Prof. Dr. Moritz Diehl, Rachel Leuthold, and Nick Harder

SOLUTIONExercise Sheet 1: Power Harvesting Factor

Consider a symmetrical, three-bladed ($B = 3$) wind turbine with rotor radius R. Assume a constant angular velocity Ω of the rotor and a uniform wind field with velocity $\boldsymbol{u}_\infty$ so that the dominant wind direction $\hat{\boldsymbol{x}}$ is along the turbine axis of rotation. We will also use a nondimensional spanwise position $\mu = r/R$ that is 0 at the blade root/rotor hub, and 1 at the blade tips.

1. What is the tip speed ratio λ of the turbine?

   The tip speed ratio $\lambda = \frac{\Omega R}{u_\infty}$ is the ratio between the blade speed u_b at the tip and the freestream wind speed u_∞.

   (We'll use the abreviation that $v = \|\boldsymbol{v}\|_2$ from here on.)

2. What is the local speed ratio λ_r at some spanwise location μ?

   The local speed ratio is the equivalent concept to the tip speed ratio, but considered at different spanwise positions μ. That is:

$$\lambda_r = \mu\lambda = \mu\frac{\Omega R}{u_\infty}.$$

3. What is the apparent velocity $\boldsymbol{u}_\text{a}$ at the position μ?

   The apparent wind $\boldsymbol{u}_\text{a}$ is the difference between the freestream wind velocity $\boldsymbol{u}_\infty$ and the blade's motion $\boldsymbol{u}_\text{b}$ at the station. That is:

$$\boldsymbol{u}_\text{a} = \boldsymbol{u}_\infty - \boldsymbol{u}_\text{b}.$$

   The velocity of the blade points in the tangential direction $\hat{\boldsymbol{t}}$, with magnitude λu_∞. That is:

$$\boldsymbol{u}_\text{b} = \mu\Omega R\hat{\boldsymbol{t}} = \mu\lambda u_\infty\hat{\boldsymbol{t}}.$$

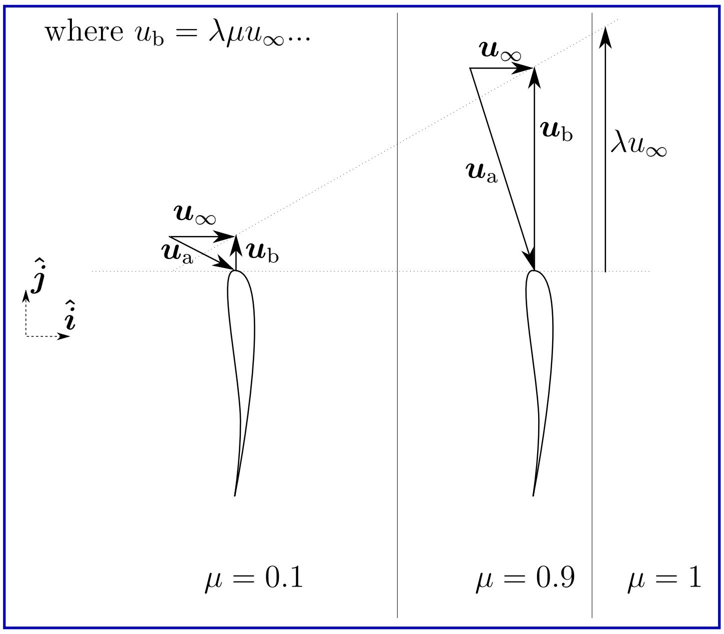
   So, the apparent velocity at position μ is:

$$\boldsymbol{u}_\text{a} = u_\infty\hat{\boldsymbol{x}} - \mu\lambda u_\infty\hat{\boldsymbol{t}}$$

4. Sketch the velocity triangles for the following positions:

   (a) $\mu = 0.1$

   (b) $\mu = 0.9$



5. Assume that the blades are uniformly pitched with an angle β, but have a 'perfect' twist distribution $\theta(\mu)$ so that α always takes its design value of 6 degrees if $\beta = 0$. What is $\theta(\mu)$?

The angle of attack α is the angle between the chord line and the apparent velocity. The flow angle ϕ is the angle between the plane of rotation and apparent wind velocity:

$$\tan \phi = \left(\frac{\Omega \mu R}{u_\infty} \right)$$

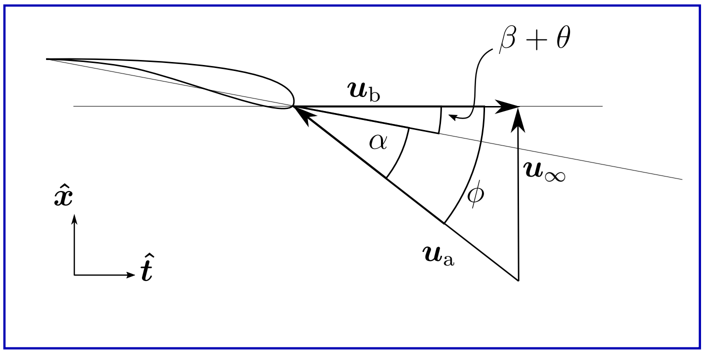
Since $\lambda = \Omega R / u_\infty$:

$$\tan \phi = (\lambda \mu)$$

Also, we define the signs of the various angles, so that the pitch angle and the twist angle help to reduce the angle of attack from the flow angle.

$$\phi = \alpha + \beta + \theta$$

The way to remember this, is to notice that the flow angle ranges between 90 degrees (at blade root, where there's almost no rotational velocity) to something much smaller (at the blade tip). Since the flow stalls (separates completely from the airfoil surface) at large angles of attack, the reason that we would pitch and twist the blades has to be to allow α to be smaller than that very large flow angle. See, also, the sketches in Section 3.5 of Burton et al.



$$\tan^{-1}(\lambda\mu) = \alpha + \beta + \theta$$

If $\beta = 0$ and $\alpha = 6\pi/180$ rad $= \pi/30$ rad, then:

$$\theta(\mu) = \tan^{-1}(\lambda\mu) - \alpha - \beta = \tan^{-1}(\lambda\mu) - \frac{\pi}{30} - 0$$

6. For arbitrary lift c_l and drag c_d coefficients, what is the aerodynamic force $\mathrm{d}\boldsymbol{F}_\mathrm{aero}$ for an infinitesimal segment of area $\mathrm{d}A$ around a position μ? Assume that the blades point straight, radially outwards.

   We know that the aerodynamic force is the sum of the lift and drag forces

$$\mathrm{d}\boldsymbol{F}_\mathrm{aero} = \mathrm{d}\boldsymbol{F}_\mathrm{L} + \mathrm{d}\boldsymbol{F}_\mathrm{D}.$$

   By using the definitions of the coefficients, we can see that:

$$\mathrm{d}\boldsymbol{F}_\mathrm{L} = c_\mathrm{l}\frac{1}{2}\rho\left\|\boldsymbol{u}_\mathrm{a}\right\|_2^2 \mathrm{d}A\hat{\boldsymbol{v}}l, \qquad \mathrm{d}\boldsymbol{F}_\mathrm{D} = c_\mathrm{d}\frac{1}{2}\rho\left\|\boldsymbol{u}_\mathrm{a}\right\|_2^2 \mathrm{d}A\hat{\boldsymbol{v}}d$$

   We know the orientations of these forces because the drag force must be along the apparent velocity, and the lift force must be perpedicular to the drag and the span.

$$\hat{\boldsymbol{v}}d = \frac{\boldsymbol{u}_\mathrm{a}}{\left\|\boldsymbol{u}_\mathrm{a}\right\|_2} = \frac{u_\infty\hat{\boldsymbol{x}} - \mu\lambda u_\infty\hat{\boldsymbol{t}}}{\left\|u_\infty\hat{\boldsymbol{x}} - \mu\lambda u_\infty\hat{\boldsymbol{t}}\right\|_2} = \frac{\hat{\boldsymbol{x}} - \mu\lambda\hat{\boldsymbol{t}}}{\sqrt{1+\mu^2\lambda^2}}$$

   To give a right-handed coordinate system $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{t}}$, $\hat{\boldsymbol{x}}$ in the sketch above: $\hat{\boldsymbol{r}}$ must point down into the page. Then:

$$\hat{\boldsymbol{v}}l = \frac{\boldsymbol{u}_\mathrm{a}\times\hat{\boldsymbol{r}}}{\left\|\boldsymbol{u}_\mathrm{a}\times\hat{\boldsymbol{r}}\right\|_2} = \frac{u_\infty\hat{\boldsymbol{t}} + \lambda\mu u_\infty\hat{\boldsymbol{x}}}{\left\|u_\infty\hat{\boldsymbol{t}} + \lambda\mu u_\infty\hat{\boldsymbol{x}}\right\|_2} = \frac{\hat{\boldsymbol{t}} + \mu\lambda\hat{\boldsymbol{x}}}{\sqrt{1+\mu^2\lambda^2}}$$

   Now we can put all of these expressions together:

$$\mathrm{d}\boldsymbol{F}_\mathrm{aero} = \frac{1}{2}\rho u_\infty^2\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\left(c_\mathrm{l}\left(\hat{\boldsymbol{t}} + \mu\lambda\hat{\boldsymbol{x}}\right) + c_\mathrm{d}\left(\hat{\boldsymbol{x}} - \mu\lambda\hat{\boldsymbol{t}}\right)\right)\mathrm{d}A$$

7. What is the mechanical power production $\mathrm{d}P(\mu)$ of that segment around position μ?

   The power is the force acting parallel to the blade's motion:

$$\mathrm{d}P = \mathrm{d}\boldsymbol{F}_\mathrm{aero}\cdot\boldsymbol{u}_\mathrm{b}.$$

   Since we know that the blade's motion is in the $\hat{\boldsymbol{t}}$ direction, we can use the above force expression:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^2\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\mathrm{d}A\left(c_\mathrm{l} - c_\mathrm{d}\mu\lambda\right)\left(\lambda\mu u_\infty\right) = \frac{1}{2}\rho u_\infty^3\lambda\mu\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\mathrm{d}A\left(c_\mathrm{l} - c_\mathrm{d}\mu\lambda\right).$$

   Just for abbreviation, let's define $\xi_n := c_\mathrm{l} - c_\mathrm{d}\mu\lambda$. Then:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^3\lambda\mu\left(1+\mu^2\lambda^2\right)^{\frac{1}{2}}\mathrm{d}A\xi_n.$$

8. If the lift c_l and drag c_d coefficients can be found with the following relations, what is the power harvested by the blade segment around position μ?

$$c_\mathrm{l}(\mu) = 1.2\mu, \qquad \frac{c_\mathrm{l}}{c_\mathrm{d}}(\mu) = 100\mu$$

Let's start with the $\mathrm{d}P$ expression from above:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^3 \lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \mathrm{d}A \left(c_\mathrm{l} - c_\mathrm{d}\mu\lambda\right).$$

If we plug the above lift and drag ($c_\mathrm{d} = c_\mathrm{l}/(c_\mathrm{l}/c_\mathrm{d}) = 1.2/100$) expressions into this power statement then, it gives the following:

$$\mathrm{d}P = \frac{1}{2}\rho u_\infty^3 \lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \mathrm{d}A \left(1.2\mu - \frac{1.2}{100}\mu\lambda\right).$$

9. What is the relationship between the power harvesting factor ζ and μ?

The power harvesting factor is the harvested power divided by the power density and the segment area. That is:

$$\zeta = \frac{\mathrm{d}P}{\frac{1}{2}\rho u_\infty^3 \mathrm{d}A} = \lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \xi_n.$$

10. How would you go about finding the total power P harvested by the entire turbine? (*Hint: just give the procedure; don't follow it yet.*)

Everything in our problem so far has been symmetrical. That means that the total power must be the sum of all segment powers for all blades:

$$P = B \int_{\mu=0}^{\mu=1} \mathrm{d}P$$

Remember that $\mathrm{d}P = \mathrm{d}P(\mathrm{d}\mu)$. So, we'll have to integrate over μ.

11. How would you go about finding the power coefficient C_P of the entire turbine? Use the following definition: $\mathrm{d}A = c(\mu)\mathrm{d}\mu R$, where $c(\mu)$ is a chord length as a function of μ.

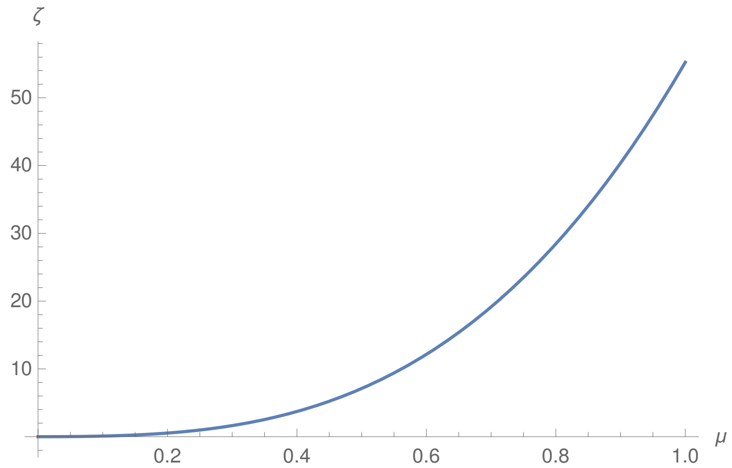
The power coefficient of the entire turbine is the total harvested power divided by the power density (at hub-height, though this is not relevant in a uniform wind field) and the total rotor area πR^2. That gives:

$$C_\mathrm{P} = \frac{P}{\frac{1}{2}\rho u_\infty^3 \; \pi R^2} = \frac{B}{\pi R} \int_{\mu=0}^{\mu=1} \lambda\mu \left(1+\mu^2\lambda^2\right)^{\frac{1}{2}} \xi_n c(\mu) d\mu$$

12. If we use the above model that we've described to this point, for some given parameter values ($\lambda = 7$, $c_0 = 0.15R$, $c_1 = 0.05R$, $u_\infty = 10$ m/s, $\rho = 1.225$ kg/m^3, $R = 50$ m and $B = 3$), can you find how much power the full turbine will extract?

    *here assume that the chord is a linear interpolation between the chord c_1 at the tip and the chord c_0 at the root: $c(\mu) = c_0 + (c_1 - c_0)\mu$, what gives us $\mathrm{d}A = (c_0 + (c_1 - c_0)\mu)\,\mathrm{d}\mu R$

    (a) plot the power harvesting factor ζ vs. μ



    (b) find how much power the full turbine will extract

When we plug in our values into the dP expression, we get the following ugly numeric expression:

$$\mathrm{d}P \approx 1.2 \times 10^6 \mu^2 (1.5 - \mu)\sqrt{49\mu^2 + 1}$$

We can integrate this expression numerically between $\mu = 0$ and $\mu = 1$ to get:

$$P = B \int_0^1 \mathrm{d}P \approx 4.5 \cdot 10^6 \text{ W} = 4.5 \text{ MW}$$