Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2023-2024

Exercise 7: Dynamic Systems

(to be returned on Jan 8th, 8:30)

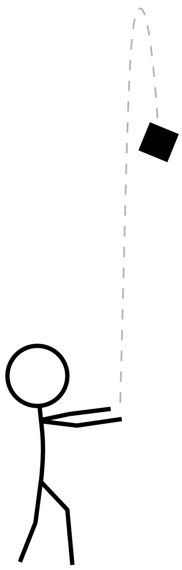
Prof. Dr. Moritz Diehl, Katrin Baumgärtner, Jakob Harzer, Yizhen Wang,
Adithya Anoop Thoniparambil, Premnath Srinivasan

Consider the trajectory of a some object thrown vertically into the air, in a gravity field for which we would like to estimate the constant of acceleration g. Using multiple cameras, we measure the position p of the object with measurement noise ϵ as $y = p + \epsilon$.

1. **(Continous and Discrete Model)** From kinematics we know that the object follows the simple ordinary differential equation

$$\ddot{p} = -g.$$

To formulate a model, let p and v correspond to the vertical position and velocity of the object, respectively.



(a) Paper: By defining a state vector $x = [p, v]^\top \in \mathbb{R}^2$, we can reformulate this kinematic model into the linear affine state-space form:

$$\dot{x} = Ax + b \tag{1a}$$

$$y = Cx + d + \epsilon \tag{1b}$$

Define the values and dimensions of the matrices A and C as well of the vectors b and d accordingly. (1 points)

(b) Paper: Show that the analytical solution to the differential equation (1a), with initial state $x_0 = x(0)$,

$$x(t) = \exp(At)x_0 + \int_0^t \exp(A(t-\tau))b \, d\tau$$

is for this model equivalent to:

$$x(t) = x_0 + (Ax_0 + b)t + \frac{1}{2}Abt^2$$

Hint: Take a close look at the high order terms of the expansion of the matrix exponential given by $\exp(At) := \mathbb{I} + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 \dots$ (2 points)

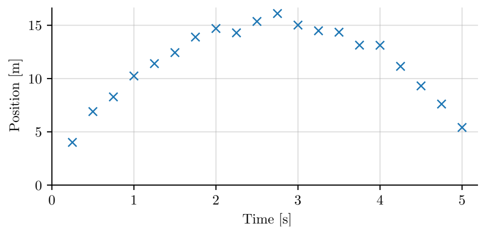
(c) Paper: Since we later will have measurements on an equidistant time grid with stepsize h, we are interested to find a discrete state-space model

$$x_{k+1} = \tilde{A}x_k + \tilde{b} \tag{2a}$$

$$y_k = \tilde{C}_k x_k + \tilde{d} + \epsilon_k \tag{2b}$$

where $x_k = x(t_k)$ is the state at some sampling time $t_k = kh$ and $x_{k+1} = x(t_{k+1}) = x(t_k + h)$ is the state at the next sampling time. Use the equations of the analytical solution above to define the matrices and vectors $\tilde{A}, \tilde{b}, \tilde{C}$ and $\tilde{d}$ of the discrete model. (2 points)

2. We assume that the position measurements (the output of our model) are subject to i.i.d. noise ϵ_k with a standard deviation of $\sigma_\epsilon = 0.3\,\text{m}$. The series of $N = 20$ measurements $y_1, y_2, \ldots, y_N$ lies on an evenly spaced time grid $t_k = kh, k = 1, \ldots, N$, with $h = 0.25\,\text{s}$.



(Physical Model) From the solution of the previous tasks we can derive a model for the measurements that is inspired from physics as

$$y_k = p_0 + v_0 t_k - \frac{1}{2} g t_k^2 + \epsilon_k \tag{3}$$

that is parameterized by the initial position $p_0 = p(t_0)$, initial velocity $v_0 = v(t_0)$ and the constant g.

(a) Paper: To estimate the unknown parameters $\theta = [p_0,\, v_0,\, g]^\top \in \mathbb{R}^5$ we want use linear least squares with the measurement model above and thus want to solve the problem

$$\theta_{\text{phy}} = \arg\min_\theta \|y_{\text{phy}} - \Phi_{\text{phy}}\theta\|_2^2\,.$$

Define the vectors y_{phy} and the matrix Φ_{phy}, along with their dimensions. (0.5 points)

(b) Code: Use linear least squares to find the solution to the estimation problem. Estimate the covariance matrix Σ_θ of the estimation result. Complete the code to plot the trajectory of the predicted output with the found parameters over time, along with a 2-sigma confidence interval of the predicted output y. For this you will need to calculate the covariance of the predicted outputs using the formula

$$\Sigma_{y_{\text{phy}}} = \Phi_{\text{phy}} \Sigma_\theta \Phi_{\text{phy}}^\top + \Sigma_\epsilon$$

(2 points)

(c) Paper: Where is the experiment taking place? (0 points)

(ARX Model) Another possible modelling choice is to use an autoregressive model

$$y_{k+1} = \theta_1 y_{k-1} + \theta_2 y_k + \theta_3 \tag{4}$$

that relates the next output to previous outputs and a constant. With this model, we can only estimate the constant g, not the initial position and velocity.

(d) Paper: Show how the discrete linear state-space model (2a) can be reformulated into the ARX model (4). How does the constant g relate to the coefficients $\theta = [\theta_1\,, \theta_2\,, \theta_3]^\top$? (1 points)

(e) Paper: Fit an ARX model to the data by formulating an LLS problem

$$\theta_{\text{ARX}} = \arg\min_\theta \|y_{\text{ARX}} - \Phi_{\text{ARX}}\theta\|_2^2\,.$$

Define the vectors y_{ARX} and the matrix Φ_{ARX}, along with their dimensions. (0.5 points)

(f) Code: Use LLS to find the coefficients of the ARX model. Use the code in the template to visualize a 'rollout' of the model. (0.5 points)

(g) Paper: Why do you think the ARX modelling approach performs so much worse? (0.5 points)

This sheet gives in total 10 points