Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2023-2024

Exercise 3: Optimality Conditions and Linear Least Squares

(to be returned on November 20th, 8:30)

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The aim of this sheet is to strengthen your knowledge in optimality conditions and least squares estimation.

Exercise Tasks

1. PAPER: Given the function $f(x) : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^\top Q\, x + c^\top x$ and fixed $c \in \mathbb{R}^n$.

   (a) Consider the not necessarily symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and compute the gradient $\nabla f(x) \in \mathbb{R}^n$ and the Hessian $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ of this function for any x.

   (b) If Q is symmetric, what properties does it have to fulfil such that the unique minimizer x^* can be computed?

   (c) Compute the unique minimizer and the minimum function value $f(x^*)$ under the correct assumptions. (3 points)

2. PAPER: We would like to find the parameters $\hat{\theta}_{\text{LS}}$ of a linear model $y(k) = \phi(k)^\top \theta + \epsilon(k)$, where $\epsilon(k) \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is an additive i.i.d. zero-mean Gaussian noise that perturbed a series of N scalar measurements $y_N = [y(1), \dots, y(N)] \in \mathbb{R}^N$. From the lecture we know that $\hat{\theta}_{\text{LS}}$ can be computed using least-squares:

$$\hat{\theta}_{\text{LS}} = \arg\min_{\theta} \frac{1}{2} \|y_N - \Phi_N \theta\|_2^2$$

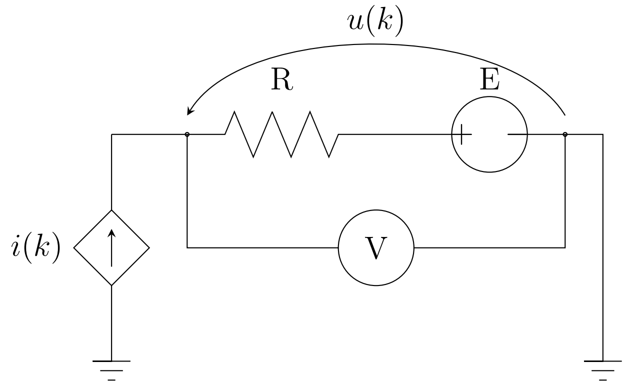
   where $\Phi_N \in \mathbb{R}^{N\times d}$. Assume that σ_ϵ^2 is known.

   (a) State the matrix Φ_N and the solution of least squares problem $\hat{\theta}_{\text{LS}}$.

   (b) Calculate the covariance of the least squares estimate $\text{cov}(\hat{\theta}_{\text{LS}})$.

   Hint: Recall from Exercise 2 that the covariance matrix of a vector-valued variable $Y = AX + b$ *for a constant* $A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$ *is given by* $\text{cov}(Y) = A\ \text{cov}(X) A^\top$. (2 points)

3. Consider the following experimental set up to estimate the values of E and R.



You obtain a dataset containing N measurements of the voltage $u(k)$ for different values of $i(k)$. We assume that the input measurements $i(k)$ is not affected by noise, but that the measurements $u(k)$ are affected by i.i.d. additive noise $n_{\mathrm{u}}(k)$. Under these assumptions the measurement model is given by:

$$u(k) = m(k) + n_{\mathrm{u}}(k) \quad \text{where} \quad m(k) = E + R \cdot i(k).$$

(a) CODE: Load the dataset containing the measurements. Plot the data with 'x' markers and add labels to both axis. (0.5 points)

(b) PAPER: Formulate the problem of estimating the parameters E and R as a least squares problem where $\theta = [E\ R]^\top \in \mathbb{R}^2$ by defining $\Phi \in \mathbb{R}^{N\times 2}$ and $y \in \mathbb{R}^N$ such that the optimizer is given by $\hat{\theta} = \arg\min_{\theta} \|y - \Phi\theta\|_2^2$. (1 points)

(c) CODE: Use the least squares estimator formulated in the previous subtask to find the experimental values of R and E for the dataset. Plot the linear fit through the measurement data. (1 points)

(d) CODE: Plot a histogram of the residuals defined as $r(k) = u(k) - m(k)$, where $m(k) = E + R \cdot i(k)$ is the voltage determined by the model, and $u(k)$ are the obtained measurements. (0.5 points)

(e) PAPER: Give an educated guess for the type of noise distribution. (0.5 points)

As we will learn later in the lecture, for i.i.d. measurement noise, the covariance matrix of the d estimated parameters $\hat{\theta}$ from a single experiment can be approximated by

$$\hat{\Sigma}_{\hat{\theta}} = \frac{\|y - \Phi\hat{\theta}\|_2^2}{N - d}(\Phi^\top \Phi)^{-1}$$

From this estimated covariance matrix, the estimated variances for the single components of the parameter vector can be extracted as the respective diagonal entry.

(f) CODE: Calculate the estimated variance of both the R and E estimate and print their values along with a 1-σ confidence estimate in the format of

$$\text{(quantity)} = \text{(value)} \pm \text{(1 standard deviation)}$$

(1.5 points)

This sheet gives 10 points in total.