Nonlinear model predictive control

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Outline

System model[1]

- We consider systems of the form

$$x^+ = f(x, u)$$

where the state x lies in $\mathbb{X} \subseteq \mathbb{R}^n$ and the control (input) u lies in $\mathbb{U} \subseteq \mathbb{R}^m$;

- In this formulation x and u denote, respectively, the current state and control, and x^+ the successor state.
- We assume in the sequel that the function $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is continuous, and the sets $\mathbb{X}$ and $\mathbb{U}$ are closed.
- Let

$$\phi(k; x, \boldsymbol{u})$$

denote the solution of $x^+ = f(x, u)$ at time k if the initial state is $x(0) = x$ and the control sequence is $\boldsymbol{u} = (u(0), u(1), u(2), \ldots)$;

- The solution exists and is unique.

[1] Most of this preliminary material is taken from Rawlings, Mayne, and Diehl (2020, Appendix B). Downloadable from engineering.ucsb.edu/~jbraw/mpc.

Existence of solutions to model

- If a state-feedback control law $u = \kappa(x)$ has been chosen, the closed-loop system is described by $x^+ = f(x, \kappa(x))$.
- Let $\phi(k; x, \kappa(\cdot))$ denote the solution of this difference equation at time k if the initial state at time 0 is $x(0) = x$; the solution exists and is unique (even if $\kappa(\cdot)$ is discontinuous).
- If $\kappa(\cdot)$ is not continuous, as may be the case when $\kappa(\cdot)$ is a model predictive control (MPC) law, then $f((\cdot), \kappa(\cdot))$ may not be continuous.
- In this case we assume that $f((\cdot), \kappa(\cdot))$ is *locally bounded*.

Definition 1 (Locally bounded)

A function $f : X \to X$ is locally bounded if, for any $x \in X$, there exists a neighborhood $\mathcal{N}$ of x such that $f(\mathcal{N})$ is a bounded set, i.e., if there exists a $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathcal{N}$.

Stability and equilibrium point

We would like to be sure that the controlled system is "stable", i.e., that small perturbations of the initial state do not cause large variations in the subsequent behavior of the system, and that the state converges to a desired state or, if this is impossible due to disturbances, to a desired set of states.

If convergence to a specified state, x^* say, is sought, it is desirable for this state to be an *equilibrium* point:

Definition 2 (Equilibrium point)

A point x^* is an equilibrium point of $x^+ = f(x)$ if $x(0) = x^*$ implies $x(k) = \phi(k; x^*) = x^*$ for all $k \geq 0$. Hence x^* is an equilibrium point if it satisfies

$$x^* = f(x^*)$$

Positive invariant set

In other situations, for example when studying the stability properties of an oscillator, convergence to a specified closed set $\mathcal{A} \subset \mathbb{X}$ is sought.
If convergence to a set $\mathcal{A}$ is sought, it is desirable for the set $\mathcal{A}$ to be *positive invariant*:

Definition 3 (Positive invariant set)

A set $\mathcal{A}$ is positive invariant for the system $x^+ = f(x)$ if $x \in \mathcal{A}$ implies $f(x) \in \mathcal{A}$.

Clearly, any solution of $x^+ = f(x)$ with initial state in $\mathcal{A}$, remains in $\mathcal{A}$.
The (closed) set $\mathcal{A} = \{x^*\}$ consisting of a (single) equilibrium point is a special case; $x \in \mathcal{A}$ ($x = x^*$) implies $f(x) \in \mathcal{A}$ ($f(x) = x^*$).

Distance to a set; set addition

- Define distance from point x to set $\mathcal{A}$

$$|x|_{\mathcal{A}} := \inf_{z \in \mathcal{A}} |x - z|$$

If $\mathcal{A} = \{x^*\}$, then $|x|_{\mathcal{A}} = |x - x^*|$ which reduces to $|x|$ when $x^* = 0$.

- Set addition: $A \oplus B := \{a + b \mid a \in A, b \in B\}$.

$\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions

Definition 4

- A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing;
- $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ and unbounded ($\sigma(s) \to \infty$ as $s \to \infty$).
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is continuous and if, for each $t \geq 0$, $\beta(\cdot, t)$ is a class $\mathcal{K}$ function and for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t\to\infty} \beta(s, t) = 0$.

Some useful properties of $\mathcal{K}$ functions

The following useful properties of these functions are established in Khalil (2002, Lemma 4.2):

- if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$ functions ($\mathcal{K}_\infty$ functions), then $\alpha_1^{-1}(\cdot)$ and $(\alpha_1 \circ \alpha_2)(\cdot) := \alpha_1(\alpha_2(\cdot))$ are $\mathcal{K}$ functions ($\mathcal{K}_\infty$ functions).
- Moreover, if $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}$ functions and $\beta(\cdot)$ is a $\mathcal{KL}$ function, then $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is a $\mathcal{KL}$ function.

Stability—Definitions

Definition 5 ((Classic) Asymptotic stability (constrained))

Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that $\mathcal{A} \subset X$ is closed and positive invariant for $x^+ = f(x)$. Then $\mathcal{A}$ is

1. locally stable in X if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $x \in X \cap (\mathcal{A} \oplus \delta\mathcal{B})$, implies $|\phi(i;x)|_{\mathcal{A}} < \varepsilon$ for all $i \in \mathbb{I}_{\geq 0}$.[a]
2. locally attractive in X if there exists a $\eta > 0$ such that $x \in X \cap (\mathcal{A} \oplus \eta\mathcal{B})$ implies $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$.
3. attractive in X if $|\phi(i;x)|_{\mathcal{A}} \to 0$ as $i \to \infty$ for all $x \in X$.
4. asymptotically stable with a region of attraction X if it is locally stable in X and attractive in X.

[a] $\mathcal{B}$ denotes the unit ball in $\mathbb{R}^n$.

Asymptotic stability—stronger definition

Definition 6 (Asymptotic stability (constrained – KL version))

Suppose $X \subset \mathbb{R}^n$ is positive invariant for $x^+ = f(x)$, that the origin is an equilibrium of $x^+ = f(x)$, and that the origin is in X. The origin is *asymptotically stable in* X for $x^+ = f(x)$ if there exists a $\mathcal{KL}$ function $\beta(\cdot)$ such that, for each $x \in X$

$$|\phi(i; x)| \leq \beta(|x|, i) \qquad \forall i \geq 0 \tag{1}$$

See Teel and Zaccarian (2006) and the "Notes on Recent MPC Literature" link on: `engineering.ucsb.edu/~jbraw/mpc` for further discussion of the differences in the two definitions.

If $f(\cdot)$ is *continuous*, the two definitions are equivalent.

Lyapunov function

Definition 7 (Lyapunov function (constrained))

Suppose that X is positive invariant and the origin is an equilibrium for $x^+ = f(x)$. A function $V : X \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function in X for the system $x^+ = f(x)$ if there exist functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for any $x \in X$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$V(f(x)) - V(x) \leq -\alpha_3(|x|)$$

Lyapunov stability theorem

Theorem 8 (Lyapunov stability theorem—constrained case)

Suppose that X is positive invariant and the origin is an equilibrium for $x^+ = f(x)$. If there exists a Lyapunov function in X for the system $x^+ = f(x)$ then the origin is asymptotically stable in X for $x^+ = f(x)$.

In other words, we don't have to analyze closed-loop stability of MPC on a case-by-case basis.

We instead establish that the optimal MPC cost function is a Lyapunov function for the closed-loop system!

Exercise B.3: A converse theorem for exponential stability

a. Assume that the origin is globally exponentially stable (GES) for the system

$$x^+ = f(x)$$

in which f is Lipschitz continuous. Show that there exists a Lipschitz continuous Lyapunov function $V(\cdot)$ for the system satisfying for all $x \in \mathbb{R}^n$

$$a_1 |x|^\sigma \leq V(x) \leq a_2 |x|^\sigma$$
$$V(f(x)) - V(x) \leq -a_3 |x|^\sigma$$

in which $a_1, a_2, a_3, \sigma > 0$.
Hint: Consider summing the solution $|\phi(i; x)|$ on i as a candidate Lyapunov function $V(x)$.

b. Establish also that in the Lyapunov function defined above, any $\sigma > 0$ is valid, and the constant a_3 can be chosen as large as one wishes.

The basic nonlinear, constrained MPC problem

- The system model is
$$x^+ = f(x, u) \tag{2}$$
- Both state and input are subject to constraints
$$x(k) \in \mathbb{X}\,, \qquad u(k) \in \mathbb{U} \qquad \text{for all } k \in \mathbb{I}_{\geq 0}$$
- Given an integer N (referred to as the finite horizon), and an input sequence $\boldsymbol{u}$ of length N, $\boldsymbol{u} = (u(0), u(1), \ldots, u(N-1))$, let $\phi(k; x, \boldsymbol{u})$ denote the solution of (2) at time k for a given initial state $x(0) = x$.
- Terminal constraint (and penalty)
$$\phi(N; x, \boldsymbol{u}) \in \mathbb{X}_f \subseteq \mathbb{X}$$

Feasible sets

- For an initial x, the corresponding set of feasible inputsequences is

$$\mathcal{U}_N(x) = \{\boldsymbol{u} \mid u(k) \in \mathbb{U},\ \phi(k; x, \boldsymbol{u}) \in \mathbb{X} \text{ for all } k \in \mathbb{I}_{0:N-1}, \\ \text{and } \phi(N; x, \boldsymbol{u}) \in \mathbb{X}_f\}$$

- The set of feasible initial states is

$$\mathcal{X}_N = \{x \in \mathbb{X} \mid \mathcal{U}_N(x) \neq \emptyset\} \tag{3}$$

Cost function and control problem

- For any state $x \in \mathbb{X}$ and input sequence $\boldsymbol{u} \in \mathbb{U}^N$, we define

$$V_N(x, \boldsymbol{u}) = \sum_{k=0}^{N-1} \ell(\phi(k; x, \boldsymbol{u}), u(k)) + V_f(\phi(N; x, \boldsymbol{u}))$$

- $\ell(x, u)$ is the stage cost; $V_f(x(N))$ is the terminal cost
- Consider the finite horizon optimal control problem

$$\mathbb{P}_N(x) : \qquad \min_{\boldsymbol{u} \in \mathcal{U}_N} V_N(x, \boldsymbol{u})$$

Control law and closed-loop system

- The control law is

$$\kappa_N(x) = u^0(0; x)$$

  The optimum may not be unique; then $\kappa_N(\cdot)$ is a point-to-set map
- Closed-loop system

$$x^+ = f(x, \kappa_N(x)) \qquad \text{difference equation}$$

$$x^+ \in f(x, \kappa_N(x)) \qquad \text{difference inclusion}$$

- Nominal closed-loop stability question; is the origin stable?
- If yes, what is the region of attraction? All of $\mathcal{X}_N$?

Basic MPC assumptions

Assumption 9 (Continuity of system and cost)

The functions $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ and $V_f : \mathbb{X}_f \to \mathbb{R}_{\geq 0}$ are continuous, $f(0,0) = 0$, $\ell(0,0) = 0$, and $V_f(0) = 0$.

Assumption 10 (Properties of constraint sets)

The set $\mathbb{U}$ is compact and contains the origin. The sets $\mathbb{X}$ and $\mathbb{X}_f$ are closed and contain the origin in their interiors, $\mathbb{X}_f \subseteq \mathbb{X}$.

Note: origin can be on boundary of $\mathbb{U}$, but origin cannot be on boundary of $\mathbb{X}_f, \mathbb{X}$.

Basic MPC assumptions

Assumption 11 (Lower bound on stage cost)

The stage cost $\ell(\cdot)$ satisfies

$$\ell(x, u) \geq \alpha_1(|x|) \quad \forall x \in \mathcal{X}_N, \ \forall u \in \mathbb{U}$$

in which $\alpha_1(\cdot)$ is a $\mathcal{K}_\infty$ function.

Remark 12 (Upper bound on terminal cost)

Because $V_f(\cdot)$ is continuous and $V_f(0) = 0$, we also have that

$$V_f(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{X}_f$$

in which $\alpha_2(\cdot)$ is a $\mathcal{K}_\infty$ function.

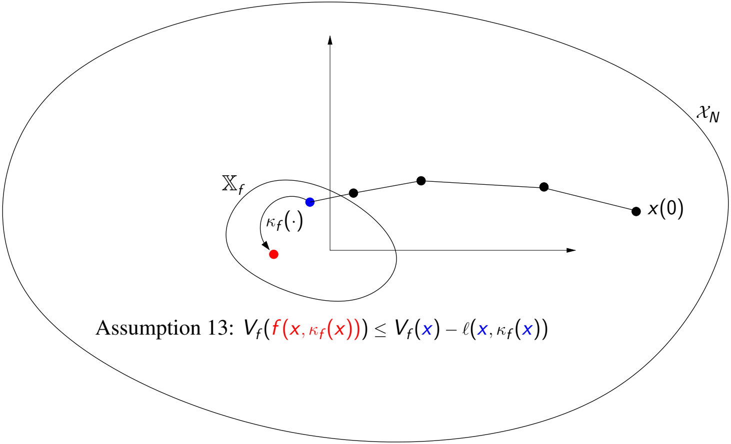
Basic MPC assumptions

Assumption 13 (Basic stability assumption)

For any $x \in \mathbb{X}_f$ there exists $u := \kappa_f(x) \in \mathbb{U}$ such that $f(x, u) \in \mathbb{X}_f$ and $V_f(f(x, u)) \leq V_f(x) - \ell(x, u)$.

Note: understanding this requirement created a big research challenge for the development of nonlinear MPC.

The MPC problem in pictures



Optimal MPC cost function as Lyapunov function

We show that the optimal cost $V_N^0(\cdot)$ is a Lyapunov function for the closed-loop system. We require three properties.

Lower bound.

$$V_N^0(x) \geq \alpha_1(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N$$

Given the definition of $V_N(x, \boldsymbol{u})$ as a sum of stage costs, we have using Assumption 11

$$V_N(x, \boldsymbol{u}) \geq \ell(x, u(0; x)) \geq \alpha_1(|x|) \quad \text{for all} \quad x \in \mathcal{X}_N, \boldsymbol{u} \in \mathbb{U}^N$$

so the first property is established.

MPC cost function as Lyapunov function – cost decrease

Next we require the **cost decrease**

$$V_N^0(f(x,\kappa_N(x))) \leq V_N^0(x) - \alpha_3(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N$$

At state $x \in \mathcal{X}_N$, consider the optimal sequence $\boldsymbol{u}^0(x) = (u(0;x), u(1;x), \ldots, u(N-1;x))$, and generate a *candidate sequence* for the successor state, $x^+ := f(x, \kappa_N(x))$

$$\tilde{\boldsymbol{u}} = (u(1;x), u(2;x), \ldots, u(N-1;x), \kappa_f(x(N)))$$

with $x(N) := \phi(N; x, \boldsymbol{u})$. This candidate is *feasible* for x^+ because $\mathbb{X}_f$ is control invariant under control law $\kappa_f(\cdot)$ (Assumption 13).
The cost is

$$\begin{aligned} V_N(x^+, \tilde{\boldsymbol{u}}) = {} & V_N^0(x) - \ell(x, u(0;x)) \\ & - \underbrace{V_f(x(N)) + \ell(x(N), \kappa_f(x(N))) + V_f(f(x(N), \kappa_f(x(N))))} \end{aligned}$$

Cost decrease (cont.)

But by Assumption 13

$$V_f(f(x, \kappa_f(x))) - V_f(x) + \ell(x, \kappa_f(x)) \leq 0 \quad \text{for all} \quad x \in \mathbb{X}_f$$

so we have that

$$V_N(x^+, \tilde{\boldsymbol{u}}) \leq V_N^0(x) - \ell(x, u(0; x))$$

The optimal cost is certainly no worse, giving

$$\begin{aligned} V_N^0(x^+) &\leq V_N^0(x) - \ell(x, u(0; x)) \\ V_N^0(x^+) &\leq V_N^0(x) - \alpha_1(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N \end{aligned}$$

which is the desired cost decrease with the choice $\alpha_3(\cdot) = \alpha_1(\cdot)$.

Upper bound

Finally we require the **upper bound.**

$$V_N^0(x) \leq \alpha_2(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N$$

Surprisingly, this one turns out to be the most involved.
First, we have the bound from Assumption 9 (Remark 12)

$$V_f(x) \leq \alpha_2(|x|) \qquad \text{for all} \quad x \in \mathbb{X}_f$$

Next we show that $V_N^0(x) \leq V_f(x)$ for $x \in \mathbb{X}_f$, $N \geq 1$.
Consider $N = 1$,

$$\begin{aligned} V_1^0(x) &= \min_{u \in \mathbb{U}} \{\ell(x,u) + V_f(f(x,u)) \mid f(x,u) \in \mathbb{X}_f\} \\ &= \ell(x, \kappa_1(x)) + V_f(f(x, \kappa_1(x))) \quad x \in \mathcal{X}_1 \\ &\leq \ell(x, \kappa_f(x)) + V_f(f(x, \kappa_f(x))) \quad x \in \mathbb{X}_f \\ &\leq V_f(x) \quad x \in \mathbb{X}_f \qquad \text{(by Assumption 13)} \end{aligned}$$

Dynamic programming recursion

Next consider $N = 2$, and optimal control law $\kappa_2(\cdot)$

$$\begin{aligned}
V_2^0(x) &= \min_{u \in \mathbb{U}} \{\ell(x,u) + V_1^0(f(x,u)) \mid f(x,u) \in \mathcal{X}_1\} \quad x \in \mathcal{X}_2 \\
&= \ell(x, \kappa_2(x)) + V_1^0(f(x, \kappa_2(x))) \quad x \in \mathcal{X}_2 \\
&\leq \ell(x, \kappa_1(x)) + V_1^0(\underbrace{f(x, \kappa_1(x))}_{\in \mathbb{X}_f}) \quad x \in \mathcal{X}_1 \\
&\leq \ell(x, \kappa_1(x)) + V_f(f(x, \kappa_1(x))) \quad x \in \mathcal{X}_1 \\
&= V_1^0(x) \quad x \in \mathcal{X}_1
\end{aligned}$$

Therefore

$$V_2^0(x) \leq V_f(x) \quad x \in \mathbb{X}_f$$

Continuing this recursion gives for all $N \geq 1$

$$V_N^0(x) \leq V_f(x) \leq \alpha_2(|x|) \quad x \in \mathbb{X}_f$$

Extending the upper bound from $\mathbb{X}_f$ to $\mathcal{X}_N$

- Question: When can we extend this bound from $\mathbb{X}_f$ to the (possibly unbounded!) set $\mathcal{X}_N$? Recall that $V_N^0(\cdot)$ is not necessarily continuous.
- Answer: The $\mathcal{K}_\infty$ upper bound of a function valid near the origin can be extended from $\mathbb{X}_f$ to the entire set $\mathcal{X}_N$ if and only if the function is locally bounded on $\mathcal{X}_N$.[2]
- We know from continuity of $f(\cdot)$ (Assumption 9) that $V_N(x, \boldsymbol{u})$ is a continuous function, hence locally bounded, and therefore so is $V_N^0(x)$.
  Therefore, there exists $\beta(\cdot) \in \mathcal{K}_\infty$ such that

$$V_N^0(x) \leq \beta(|x|) \qquad \text{for all} \quad x \in \mathcal{X}_N$$

- Be aware that the MPC literature has been confused about the requirements for this last result.

[2] See Proposition 11 of "Notes on Recent MPC Literature" link on: `engineering.ucsb.edu/~jbraw/mpc`. Thanks also to Andy Teel.

Asymptotic stability of constrained nonlinear MPC

Why you want a Lyapunov function

- We have established that the optimal cost $V_N^0(\cdot)$ is a Lyapunov function on $\mathcal{X}_N$ for the closed-loop system.
- Therefore, the origin is asymptotically stable (KL version) with region of attraction $\mathcal{X}_N$.
- We can also establish robust stability, but we'll do that later.
- If we strengthen the properties of $\ell(\cdot)$, we can strengthen the conclusion to exponential stability.
- Notice the essential role that $V_N^0(\cdot)$ plays in the stability analysis of MPC.
- In economic MPC we lose this Lyapunov function and have to do some work to bring it back.

A nice example (Example 2.6)

- System is linear (unstable, scalar)

$$x^+ = f(x,u) := x + u$$

- The stage cost and terminal cost are

$$\ell(x,u) := (1/2)(x^2 + u^2) \qquad V_f(x) := (1/2)x^2$$

- The control constraint is

$$u \in \mathbb{U} = [-1, 1]$$

- The horizon is $N = 2$. The feasible set is $\mathcal{U}_2 = \mathbb{U} \times \mathbb{U}$.

Nice example

- The cost function

$$V_N(x, \boldsymbol{u}) = (1/2)\big(x^2 + (x + u(0))^2 + (x + u(0) + u(1))^2 + u(0)^2 + u(1)^2\big)$$
$$= (3/2)x^2 + \begin{bmatrix} 2x & x \end{bmatrix} \boldsymbol{u} + (1/2)\boldsymbol{u}' H \boldsymbol{u}$$

  in which

$$H = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

- The optimal control problem

$$\min_{\boldsymbol{u} \in \mathcal{U}_2} V_N(x, \boldsymbol{u})$$

- The optimal control problem is a quadratic program

The quadratic program as x varies

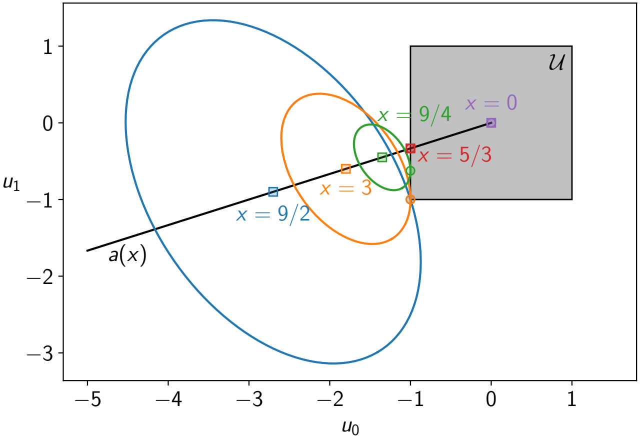


Figure 1: Feasible region $\mathcal{U}_2$, elliptical cost contours, and ellipse center, $a(x)$, and constrained minimizers for different values of x.

The simplest possible constrained control law

- The control law is piecewise affine ($u = Kx + b$) and continuous
- There are three regions: $x \leq -5/3$, $-5/3 \leq x \leq 5/3$, $5/3 \leq x$

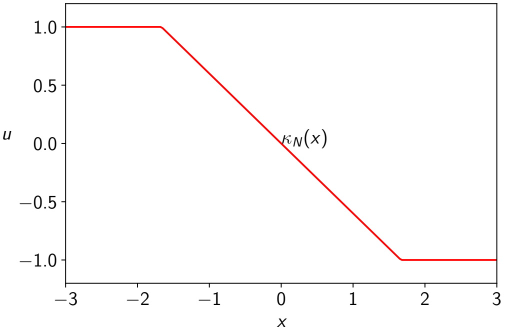


Figure 2: The optimal control law for $x^+ = x + u$, $N = 2$, $Q = R = 1$, $u \in [-1, 1]$.

The constrained control law can be complex

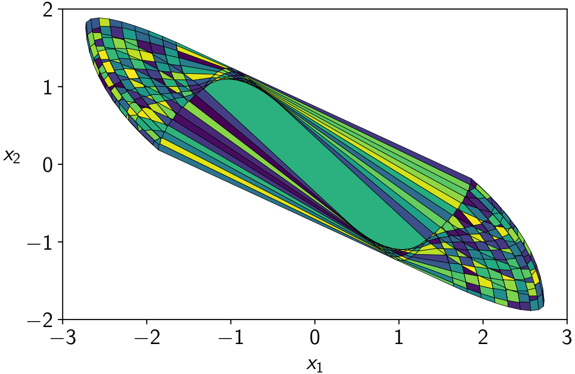


Figure 3: Regions with different linear (affine) control laws for a second-order example. (Rawlings et al., 2020, p.462)

- The number of regions increases exponentially with system order n, number of inputs, m, and horizon length N.
- Another example of Bellman's curse of dimensionality. It's difficult to store $\kappa_N(x)$, $x \in \mathbb{R}^n$, as n increases.

A troublesome example (Example 2.8)

$$x^+ = f(x, u)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u \\ u^3 \end{bmatrix}$$

- Two state, single input example. The origin is the desired steady state: $u = 0$ at $x = 0$.
- Cannot be stabilized with continuous feedback $u = \kappa(x)$.
- Because (u, u^3) have the same sign, must use negative u to stabilize first quadrant.
- Must use positive u to stabilize third quadrant.
- But u cannot pass through zero or that point is a closed-loop steady state.
- Therefore discontinuous feedback.

And its troubled history

- Introduced by Meadows, Henson, Eaton, and Rawlings (1995) to show MPC control law and optimal cost can be discontinuous.
- Based on a CT example by Coron (1990).
- Grimm, Messina, Tuna, and Teel (2005) established robustness for MPC with horizon $N \geq 4$ with a terminal cost and no terminal region constraint.

MPC with terminal equality constraint

- Because we do not know even a local controller, we try a terminal constraint $x(N) = 0$ in the MPC controller.
- For what initial x is this constraint feasible?

$$
\begin{aligned}
(x_1(1), x_2(1)) &= (x_1(0), x_2(0)) + (u_0, u_0^3) \\
(x_1(2), x_2(2)) &= (x_1(1), x_2(1)) + (u_1, u_1^3) \\
(x_1(3), x_2(3)) &= (x_1(2), x_2(2)) + (u_2, u_2^3)
\end{aligned}
$$

- For $N = 1$, the feasible set $\mathcal{X}_1$ is only the line $x_2 = x_1^3$.
- For $N = 2$, to have real roots u_0, u_1, we require $-x_1^4 + 4x_1x_2 \geq 0$ which defines $\mathcal{X}_2$
- For $N = 3$, we have $\mathcal{X}_3$ is all of $\mathbb{R}^2$.
- So the shortest horizon that can globally stabilize the system is $N = 3$.

Feasibility sets $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$

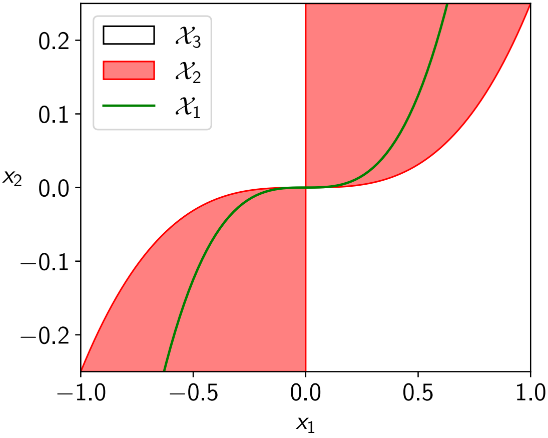
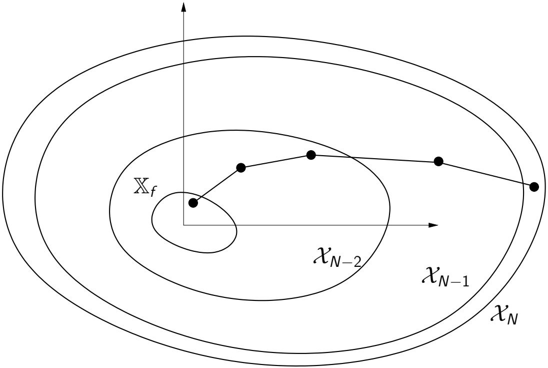


Figure 4: Feasibility sets $\mathcal{X}_N$ for $N = 1, 2, 3$.

Structure of Feasibility Sets



- The feasibility sets are nested: $\mathcal{X}_N \supseteq \mathcal{X}_{N-1} \supseteq \mathcal{X}_{N-2} \cdots \supseteq \mathbb{X}_f$
- The set $\mathcal{X}_N$ is forward invariant. Important for recursive feasibility of controller.
- The set $\mathcal{X}_{N-1}$ is also forward invariant!
- The sets $\mathcal{X}_{N-2}, \mathcal{X}_{N-3}, \ldots, \mathbb{X}_f$ are not necessarily forward invariant.

Optimal MPC with $N = 3$

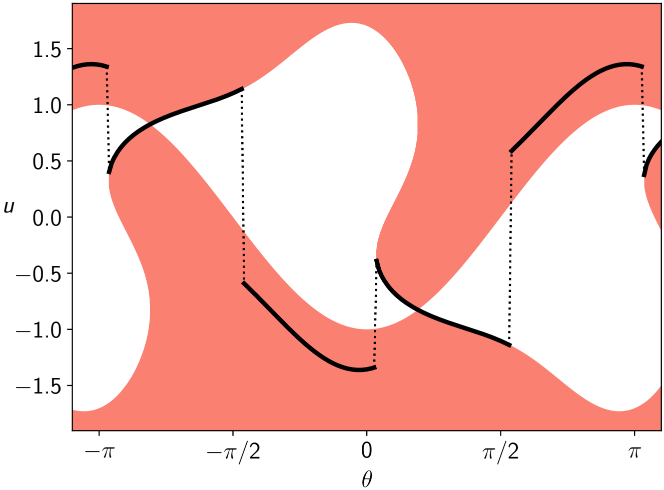


Figure 5: The control constraint set $\mathcal{U}_N(x)$ and optimal control $\kappa_N(x)$ for x on the unit circle (Rawlings et al., 2020, p. 106).

Optimal cost function with $N = 3$

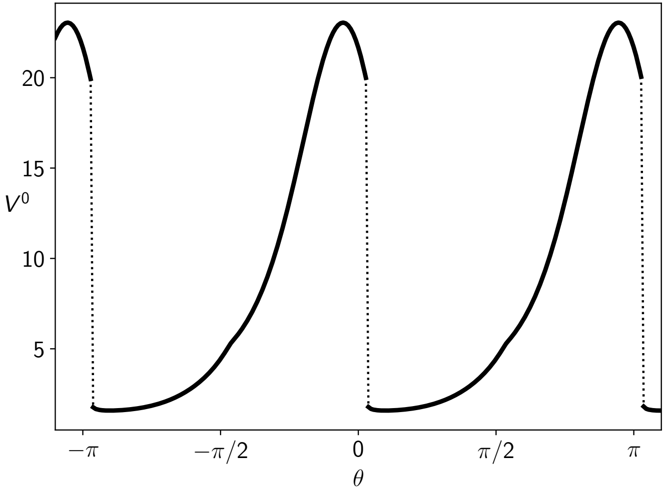


Figure 6: The discontinuity in the optimal cost for x on the unit circle

Discontinuities in optimal solution and value function

Optimal solution and parameter dependence

- Consider the general constrained optimization problem with parameter x

$$\min_{u \in \mathcal{U}(x)} V(u, x)$$

and optimal solution and value function

$$u^0(x) \qquad V^0(x)$$

- What does it take for $u^0(x)$ to be discontinuous?
- What does it take for $V^0(x)$ to be discontinuous?

Discontinuous optimal solution $u^0(x)$

It is easy to generate a smooth $V(x,u)$ and continuous constraint set $\mathcal{U}(x)$ that has a discontinuous solution $u^0(x)$ (but continuous optimal value function $V^0(x)$). Consider the following nonconvex $V(x,u)$ with the constant constraint set $\mathcal{U}(x) = \mathbb{R}$.

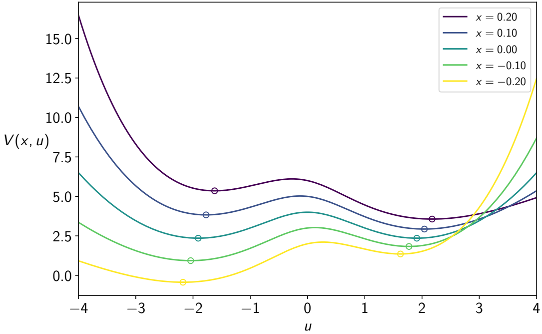


Figure 7: Smooth, nonconvex value function $V(x,u)$. There are two branches of local solutions and the optimal solution changes branches at $x = 0$.

Discontinuous optimal solution $u^0(x)$

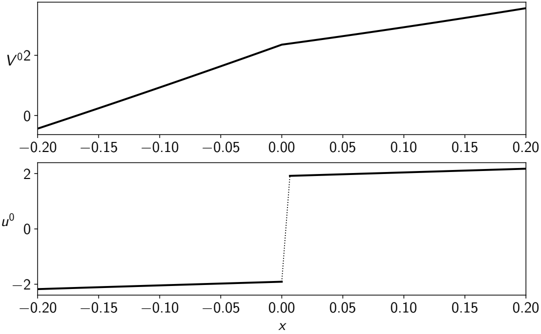


Figure 8: Smooth example with discontinuous solution and continuous value function. Note that the derivative of $V^0(x)$ is discontinuous.

Discontinuous optimal value function $V^0(x)$

To obtain a discontinuous optimal value function from a smooth $V(x, u)$, we have to make the constraint set $\mathcal{U}(x)$ discontinuous. The objective function $V(x, u)$ can be convex in this case. Consider

$$\mathcal{U}(x) = \{u \mid 1 \leq u \leq 3, \text{ or } \max(x, -1) \leq u \leq \min(-x, 1)\}$$

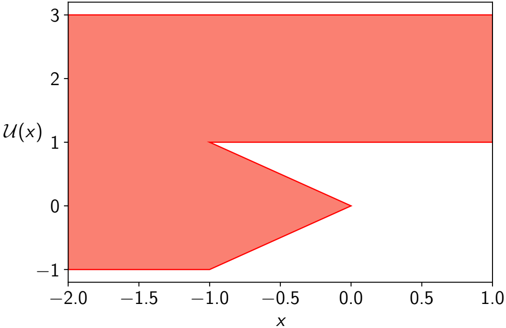


Figure 9: Discontinuous constraint set $\mathcal{U}(x)$. Note that $\mathcal{U}(x)$ at $x = 0^+$ contains no value near the point $0 \in \mathcal{U}(x)$ at $x = 0$.

Discontinuous optimal value function $V^0(x)$

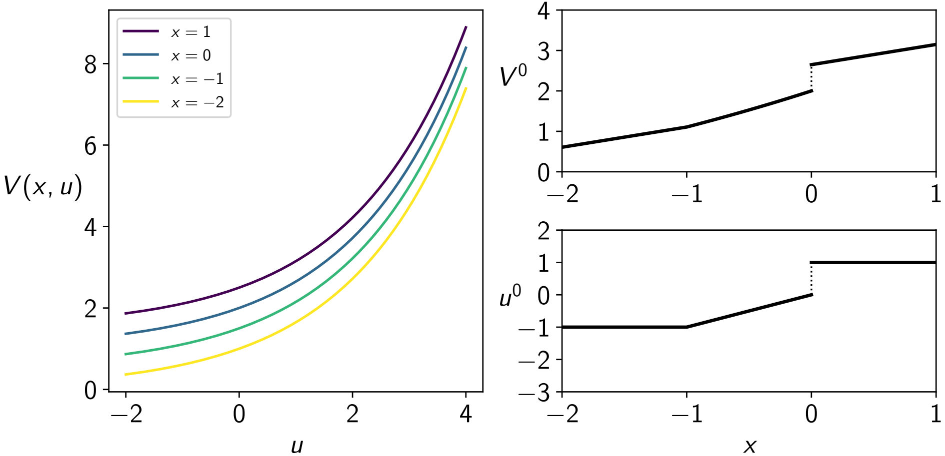


Figure 10: Smooth, convex value function $V(x, u)$ (left) and discontinuous optimal value function $V^0(x)$ and solution $u^0(x)$ (right).

Further reading I

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Further reading II

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