Exercises for Lecture Course on Numerical Optimization (NUMOPT)
Albert-Ludwigs-Universität Freiburg – Summer Semester 2025

Exercise 5: Exam Type questions and last chapters

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The solutions for these exercises will be given and discussed during the exercise session on July 1st.
To receive feedback on your solutions, please hand it in during the exercise session on July 1st, or by e-mail to leo.simpson@imtek.uni-freiburg.de before the same date.

I Hanging chain, the last episode: the Interior Point Method

Consider once again the hanging chain problem, with zero rest length, and ground constraints:

$$\begin{aligned} \underset{y \in \mathbb{R}^{N-1}, z \in \mathbb{R}^{N-1}}{\text{minimize}} \quad & \frac{1}{2}\sum_{i=0}^{N-1} D\left((y_i - y_{i+1})^2 + (z_i - z_{i+1})^2\right) + g \sum_{i=0}^{N} m\, z_i \\ \text{subject to} \quad & z_i \geq 0.5 \qquad\qquad\; \text{for } i = 1, \dots, N-1, \\ & z_i \geq 0.5 + 0.1 y_i \quad \text{for } i = 1, \dots, N-1, \\ & z_i \geq -1 - y_i \qquad \text{for } i = 1, \dots, N-1 \end{aligned} \tag{1}$$

with the same parameters as before. In this exercise, we will treat the problem in its standard form:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) = \frac{1}{2} x^\top Q x + c^\top x \\ \text{subject to} \quad & a_j^\top x + b_j \geq 0 \quad \text{for } j = 1, \dots, m \end{aligned} \tag{2}$$

where $x = \begin{bmatrix} y & z \end{bmatrix}^\top$ is the decision variable. You do not need to compute Q, c, a_j and b_j explicitly, this is already implemented for you in the file `hanging_chain_ip_matrices.py`.
Here, we will implement an interior point method to solve this problem.

1. What type of problem is (2)?
2. A popular approach to solve this problem is *the interior point method.* In this algorithm, we choose a decreasing sequence of barrier parameters $\tau^{[k]}$, solve iteratively the following problems:

   $$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f^{[k]}(x) := f(x) - \tau^{[k]} \sum_{j=1}^{m} \log\left(a_j^\top x + b_j\right) \tag{3}$$

   Are these problems convex?
3. Compute the gradient and the Hessian of the function $f^{[k]}(x)$.
4. A popular variant of this algorithm is to perform only one Newton step at each iteration to find the next iterate $x^{[k+1]}$. Write the update rule for $x^{[k+1]}$.

   Remark: *In practice, we use backtracking line-search as a globalization strategy, but you do not have to write the rules of the line-search here.*
5. Complete the code in the file `hanging_chain_ip.py` to implement the interior point method.

   Like in exercise 3, a visualization of the iterates is provided.

   Comment what you see.

II Sequential Quadratic Programming

In this exercise, we study the Sequential Quadratic Programming (SQP) method for a general nonlinear programming (NLP) problem:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0 \end{aligned} \tag{4}$$

This method consists of solving an approximate problem at each iteration, where the objective is approximated by a quadratic function and the constraints are approximated by linear functions. These approximations are based on the Taylors expansion at the current solution points.

1. Write down the KKT conditions of the generic NLP (4).
2. Formulate the generic QP subproblem that would result if your current iterate is $x^{[k]}$.

   Hint: You can formulate it with the decision variable $p = x - x^{[k]}$.
3. Prove that if $x^{[k]}$ is a KKT point (for some multipliers $\lambda^{[k]}$ and $\mu^{[k]}$) of the NLP problem, and that the Hessian is positive (i.e. $\nabla^2 f\left(x^{[k]}\right) \succ 0$), then it is a solution to the QP subproblem.

   Hint: Meaning that $p = 0$ is a solution if you choose $p = x - x^{[k]}$ as the decision variable.
4. Show the converse, i.e. if $x = x^{[k]}$ is the solution of the QP subproblem, then $x^{[k]}$ is a KKT point.

   Hint: You can use the fact that for two matrices A and B, if $\operatorname{Ker}\left(A^\top\right) \subset \operatorname{Ker}\left(B^\top\right)$ then $\operatorname{Im}\left(B\right) \subset \operatorname{Im}\left(A\right)$

III A sample exam question

Regard the following minimization problem:

$$\begin{aligned} \underset{x \in \mathbb{R}^2}{\text{minimize}} \quad & x_2^4 + (x_1+2)^4 \\ \text{subject to} \quad & x_1 - x_2 = 0, \\ & x_1^2 + x_2^2 \leq 8 \end{aligned}$$

1. How many scalar decision variables, how many equality, and how many inequality constraints does this problem have?

2	

2. Sketch the feasible set $\Omega \in \mathbb{R}^2$ of this problem.

3	

3. Bring this problem into the NLP standard form

$$\begin{aligned} \underset{x \in \mathbb{R}^2}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0, \\ & h(x) \geq 0 \end{aligned}$$

   by defining the functions f, g, h appropriately.

3	

FROM NOW ON UNTIL THE END TREAT THE PROBLEM IN THIS STANDARD FORM.

4. Is this optimization problem convex? Justify your answer.

2

5. Write down the Lagrangian function of this optimization problem.

2

6. A feasible solution of the problem is $\bar{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. What is the active set $\mathcal{A}(\bar{x})$ at this point?

2

7. Is the *linear independence constraint qualification (LICQ)* satisfied at $\bar{x}$? Justify.

3

8. An optimal solution of the problem is $x^\star = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. What is the active set $\mathcal{A}(x^\star)$ at this point?

1

9. Is the *linear independence constraint qualification (LICQ)* satisfied at $x^\star$? Justify.

2

10. Describe the tangent cone $T_\Omega(x^\star)$ (the set of feasible directions) to the feasible set at this point $x^\star$, <u>by a set definition formula with explicitly computed numbers</u>.

2

11. Write down the KKT conditions for the point $x^\star$ and solve them to find the multipliers $\lambda^\star$ and $\mu^\star$.

4	

12. Formulate the second order necessary conditions for optimality (SONC) for this problem and test if they are satisfied at $(x^\star, \lambda^\star, \mu^\star)$. Can you prove whether $x^\star$ is a local or even global minimizer?

4	