MinMax MPC of linear models: fundamental tools

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Outline

The basics—linear algebra

We assume throughout that the parameters $D \in \mathbb{R}^{n\times n} \geq 0$, $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, d \in \mathbb{R}^n$. [1]

- Let $A^+ \in \mathbb{R}^{n\times m}$ denote the pseudoinverse of matrix $A \in \mathbb{R}^{m\times n}$.
- Let $N(A)$ and $R(A)$ denoted the null space and range space of matrix A, respectively.
- We will also make use of the singular value decomposition (SVD) of A given by

$$A = USV$$

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = U_1 \Sigma_r V_1' \tag{1}$$

  and r is the rank of A.
- We also have that $A^+ = V_1 \Sigma_r^{-1} U_1'$. (mnemonic)

[1] This material is taken from Rawlings, Mannini, and Kuntz (2024).

The basics—linear algebra

- The properties of the SVD and the fundamental theorem of linear algebra imply that
  - the orthonormal columns of U_1 and U_2 are bases for $R(A)$ and $N(A')$
  - the orthonormal columns of V_1 and V_2 are bases for $R(A')$ and $N(A)$
- Edge cases: $A = 0$ has $r = 0$ and empty U_1, V_1, Σ_r matrices, so $U = U_2, V = V_2$ and $R(A) = \{0\}, R(A') = \{0\}, N(A) = \mathbb{R}^n, N(A') = \mathbb{R}^m$.
- Other extreme: if A is square and invertible, $r = m = n$ and U_2, V_2 are empty so $U = U_1, V = V_1$, and $R(A) = \mathbb{R}^n, R(A') = \mathbb{R}^m, N(A) = \{0\}, N(A') = \{0\}$. [a]

[a] In this case, don't say (as I often do) that $N(A)$ is empty. Why not?

Solving linear algebra problems

We require solutions to linear algebra problems when such solutions exist.

Proposition 1 (Solving linear algebra problems.)

Consider the linear algebra problem

$$Ax = b$$

1. *A solution exists if and only if $b \in R(A)$.*
2. *For $b \in R(A)$, the solution (set of solutions) is given by*[a]

$$x^0 \in A^+ b + N(A) \tag{2}$$

[a] We overload the addition symbol to mean set addition when adding singletons ($A^+ b$) and sets ($N(A)$).

Derivation or proof, which matters more? [2]

- If one is interested in *deriving* (2) use the two orthogonal coordinate systems provided by the SVD of A
- Let $x = V\alpha$, $b = U\beta$, and solve that simpler *decoupled* linear algebra problem for α^0 as a function of β,
- Convert back to x^0 in terms of b.
- If $b \notin R(A)$, x^0 is still well-defined, but $Ax^0 - b = (AA^+ - I)b = -U_2 U_2' b \neq 0$.
- In this case, the x^0 given in (2) solves $\min_x |Ax - b|$ (least-squares solution), and achieves value $\left|Ax^0 - b\right| = |U_2' b|$.

[1] In my experience derivation matters when you want to know the answer. Proof matters when you want to know when the answer is valid.

Proposition 2 (Minimum of quadratic functions)

Consider the quadratic function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ *with* $D \in \mathbb{R}^{n \times n} \geq 0$

$$V(u) := (1/2)u'Du + u'd$$

1. *A solution to* $\min_u V$ *exists if and only if* $d \in R(D)$.
2. *For* $d \in R(D)$, *the optimal solution and value function are*

$$u^0 \in -D^+d + N(D) \qquad V^0 = -(1/2)d'D^+d \tag{3}$$

and $(d/du)V(u) = 0$ *at* u^0.

For maximization problems, we can replace V with $-V$ and min with max.

Minmax and Maxmin

- We are interested in a function $V(u, w)$ $V : \mathbb{U} \times \mathbb{W} \to \mathbb{R}$ and the optimization problems

$$\inf_{u \in \mathbb{U}} \sup_{w \in \mathbb{W}} V(u, w) \qquad \sup_{w \in \mathbb{W}} \inf_{u \in \mathbb{U}} V(u, w)$$

- We assume in the following that the inf and sup are achieved on the respective sets and replace them with min and max.

$$\min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} V(u, w) \qquad \max_{w \in \mathbb{W}} \min_{u \in \mathbb{U}} V(u, w)$$

Continuous functions

Let's start here. According to Wikipedia, von Neumann's minimax theorem states (von Neumann, 1928)

Theorem 3 (Minimax Theorem)

Let $\mathbb{U} \subset \mathbb{R}^m$ and $\mathbb{W} \subset \mathbb{R}^n$ be compact convex sets. If $V : \mathbb{U} \times \mathbb{W} \to \mathbb{R}$ is a continuous function that is convex-concave, i.e., $V(\cdot, w) : \mathbb{U} \to \mathbb{R}$ is convex for all $w \in \mathbb{W}$, and $V(u, \cdot) : \mathbb{W} \to \mathbb{R}$ is concave for all $u \in \mathbb{U}$ Then we have that

$$\min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} V(u, w) = \max_{w \in \mathbb{W}} \min_{u \in \mathbb{U}} V(u, w)$$

Note that existence of min and max is guaranteed by compactness of $\mathbb{U}, \mathbb{W}$ (closed, bounded). And that's the Weierstrass (extreme value) theorem: a continuous function on a closed and bounded set attains its min and max.

Weak duality

- Also note that the following holds for any continuous function V

$$\min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} V(u, w) \geq \max_{w \in \mathbb{W}} \min_{u \in \mathbb{U}} V(u, w)$$

  (mnemonic)
- This is often called *weak duality*. It's easy to establish.[a]
- We are regarding the switching of the order of min and max as a form of duality. (Think of observability and controllability as duals of each other.)

[a] Never trust authors when they say something is easy to establish!

Strong duality and duality gap

$$\min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} V(u, w) \geq \max_{w \in \mathbb{W}} \min_{u \in \mathbb{U}} V(u, w)$$

- So when this inequality achieves equality, that's often called *strong duality.*
- So the minimax theorem says that continuous functions that are convex-concave on compact sets satisfy strong duality.
- When strong duality is not achieved, we refer to the difference as the *duality gap*, which is positive due to weak duality

$$\min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} V(u, w) - \max_{w \in \mathbb{W}} \min_{u \in \mathbb{U}} V(u, w) > 0$$

Saddle Points

In characterizing solutions of these problems, it is useful to define a saddle point of the function $V(u, w)$.

Definition 4 (Saddle point)

The point $(u^*, w^*) \in \mathbb{U} \times \mathbb{W}$ is called a saddle point of $V(\cdot)$ if

$$V(u^*, w) \leq V(u^*, w^*) \leq V(u, w^*) \quad \text{for all } u \in \mathbb{U}, w \in \mathbb{W} \tag{4}$$

Why are saddle points useful?

Proposition 5 (Saddle-point theorem)

The point $(u^*, w^*) \in \mathbb{U} \times \mathbb{W}$ is a saddle point of function $V(\cdot)$ if and only if strong duality holds and (u^*, w^*) is a solution to the two problems

$$\min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} V(u, w) = \max_{w \in \mathbb{W}} \min_{u \in \mathbb{U}} V(u, w) = V(u^*, w^*) \tag{5}$$

$$u^* = \arg \min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} V(u, w) \qquad w^* = \arg \max_{w \in \mathbb{W}} \min_{u \in \mathbb{U}} V(u, w) \tag{6}$$

The inner problems

In the following development it is convenient to define the solutions to the inner minimization and maximization problems

$$\underline{u}^0(w) := \arg\min_{u \in \mathbb{U}} V(u, w), \quad w \in \mathbb{W} \tag{7}$$

$$\overline{w}^0(u) := \arg\max_{w \in \mathbb{W}} V(u, w), \quad u \in \mathbb{U} \tag{8}$$

Note that these inner solution sets are too "large" in the following sense. Even if we evaluate them at the optimizers of their respective outer problems, we know only that

$$u^* \subseteq \underline{u}^0(w^*) \qquad w^* \subseteq \overline{w}^0(u^*)$$

and these subsets may be strict. So we have to exercise some care when we exploit strong duality and want to extract the optimizer from a dual problem. We shall illustrate this issue in the upcoming results.

Audience participation

Let $V(u,w) = (u-w)^2$, $\mathbb{U} = \mathbb{R}, \mathbb{W} = \{w \mid w^2 = 1\}$, i.e. $\mathbb{W} = \{\pm 1\}$

- Solving $\min_{u \in \mathbb{U}} \max_{w \in W} V(u,w)$
  1. Solve inner $\max_{w \in \mathbb{W}} V(u,w)$ and find $\overline{w}^0(u)$ and $V(u, \overline{w}^0(u))$
  2. Sketch the function $V(u, \overline{w}^0(u))$ vs. u
  3. Now solve $\min_u V(u, \overline{w}^0(u))$ for u^0 and V^0
- Find the stationary points (u_s, w_s) of $V(u,w)$ and evaluate $V(u_s, w_s)$
- Solving $\max_{w \in \mathbb{W}} \min_{u \in \mathbb{U}} V(u,w)$
  1. Solve $\min_u V(u,w)$ and find $\underline{u}^0(w)$ and $V(\underline{u}^0(w), w)$
  2. Now solve $\max_{w \in \mathbb{W}} V(\underline{u}^0(w), w)$ for w^0 and V^0
- Is this problem strongly dual? If not, what is the duality gap? What is the interpretation of the stationary points in this problem?

Quadratic functions

- In control problems, we will min and max over possibly unbounded sets
- So we'll need something other than compactness to guarantee existence of solutions.
- When we have linear dynamic models and quadratic stage cost (LQ), we develop the following results for quadratic functions.
- First up. A saddle-point theorem for quadratic functions.

Partitioned semidefinite matrices

For quadratic functions, we shall make extensive use of partitioned matrices

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{bmatrix}$$

We have the following result for positive semidefinite partitioned matrices (Boyd and Vandenberghe, 2004, p.651).

Proposition 6 (Positive semidefinite partitioned matrices)

The matrix $M \geq 0$ if and only if $M_{11} \geq 0$, $M_{22} - M_{12}' M_{11}^{+} M_{12} \geq 0$, and $R(M_{12}) \subseteq R(M_{11})$.

Note also that given the partitioning in M, we define

$$\begin{aligned} \tilde{M}_{11} &:= M_{22} - M_{12}' M_{11}^{+} M_{12} \\ \tilde{M}_{22} &:= M_{11} - M_{12} M_{22}^{+} M_{12}' \end{aligned} \tag{9}$$

and $\tilde{M}_{11}$ is known as the Schur complement of M_{11}, and $\tilde{M}_{22}$ is known as the Schur complement of M_{22}.

Proposition 7 (Saddle-Point Theorem for Quadratic Functions)

Consider the quadratic function $V(\cdot) : \mathbb{R}^{n+m} \to \mathbb{R}$

$$V(u,w) := (1/2) \begin{bmatrix} u \\ w \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

with M *symmetric,* $M_{11} \in \mathbb{R}^{m\times m}$*,* $M_{22} \in \mathbb{R}^{n\times n}$*,* $M_{12} \in \mathbb{R}^{m\times n}, d \in \mathbb{R}^{m+n}$*.*
For $d \in R(M)$*, define stationary points* (u^*, w^*) *of function* $V(\cdot)$ *as* $dV(u,w)/d(u,w) = 0$ *at* (u^*, w^*)*, satisfying*

$$M \begin{bmatrix} u^* \\ w^* \end{bmatrix} = -d \qquad \begin{bmatrix} u^* \\ w^* \end{bmatrix} \in -M^+ d + N(M) \tag{10}$$

with cost

$$V(u^*, w^*) = -(1/2) d' M^+ d \tag{11}$$

Denote the solutions to the inner optimizations, when they exist, by $\underline{u}^0(w) := \arg\min_u V(u,w)$ *and* $\overline{w}^0(u) := \arg\max_w V(u,w)$*.*

Proposition 7 (cont.)

1. Solutions to $\min_u \max_w V$ exist if $d \in R(M)$, $M_{22} \leq 0$, and $\tilde{M}_{22} \geq 0$ and satisfy

$$\arg\min_u \max_w V(u, w) = u^* \qquad M_{12}' u^* + M_{22} \overline{w}^0(u^*) = -d_2$$
$$V(u^*, \overline{w}^0(u^*)) = -(1/2) d' M^+ d$$

2. Similarly, solutions to $\max_w \min_u V$ exist if $d \in R(M)$, $M_{11} \geq 0$, and $\tilde{M}_{11} \leq 0$, and satisfy

$$\arg\max_w \min_u V(u, w) = w^* \qquad M_{11} \underline{u}^0(w^*) + M_{12} w^* = -d_1$$
$$V(\underline{u}^0(w^*), w^*) = -(1/2) d' M^+ d$$

3. Strong duality holds if and only if $d \in R(M)$, $M_{11} \geq 0$, and $M_{22} \leq 0$

$$\min_u \max_w V(u, w) = \max_w \min_u V(u, w) = V(u^*, w^*)$$

In this case both inner optimizations exist and $u^* \subseteq \underline{u}^0(w^*)$ and $w^* \subseteq \overline{w}^0(u^*)$.

Useful intermediate result in the proof of Proposition 7

Expand $V(\cdot)$ as

$$V(u,w) = (1/2)w'M_{22}w + w'(M_{12}'u + d_2) + (1/2)u'M_{11}u + u'd_1 \quad (12)$$

From Proposition 2, $\max_w V$ exists if and only $M_{22} \leq 0$ and $M_{12}'u + d_2 \in R(M_{22})$. This condition is satisfied for some nonempty set of u by the bottom half of $d \in R(M)$. For such u we have the necessary and sufficient condition for the optimum

$$M_{22}\overline{w}^0 + M_{12}'u + d_2 = 0$$

which defines an implicit function $\overline{w}^0(u)$, and optimal value given by (3)

$$\begin{aligned} \overline{w}^0(u) &= -M_{22}^+(M_{12}'u + d_2) + N(M_{22}) \\ V(u,\overline{w}^0(u)) &= (1/2)u'\tilde{M}_{22}u + u'(d_1 - M_{12}M_{22}^+d_2) \\ &\quad - (1/2)d_2M_{22}^+d_2 \end{aligned} \quad (13)$$

We'll use $V(u,\overline{w}^0(u))$ later

Lagrangian functions

The connections between constrained optimization problems via the use of Lagrange multipliers and game theory problems are useful (Rockafellar, 1993).

> *For optimization problems of convex type, Lagrange multipliers take on a game-theoretic role that could hardly even have been imagined before the creative insights of von Neumann [32], [33], in applying mathematics to models of social and economic conflict.*
>
> *–T.A. Rockafellar*

Lagrangian functions—motivation

- In optimal control problems with linear models, we often use quadratic stage cost

$$x^+ = Ax + Bu + Gw, \qquad \ell(x,u) = (1/2)(x'Qx + u'Ru)$$

- So $V(\boldsymbol{x}, \boldsymbol{u})$ typically has a positive definite penalty on $\boldsymbol{w}$ so $V(\boldsymbol{x}, \boldsymbol{u}) \to +\infty$ as $\boldsymbol{w} \to \infty$.
- So we must constrain w to even have a solution to $\max_w V$.
- Consider $w'w \leq 1$ as a standard (scaled) constraint. Typically this constraint is always active at the solution, so we consider

$$\max_{w'w=1} V(x,u) \qquad L(x,u,w) = V(x,u) - \lambda(w'w - 1)$$

Lagrangian functions

So we are interested in the Lagrangian function $L(\cdot) : \mathbb{R}^{n+m+1} \to \mathbb{R}$

$$L(u, w, \lambda) := (1/2) \begin{bmatrix} u \\ w \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d - (1/2)\lambda(w'w - 1)$$
$$= (1/2) \begin{bmatrix} u \\ w \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} - \lambda I \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d + \lambda/2$$

with $M \in \mathbb{R}^{m+n \times m+n} \geq 0$, $M_{22} \in \mathbb{R}^{n \times n}$, $M_{11} \in \mathbb{R}^{m \times m}$, $M_{12} \in \mathbb{R}^{m \times n}$.

- Note that $M \geq 0$ implies $M_{22} \geq 0$ so that $\max_w$ is not bounded unless we choose λ large enough to make $M_{22} - \lambda I \leq 0$.
- The following Schur complements are useful for expressing the solution.

$$\tilde{M}_{11}(\lambda) := (M_{22} - \lambda I) - M_{12}' M_{11}^{+} M_{12}$$
$$\tilde{M}_{22}(\lambda) := M_{11} - M_{12}(M_{22} - \lambda I)^{+} M_{12}'$$

Proposition 8 (Minmax and maxmin of a quadratic function with a parameter)

Consider the Lagrangian function $L(\cdot) : \mathbb{R}^{m+n+1} \to \mathbb{R}$

$$L(u, w, \lambda) \coloneqq (1/2) \begin{bmatrix} u \\ w \end{bmatrix}' M(\lambda) \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d + \lambda/2$$

with $M_{11} > 0$, $M(0) \geq 0$, *and the two problems*

$$\min_u \max_w L(u, w, \lambda) \qquad \max_w \min_u L(u, w, \lambda) \tag{14}$$

We characterize existence of solutions as a function of (decreasing) parameter λ.

1. *For* $\lambda > |M_{22}|$*: Solutions to both problems exist for all* $d \in \mathbb{R}^{m+n}$.
2. *For* $\lambda = |M_{22}|$*, we have the following two cases:*
    1. *For* $d \in R(M(|M_{22}|))$*: The solutions to both problems exist.*
    2. *For* $d \notin R(M(|M_{22}|))$*: Neither problem has a solution.*

If $M(0)$ *is such that* $|\tilde{M}_{11}| < |M_{22}|$*, then we have the following cases.*

Proposition 8 (cont.)

3. For $|\tilde{M}_{11}| < \lambda < |M_{22}|$: Only the solution to the $\max_w \min_u L$ problem exists, and it exists for all $d \in \mathbb{R}^{m+n}$.

4. For $\lambda = |\tilde{M}_{11}|$, we have the following two cases:
   1. For $d \in R(M(|\tilde{M}_{11}|))$: Only the solution to the $\max_w \min_u L$ problem exists.
   2. For $d \notin R(M(|\tilde{M}_{11}|))$: Neither problem has a solution.

5. For $\lambda < |\tilde{M}_{11}|$: Neither problem has a solution.

If $M(0)$ is such that $|\tilde{M}_{11}| = |M_{22}|$, then cases 3 and 4 do not arise.
For $d \in R(M(\lambda))$ denote the stationary points $(u^*(\lambda), w^*(\lambda))$ by

$$\begin{bmatrix} u^* \\ w^* \end{bmatrix} (\lambda) \in -M^+(\lambda)d + N(M(\lambda))$$

When solutions to the respective problems exist, we have that

$$u^*(\lambda) = \arg\min_u \max_w L(u, w, \lambda) \qquad w^*(\lambda) = \arg\max_w \min_u L(u, w, \lambda)$$
$$L^0(\lambda) = -(1/2)d' M^+(\lambda)d + \lambda/2$$

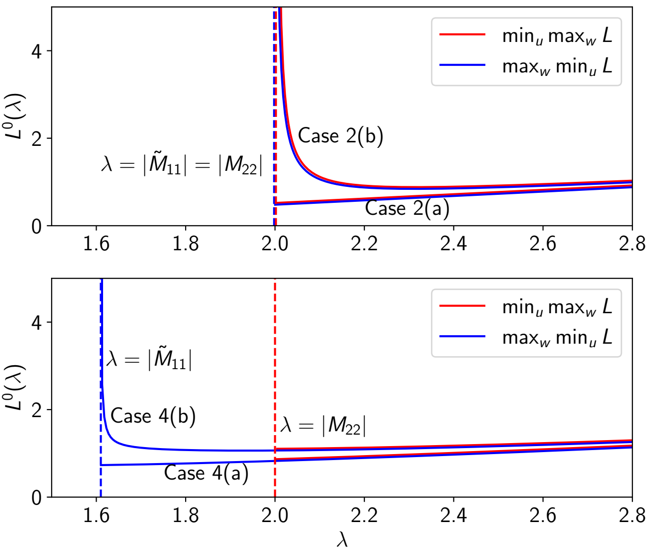

Figure 1: The optimal value function L^0 for $\min_u \max_w L$ and $\max_w \min_u L$ versus parameter λ.

Constrained quadratic optimization

A mysterious piece of information has been uncovered. In our innocence we thought we were engaged straightforwardly in solving a single problem (P). But we find we've assumed the role of Player 1 in a certain game in which we have an adversary, Player 2, whose interests are diametrically opposed to ours!

–T.A. Rockafellar

We next address the *maximization* of a *convex* function so that a constraint is required for existence of a solution.

Constrained minmax and maxmin

Consider quadratic function $V(\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}$, compact constraint set $\mathbb{W}$, and Lagrangian function $L(\cdot) : \mathbb{R}^{m+n+1} \to \mathbb{R}$

$$V(u,w) = (1/2) \begin{bmatrix} u \\ w \end{bmatrix}' M(0) \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d \qquad \mathbb{W} := \{w \mid w'w = 1\}$$

$$L(u,w,\lambda) = (1/2) \begin{bmatrix} u \\ w \end{bmatrix}' M(\lambda) \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d + \lambda/2$$

We consider the two constrained optimization problems

$$\min_u \max_{w \in \mathbb{W}} V(u,w) \quad \text{robust control} \tag{15}$$

$$\max_{w \in \mathbb{W}} \min_u V(u,w) \quad \text{worst-case feedforward control} \tag{16}$$

Assume $M(0) \geq 0$ and $M_{11} > 0$.

Constrained minmax and maxmin (cont.)

For $d \in R(M(\lambda))$ denote stationary points by $(u^*(\lambda), w^*(\lambda))$ and evaluated Lagrangian function

$$\begin{bmatrix} u^*(\lambda) \\ w^*(\lambda) \end{bmatrix} \in -M^+(\lambda)d + N(M(\lambda))$$

$$L(\lambda) = V(u^*, w^*) - (1/2)\lambda((w^*)'w^* - 1) = -(1/2)d'M^+(\lambda)d + \lambda/2$$

This is the one that we need!
(15) is the robust control problem;
(16) is the worst-case feedforward control problem.

Robust control, constrained minmax

Theorem 9 (Constrained Minmax and Maxmin)

1. *The solution to problem* (15) *exists for all* $d \in \mathbb{R}^{m+n}$ *and is given by*

$$u_r^0 = u^*(\lambda_r^0) \qquad w_r^0 = \overline{w}^0 \cap \mathbb{W}$$

where λ_r^0 *denotes the solution to the following optimization, which exists for all* $d \in \mathbb{R}^{m+n}$

$$\lambda_r^0 = \arg \min_{\lambda \geq |M_{22}|} L(\lambda) \tag{17}$$

and $\overline{w}^0$ *is all solutions to*

$$M_{11} u^*(\lambda_r^0) + M_{12} \overline{w}^0 = -d_1$$

The optimal cost is given by $V(u_r^0, w_r^0) = L(\lambda_r^0)$.

Worst-case feedforward control, constrained maxmin

Theorem 9 (cont.)

2. The solution to problem (16) exists for all $d \in \mathbb{R}^{m+n}$ and is given by

$$u_f^0 = \underline{u}^0 \qquad w_f^0 = w^*(\lambda_f^0) \cap \mathbb{W}$$

where λ_f^0 denotes the solution to the following optimization, which exists for all $d \in \mathbb{R}^{m+n}$

$$\lambda_f^0 = \arg \min_{\lambda \geq |\tilde{M}_{11}|} L(\lambda) \tag{18}$$

and $\underline{u}^0$ is all solutions to

$$M_{12}' \underline{u}^0 + (M_{22} - \lambda_f^0 I) w^*(\lambda_f^0) = -d_2$$

The optimal cost is given by $V(u_f^0, w_f^0) = L(\lambda_f^0)$.

Finally, the constrained minmax also with *input* constraints

In MPC, we will also want to enforce input constraints, $u \in \mathbb{U}$ with $\mathbb{U}$ polyhedral.

Consider quadratic function $V(\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}$, compact constraint set $\mathbb{W}$, and Lagrangian function $L(\cdot) : \mathbb{R}^{m+n+1} \to \mathbb{R}$

$$V(u, w) = (1/2) \begin{bmatrix} u \\ w \end{bmatrix}' M(0) \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d \qquad \mathbb{W} := \{w \mid w'w = 1\}$$

$$L(u, w, \lambda) = (1/2) \begin{bmatrix} u \\ w \end{bmatrix}' M(\lambda) \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} u \\ w \end{bmatrix}' d + \lambda/2$$

and the fully constrained optimization problems

$$\min_{u \in \mathbb{U}} \max_{w \in \mathbb{W}} V(u, w) \quad \text{Input constrained robust control} \tag{19}$$

Assume $M(0) \geq 0$ and $M_{11} > 0$.
This is the last one that we need!

Theorem 10 (Constrained minmax with input and disturbance constraints)

Solving (19) *is equivalent to solving the NLP*

$$\min_{u \in \mathbb{U}, \lambda \geq |M_{22}|} (1/2)\ u'(\tilde{M}_{22}(\lambda))u + u'(d_1 - M_{12}(M_{22} - \lambda I)^+ d_2) - \\ (1/2) d_2 (M_{22} - \lambda I)^+ d_2 + \lambda/2$$

where $\tilde{M}_{22}(\lambda) := M_{11} - M_{12}(M_{22} - \lambda I)^+ M_{12}'$.

Here we finally need to solve an *NLP* for the optimal control with input constraints.

Note: in this notation the *nominal* MPC problem is the QP

$$\min_{u \in \mathbb{U}} (1/2) u' M_{11} u + u' d_1$$

Derivation of the NLP in Theorem 10

Apply Proposition 8 to the constrained minmax

$$\min_{u\in\mathbb{U}} \max_{w\in\mathbb{W}} V(u,w) = \min_{u\in\mathbb{U}} \max_{w} \min_{\lambda} L(u,w,\lambda) =$$

$$\min_{u\in\mathbb{U}} \min_{\lambda} \max_{w} L(u,w,\lambda) = \min_{u\in\mathbb{U},\lambda\geq|M_{22}|} L(u,\overline{w}^0(u,\lambda),\lambda)$$

and use (13) in the proof of Proposition 7 for $L(u,\overline{w}^0(u,\lambda),\lambda)$.

A student comes to your office hours. . .

A student taking your class presents this argument for $V(u, w)$ quadratic

Robust control	Worst-case FF control
$\min_{u} \max_{w \in \mathbb{W}} V(u, w)$	$\max_{w \in \mathbb{W}} \min_{u} V(u, w)$
$=$	$=$
$\min_{u} \max_{w} \min_{\lambda} L(u, w, \lambda)$	$\max_{w} \min_{\lambda} \min_{u} L(u, w, \lambda)$
$=$	$=$
$\min_{u} \min_{\lambda} \max_{w} L(u, w, \lambda)$	$\min_{\lambda} \max_{w} \min_{u} L(u, w, \lambda)$
$=$	$=$
$\min_{\lambda} \min_{u} \max_{w} L(u, w, \lambda)$	$\min_{\lambda} \min_{u} \max_{w} L(u, w, \lambda)$

The student says that each line follows from the previous due to strong duality.
The last two expressions are obviously equal.
So robust control achieves the same cost as worst-case feedforward control?!

How do you respond to this student?

Further reading I

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