Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2023-2024

Bonus Exercise

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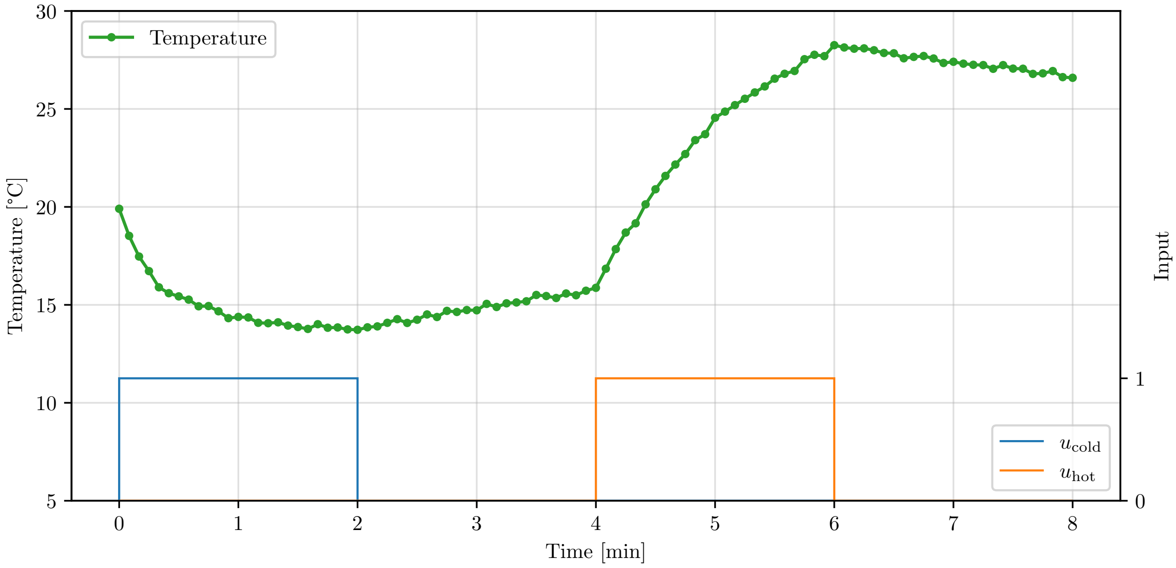
This is a bonus exercise, to apply the learned knowledge of the MSI course.

Consider a bathtub is filling up with water from two faucets, one for cold water and one for warm water. Unfortunately, we don't know the temperatures of water from the faucets. The (water) mass flow rate of both faucets is known as $F = 0.2\,\mathrm{kg\,s^{-1}}$, and they can be either turned on or off, which we denote by the controls $u_{\text{cold}}(t) \in \{0, 1\}$ and $u_{\text{hot}}(t) \in \{0, 1\}$.

The temperature of the water will change over time, depending on the controls applied. We have noisy measurements of the temperature as well as exact knowledge when the faucets were turned on and off, shown in the figure below. Initially, there is $m_0 = 5\,\mathrm{kg}$ of water in the bathtub, at the environment temperature of $T_0 = 20\,°\mathrm{C}$.



We can model the temperature in the bathtub by considering both the total heat energy $Q(t)$ (relative to $0\,°\mathrm{C}$) and the mass $m(t)$ (in kg) of the water inside the tub. These then relate to the temperature $T(t)$ (in °C) of the water by

$$Q(t) = cm(t)T(t)$$

where $c = 4200\,\mathrm{J\,kg^{-1}\,K^{-1}}$ is the heat capacity of water. The heat energy of the water changes because of heat flows from incoming hot and cold streams from the faucets as well as heat flows from and to the surrounding environment with temperature $T_0 = 20\,°\mathrm{C}$, depending on an unknown conduction constant k. We can summarize this as a differential equation for the heat energy Q given by

$$\frac{d}{dt}Q(t) = -k(T(t) - T_0) + F\left(T_{\text{cold}}u_{\text{cold}}(t) + T_{\text{hot}}u_{\text{hot}}(t)\right)$$

The mass of the water changes due to the controls as

$$\frac{d}{dt}m(t) = F\left(u_{\text{cold}}(t) + u_{\text{hot}}(t)\right)$$

We can summarize these equations as a nonlinear continuous state-space model, with state $x(t) = [Q(t),\ m(t)]^\top$

$$\begin{aligned}
\dot{x} &= f(x(t), u(t)) = \begin{bmatrix} -k(T(t) - T_0) + F\left(T_{\text{cold}} u_{\text{cold}}(t) + T_{\text{hot}} u_{\text{hot}}(t)\right) \\ F\left(u_{\text{cold}}(t) + u_{\text{hot}}(t)\right) \end{bmatrix} \\
y &= g(x(t), u(t)) = T(t) + \epsilon(t)
\end{aligned}$$

where $\epsilon(t)$ is i.i.d. Gaussian measurement noise with a standard deviation of $\sigma_\epsilon = 0.2\,°\text{C}$.

Your Task Estimate the unknown temperatures of the faucets $T_{\text{cold}}, T_{\text{hot}}$ (both in °C) and the constant k (in WK^{-1}) from the given measurement data. Also give the 1-sigma confidence interval of the estimate. You can find the data along with this sheet on the course website.

Hints *Here are two ways to do this:*

1. *Since this problem only contains measurement noise, you can use Nonlinear least squares. Discretize the dynamics, construct a residual function $R(\theta)$ and minimize the squared residuals.*

2. *Apply an EKF to estimate the parameters online. For this you will need to extend the state vector, discretize the dynamics, linearize both the dynamics and the output, and iterate the prediction and update steps of the Kalman filter.*