):173–180, 2024

[2] P. J. Goulart, E. C. Kerrigan, and D. Ralph. Efficient robust optimization for robust control with constraints. *Math. Program.*, 114, 2008

Computation times fast-SLS



Code available at `www.github.com/antoineleeman/fast-SLS`.

Summary / Conclusions

- We focused on the first two of three challenges for robust optimal control: Robust constraint satisfaction and feedback optimization.
- We used three perspectives on robust OCP: Robust optimization, OCP with set valued trajectories, Robust Dynamic Programming.
- Robust Dynamic Programming (RDP) conceptually solves the robust OCP exactly.
- Scenario-trees allow to exactly solve the problem for finite uncertainties and polytopic systems, but suffer from exponential growth
- Normbounded disturbances can be elegantly treated via the dual norm.
- State feedback leads to nonconvex optimization problems.
- Affine disturbance feedback yields convex but higher dimensional problems.
- Robust nonlinear MPC problems can be addressed by linearization, yielding approximate robustness if the linearization errors are not overbounded.
- Family of algorithms to reduce computational complexity: zoRO, SIRO, fastSLS.

Some references I

- J. Anderson, J. C. Doyle, S. H. Low, and N. Matni. System level synthesis. *Annual Reviews in Control*, 47, 2019.
- A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004.
- J. Björnberg and M. Diehl. Approximate dynamic programming for generation of robustly stable feedback controllers. In H. G. Bock, editor, *Modeling, Simulation and Optimization of Complex Processes: Proceedings of the Third International Conference on High Performance Scientific Computing*, pages 69–86. Springer Berlin Heidelberg, 2008.
- M. Diehl. Formulation of closed loop min-max MPC as a quadratically constrained quadratic program. *IEEE Transactions on Automatic Control*, 52(2):339–343, 2007.
- M. Diehl, H. Bock, and E. Kostina. An approximation technique for robust nonlinear optimization. *Mathematical Programming*, 107:213–230, 2006.
- J. Frey, Y. Gao, F. Messerer, A. Lahr, M. N. Zeilinger, and M. Diehl. Efficient zero-order robust optimization for real-time model predictive control with acados. In *Proceedings of the European Control Conference (ECC)*, 2024.
- P. J. Goulart, E. C. Kerrigan, and J. M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42:523–533, 2006.

Some references II

- P. J. Goulart, E. C. Kerrigan, and D. Ralph. Efficient robust optimization for robust control with constraints. *Math. Program.*, 114, 2008.
- B. Houska. *Robust Optimization of Dynamic Systems*. PhD thesis, KU Leuven, 2011. (ISBN: 978-94-6018-394-2).
- A. Kurzhanski and P. Valyi. *Ellipsoidal Calculus for Estimation and Control*. Birkhäuser Boston, 1997.
- A. P. Leeman, J. Kohler, F. Messerer, A. Lahr, M. Diehl, and M. N. Zeilinger. Fast system level synthesis: Robust model predictive control using riccati recursions. *IFAC-PapersOnLine*, 58(18):173–180, 2024.
- F. Messerer and M. Diehl. An efficient algorithm for tube-based robust nonlinear optimal control with optimal linear feedback. In *Proceedings of the IEEE Conference on Decision and Control (CDC)*, 2021.
- Z. Nagy and R. Braatz. Open-loop and closed-loop robust optimal control of batch processes using distributional and worst-case analysis. *Journal of Process Control*, 14:411–422, 2004.
- A. Zanelli, J. Frey, F. Messerer, and M. Diehl. Zero-order robust nonlinear model predictive control with ellipsoidal uncertainty sets. *Proceedings of the IFAC Conference on Nonlinear Model Predictive Control (NMPC)*, 2021.