Exercises for Lecture Course on Numerical Optimization (NUMOPT)
Albert-Ludwigs-Universität Freiburg – Winter Term 2023/2024

Exercise 5: Exam Type Question

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Exercise Tasks

1. **A sample exam question.**

   Regard the following minimization problem:

$$\min_{x \in \mathbb{R}^2} \quad x_2^4 + (x_1+2)^4 \quad \text{s.t.} \quad \begin{cases} x_1^2 + x_2^2 & \leq & 8 \\ x_1 - x_2 & = & 0. \end{cases}$$

   (a) How many scalar decision variables, how many equality, and how many inequality constraints does this problem have?

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   two scalar decision variables, 1 equality constraint, 1 inequality constraint

   (b) Sketch the feasible set $\Omega \in \mathbb{R}^2$ of this problem.

3	

   (c) Bring this problem into the NLP standard form

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \begin{cases} g(x) & = & 0 \\ h(x) & \geq & 0 \end{cases}$$

   by defining the functions f, g, h appropriately.

$$f(x) = x_2^4 + (x_1+2)^4$$
$$g(x) = x_1 - x_2$$
$$h(x) = 8 - x_1^2 - x_2^2$$

FROM NOW ON UNTIL THE END TREAT THE PROBLEM IN THIS STANDARD FORM.

(d) Is this optimization problem convex? Justify. $f(x)$ is convex, $g(x)$ is affine, $h(x)$ is concave $\Rightarrow$ the problem is convex

2	

(e) Write down the Lagrangian function of this optimization problem.

$$\begin{aligned}\mathcal{L}(x,\lambda,\mu) &= f(x) - \lambda^\top g(x) - \mu^\top h(x) \\ &= x_2^4 + (x_1+2)^4 - \lambda(x_1 - x_2) - \mu(8 - x_1^2 - x_2^2)\end{aligned}$$

where $\lambda, \mu \in \mathbb{R}$.

2	

(f) A feasible solution of the problem is $\bar{x} = (2,2)^T$. What is the active set $\mathcal{A}(\bar{x})$ at this point? $h(\bar{x}) = 8 - 2^2 - 2^2 = 0 \Rightarrow$ the constraint is active, $\mathcal{A}(\bar{x}) = \{1\}$ (This notation interprets $h(x)$ as vector valued function with only one dimension, i.e. a "scalar vector")

2	

(g) Is the *linear independence constraint qualification (LICQ)* satisfied at $\bar{x}$? Justify.
Check linear independence of $\nabla g(\bar{x})$ and $\nabla h_i(\bar{x})$, $i \in \mathcal{A}$ or whether $\begin{bmatrix}\nabla g(\bar{x}) & \nabla h_1(\bar{x})\end{bmatrix}$ is full rank.

$$\nabla g(x) = \begin{bmatrix}1\\-1\end{bmatrix} = \nabla g(\bar{x}) \qquad \nabla h_1(x) = \begin{bmatrix}-2x_1\\-2x_2\end{bmatrix},\ \nabla h_1(\bar{x}) = \begin{bmatrix}-4\\4\end{bmatrix}$$

$$\det\begin{bmatrix}\nabla g(\bar{x}) & \nabla h_1(\bar{x})\end{bmatrix} = \det\begin{bmatrix}1 & -4\\-1 & -4\end{bmatrix} = 6 > 0 \Rightarrow \text{full rank} \Rightarrow \text{LICQ satisfied}$$

3	

(h) An optimal solution of the problem is $x^* = (-1,-1)^T$. What is the active set $\mathcal{A}(x^*)$ at this point? $h(x^*) = 6 > 0 \Rightarrow \mathcal{A}(x^*) = \{\}$ (no active inequality constraints)

1	

(i) Is the *linear independence constraint qualification (LICQ)* satisfied at x^* ? Justify.

$$\nabla g(x^*) = \begin{bmatrix}1\\-1\end{bmatrix} \Rightarrow \text{full rank} \Rightarrow \text{LICQ satisfied}$$

2	

(j) Describe the tangent cone $T_\Omega(x^*)$ (the set of feasible directions) to the feasible set at this point x^*, by a set definition formula with explicitly computed numbers.

LICQ holds at x^*, so the tangent cone and the linearized feasible cone coincide:

$$T_\Omega(x^*) = \mathcal{F}(x^*) = \{p \in \mathbb{R}^n \mid \nabla g_i(x^*)^\top p = 0, i = 1, \ldots, m \;\&\; \nabla h_i(x^*)^\top p = 0, i \in \mathcal{A}(x^*)\}$$

Here:

$$\begin{aligned}\mathcal{F}(x^*) &= \{p \in \mathbb{R}^2 \mid \nabla g(x^*)^\top p = 0\} = \{p \in \mathbb{R}^2 \mid \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix} = 0\} \\ &= \{p \in \mathbb{R}^2 \mid p_1 = -p_2\} = \{t\begin{bmatrix}1\\-1\end{bmatrix} \mid t \in \mathbb{R}\}\end{aligned}$$

2	

(k) Compute the Lagrange gradient and find the multiplier vectors λ^*, μ^* so that the above point x^* satisfies the KKT conditions.

general KKT conditions for inequality constraint optimization

$$\begin{aligned}\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = \nabla f(x^*) - \nabla g(x^*)\lambda^* - \nabla h(x^*)\mu &= 0 \\ g(x^*) &= 0 \\ h(x^*) &\geq 0 \\ \mu^* &\geq 0 \\ \mu_i^* h_i(x^*) &= 0, \quad i = 1, \ldots, q\end{aligned}$$

Here:

$h(x^*) > 0 \Rightarrow \underline{\mu^* = 0}$

$g(x^*) = 0 \quad \checkmark$

$\nabla_x \mathcal{L}(x, \lambda, \mu) = \begin{bmatrix}4(x_1+2)^3\\4x_2^3\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix}\lambda - \begin{bmatrix}-2x_1\\-2x_2\end{bmatrix}\mu$

$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = \begin{bmatrix}4-\lambda^*\\-4+\lambda^*\end{bmatrix} = 0 \Leftrightarrow \underline{\lambda^* = 4}$

4	

(l) Describe the critical cone $C(x^*, \mu^*)$ at the point (x^*, λ^*, μ^*) <u>in a set definition using explicitly computed numbers</u>

$$\begin{aligned}\mathcal{C}(x^*, \mu^*) = \{p \in \mathbb{R}^n \mid \quad & \nabla g_i(x^*)^\top p = 0, i = 1, \dots, m \\ & \& \nabla h_i(x^*)^\top p = 0, i \in \mathcal{A}_+(x^*) \\ & \& \nabla h_i(x^*)^\top p \geq 0, i \in \mathcal{A}_0(x^*)\}\end{aligned}$$

Here ($\mathcal{A} = \{\}$):

$$\mathcal{C}(x^*, \mu^*) = \{p \in \mathbb{R}^2 \mid \nabla g(x^*)^\top p = 0\} = \mathcal{F}(x^*) = \{t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mid t \in \mathbb{R}\}$$

3	

(m) Formulate the second order necessary conditions for optimality (SONC) for this problem and test if they are satisfied at (x^*, λ^*, μ^*). Can you prove whether x^* is a local or even global minimizer?

SONC: Regard x^* with LICQ. If x^* is a local minimizer of the NLP, then

i. $\exists \lambda^*, \mu^*$ such that KKT conditions hold

ii. $\forall p \in \mathcal{C}(x^*, \mu^*)$ holds $p^\top \nabla_x^2 \mathcal{L}(x^*, \lambda^*, \mu^*) p \geq 0$

Here:

$$\nabla_x^2 \mathcal{L}(x, \lambda, \mu) = \begin{bmatrix} 12(x_1+2)^2 + 2\mu & 0 \\ 0 & 12x_2^2 + 2\mu \end{bmatrix}, \quad \Lambda^* := \nabla_x^2 \mathcal{L}(x^*, \lambda^*, \mu^*) = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$$

check SONC

i. holds due to task (1k)

ii. $\Lambda^* \succ 0 \Rightarrow p^\top \Lambda^* p \geq 0 \; \forall p \in \mathbb{R}^n$, therefore this specifically holds also for $\forall p \in \mathcal{C}(x^*, \mu^*)$

$\Rightarrow$ SONC are satisfied

Due to $\Lambda^* \succ 0$ we furthermore have $p^\top \Lambda^* p > 0 \; \forall p \in \mathbb{R}^n \setminus \{0\}$, and therefore specifically $\forall p \in \mathcal{C}(x^*, \mu^*) \setminus \{0\}$. Thus SOSC also holds, and x^* is a local minimizer. Due to convexity of the NLP this is equivalent to x^* being a global minimizer.

Alternative: Theorem 13.6. *For convex NLP and x^* with LICQ holds:*
x^* is a global minimizer $\Leftrightarrow \exists \lambda, \mu$ such that KKT conditions hold.
We know the righthandside to be true, so x^* is a global minimizer.