Real-Time Algorithms for Nonlinear Model Predictive Control

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MPC and RL Summer School
October 9, 2023

Complex Sensor Actuator Systems

SENSORS

- GPS
- acceleration
- radar
- vision
- ...

How to connect ?

ACTUATORS

- flight surfaces
- steering wheel
- motor speeds
- joint torques
- ...



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Classical Linear Controllers / Linear Filters

Map from one time series into another

$$\ldots, y_{i-1}, y_i, y_{i+1}, \ldots \longrightarrow \ldots, u_{i-1}, u_i, u_{i+1}, \ldots$$

Classical Linear Controllers / Linear Filters

Map from one time series into another

$$\boxed{\ldots, y_{i-1}, y_i}, y_{i+1}, \ldots \longrightarrow \ldots, u_{i-1}, \boxed{u_i}, u_{i+1}, \ldots$$

Special case: linear time invariant (LTI) filters

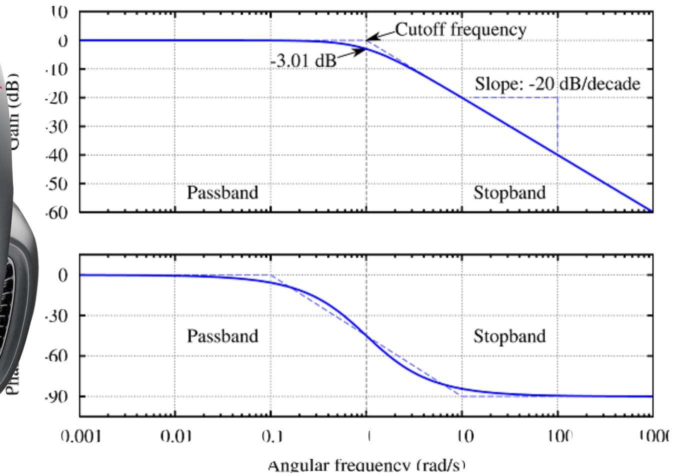
$$u_k = \sum_{i=0}^{N} a_i y_{k-i}$$

action output = weighted sum of past measurement inputs

Linear Filters are Everywhere…

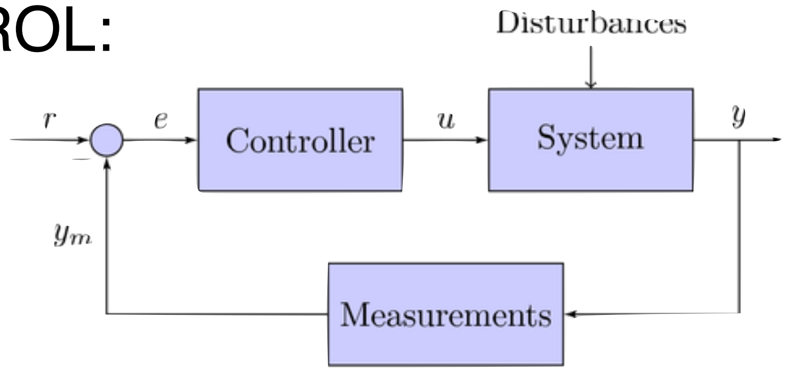
AUDIO SYSTEMS:
- Dolby,
- Echo and other effects
- active noise control

...



FEEDBACK CONTROL:
- PID controller,
- Kalman filter,
- LQR,

...



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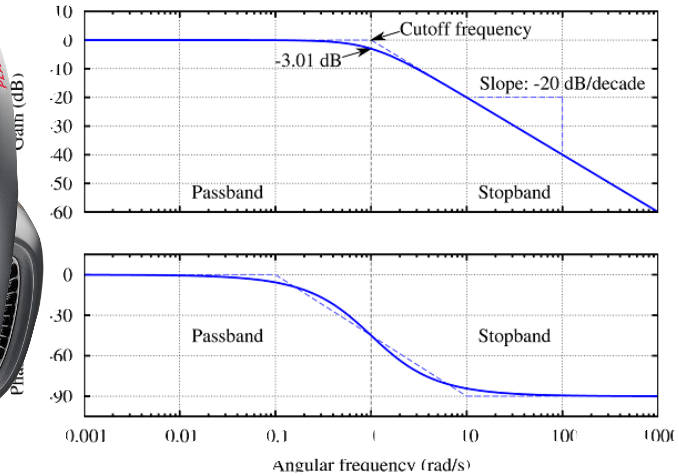

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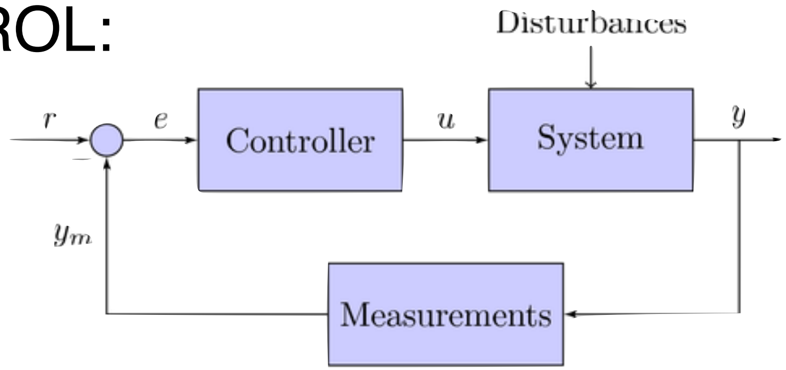

…but they need lots of tuning to cope with constraints and nonlinearities.

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Complex Sensor Actuator Systems

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- acceleration
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How to connect ?
linear controllers,
fuzzy logic,
neural networks, or:

ACTUATORS

- flight surfaces
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- ...


Complex Sensor Actuator Systems

SENSORS

- GPS
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How to connect ?
linear controllers,
fuzzy logic,
neural networks, or:
embedded optimisation

ACTUATORS

- flight surfaces
- steering wheel
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- ...


Embedded Optimization: a CPU-Intensive Map

x → EMBEDDED OPTIMIZATION → u

$$u = \arg\min_{u} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} g \\ h \end{bmatrix} \quad \text{s.t. } Au + Bx \leq b$$



Solve, in real-time and repeatedly, an optimization problem that depends on the incoming stream of input data, to generate a stream of output data.

The ubiquity of parametric convex optimization

THEOREM [Baes, D., Necoara, 2008]
Every continuous map

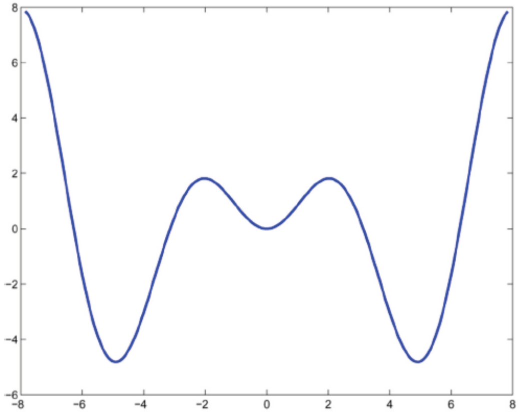
$$\begin{aligned} \mu : \mathbb{R}^{n_x} &\to \mathbb{R}^{n_u} \\ x &\mapsto u = \mu(x) \end{aligned}$$

can be represented as parametric convex program (PCP):

$$\mu(x) = \arg\min_{u} g(u, x) \quad \text{s.t.} \quad (u, x) \in \Gamma$$

PCP: objective and feasible set jointly convex in parameters and variables (x, u).

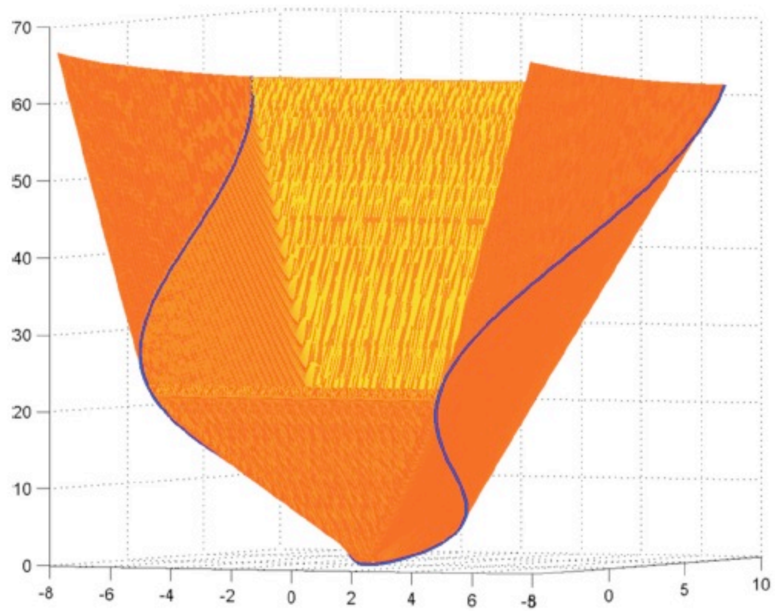
(Sketch of Proof)



Given: graph of $\mu(x)$

Construct epigraph E of $g(u, x)$

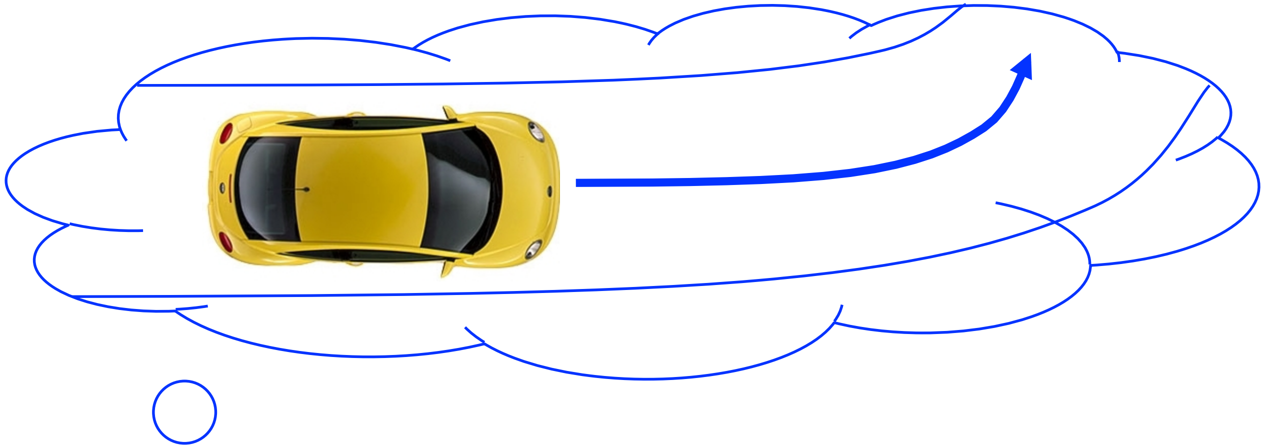
1. "Bend" graph of $\mu(x)$ using strictly convex $g^0(x)$
2. Add upward rays.
3. Take convex hull.
4. Show that minima are preserved.



$$
\begin{aligned}
S &:= \{(x, \mu(x), t) | x \in \Omega, g^0(x) \leq t\} \\
E &:= \mathrm{conv}(S)
\end{aligned}
$$

Prime Example: Model Predictive Control (MPC)

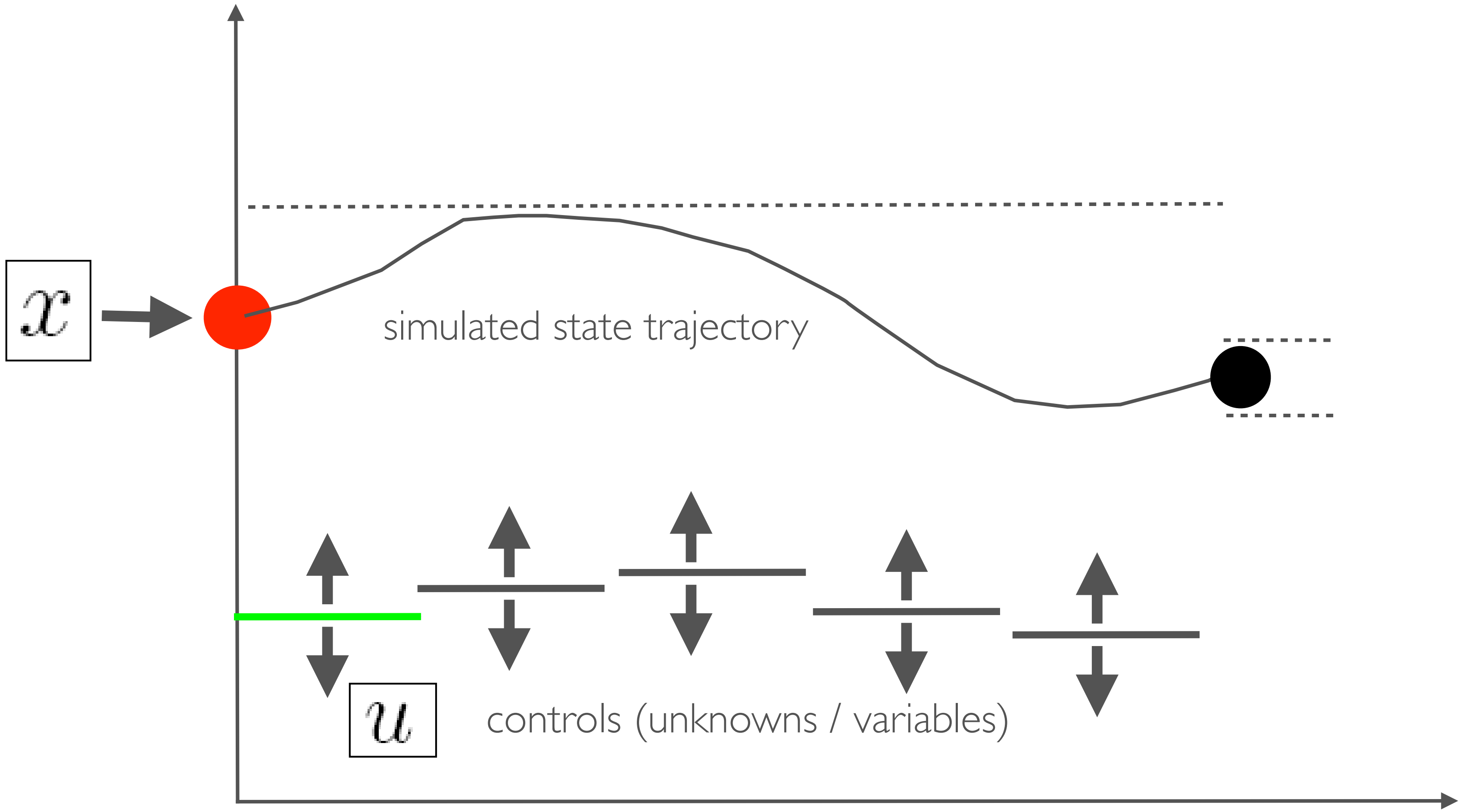
Always look a bit into the future



Example: driver predicts and optimizes, and therefore slows down before a curve

Open Loop Optimal Control Problem in MPC

For given system state $\boldsymbol{x}$, which controls $\boldsymbol{u}$ lead to the best objective value without violation of constraints ?



prediction horizon (length also unknown for time optimal MPC)

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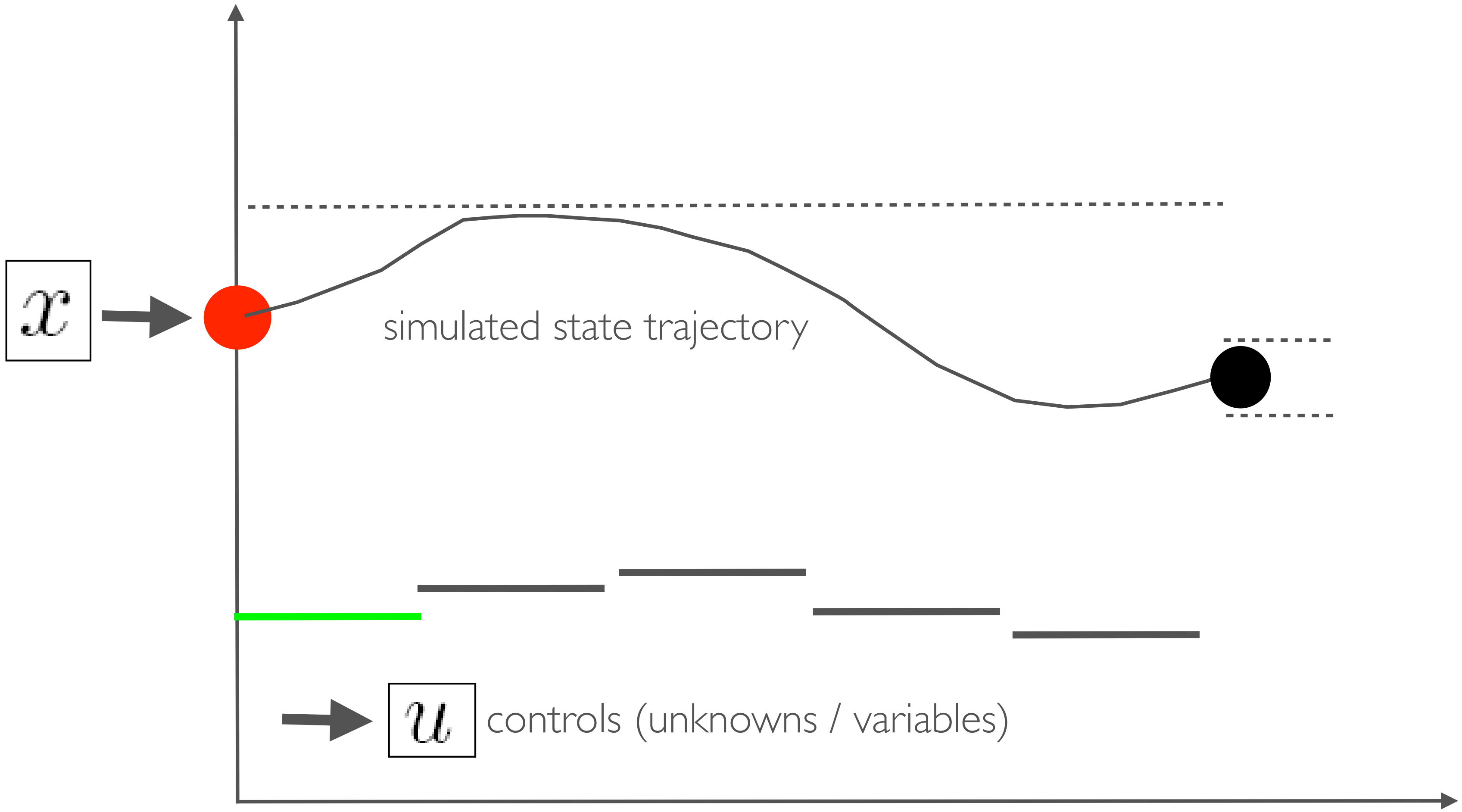


prediction horizon (length also unknown for time optimal MPC)

MPC creates a map from the initial value $\boldsymbol{x}$ to the first control $\boldsymbol{u}$
(which in fact approximates the optimal feedback control from the HJB equation)

MPC Example: Point-To-Point Motions [PhD Vandenbrouck 2012]

Fast oscillating systems (cranes, plotters, wafer steppers, …)

Control aims:

- reach end point as fast as possible
- do not violate constraints
- no residual vibrations

Idea: formulate as embedded optimization problem
in form of Model Predictive Control (MPC)

Time Optimal MPC of a Crane

SENSORS

- line angle
- cart position

→

MPC

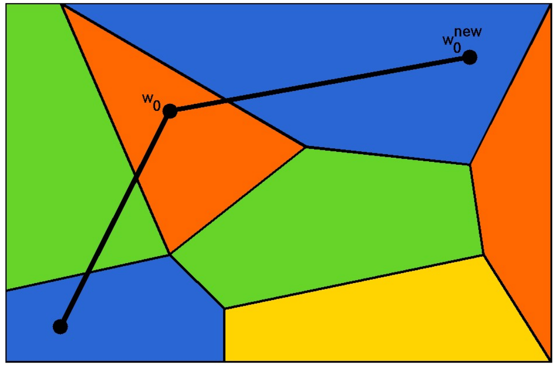
→

ACTUATOR

- cart motor

Hardware: xPC Target.   Software: qpOASES [Ferreau, D., Bock, 2008]

qpOASES



- "Quadratic Programming with the Online Active Set Strategy"
- Implements an parametric active set method with dense or sparse linear algebra in C/C++
- Open-source (LGPL): https://projects.coin-or.org/qpOASES, developed by Hans Joachim Ferreau with Christian Kirches, Andreas Potschka, …

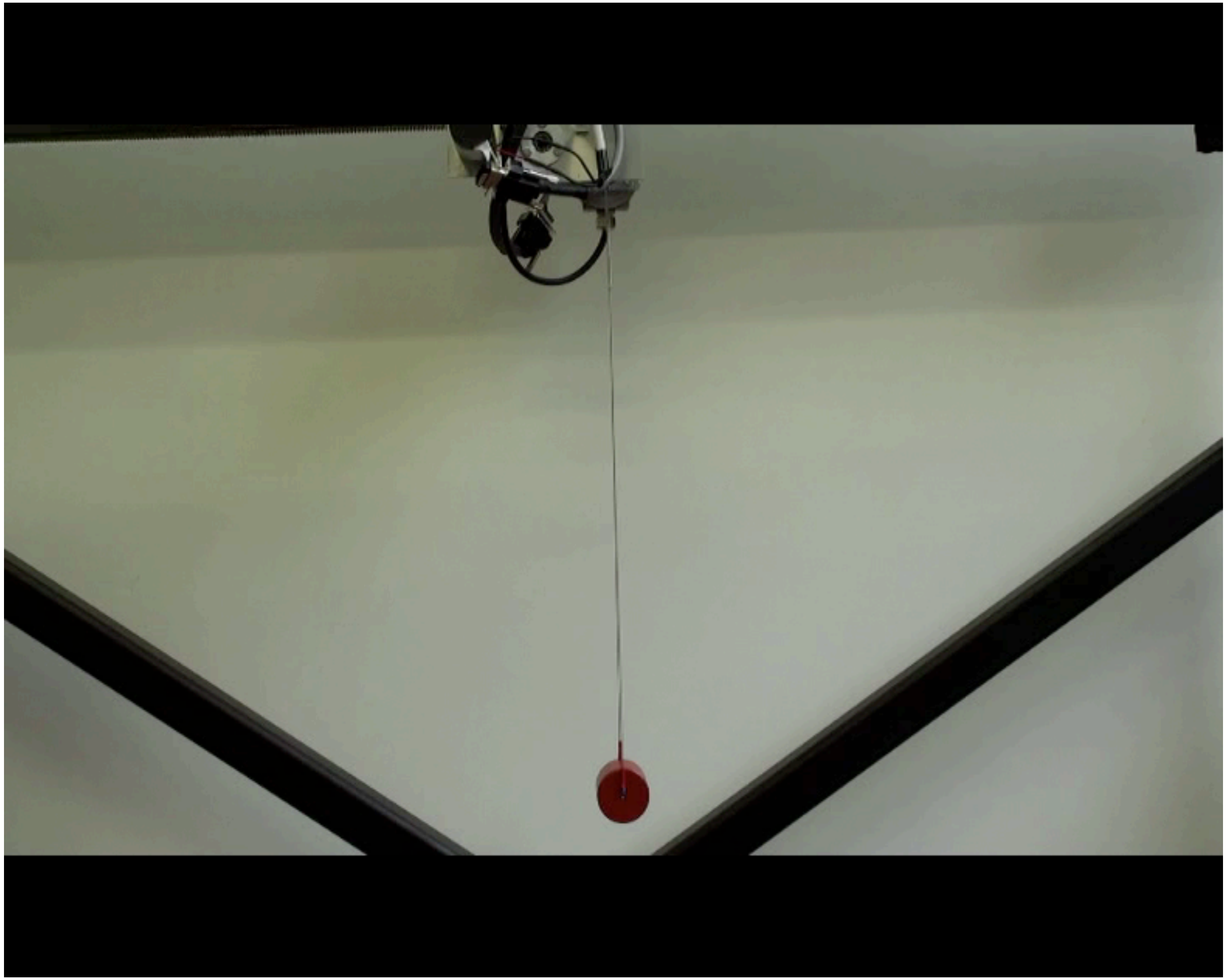


- Interfaced to C++, MATLAB, Simulink, CasADi, …

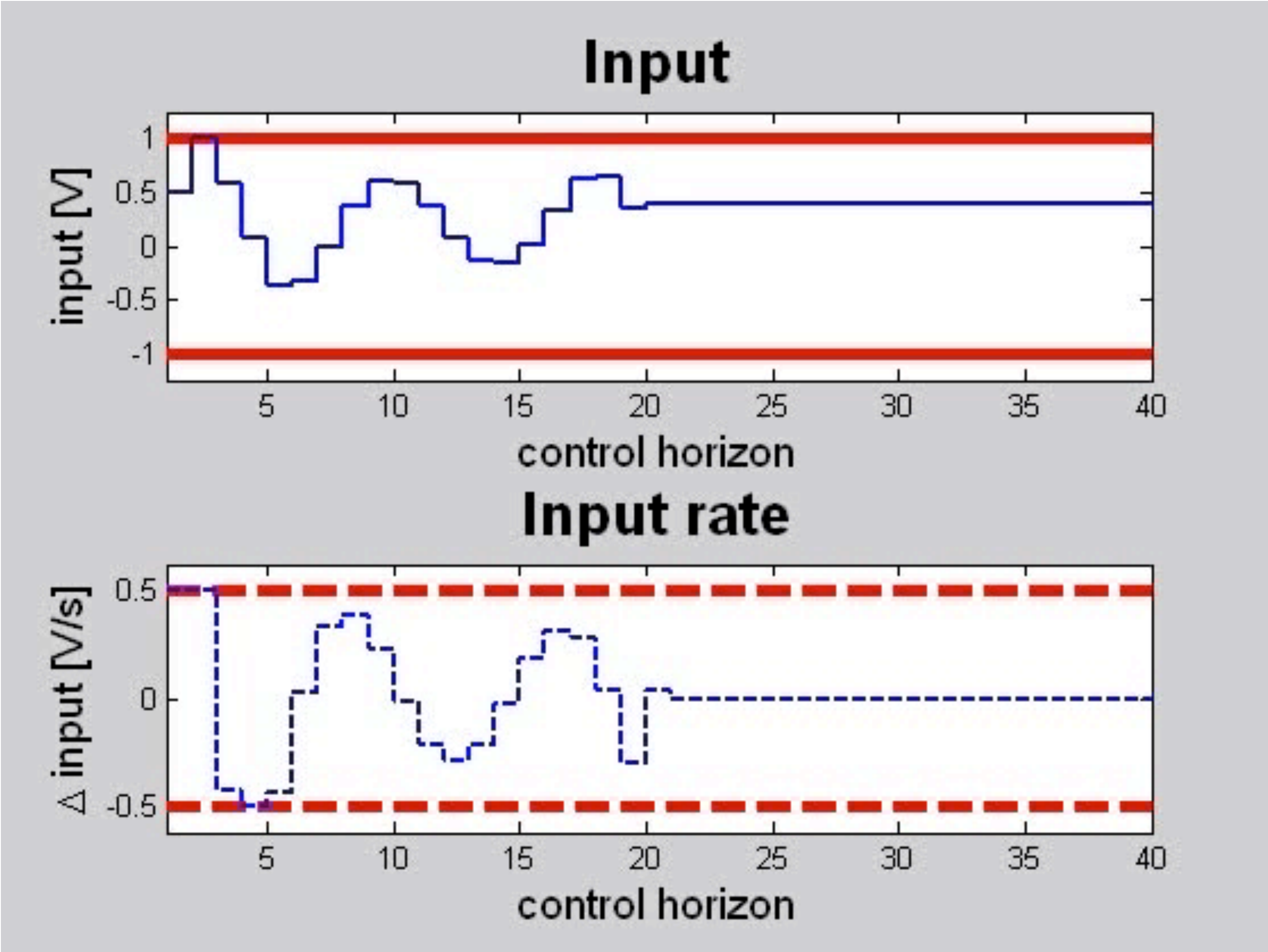
[Ferreau, Kirches, Potscka, Bock, D., Math. Prog. C, 2014]

Time Optimal MPC of a Crane

Univ. Leuven [Vandenbrouck, Swevers, D.]



Optimal solutions varying in time (inequalities matter)

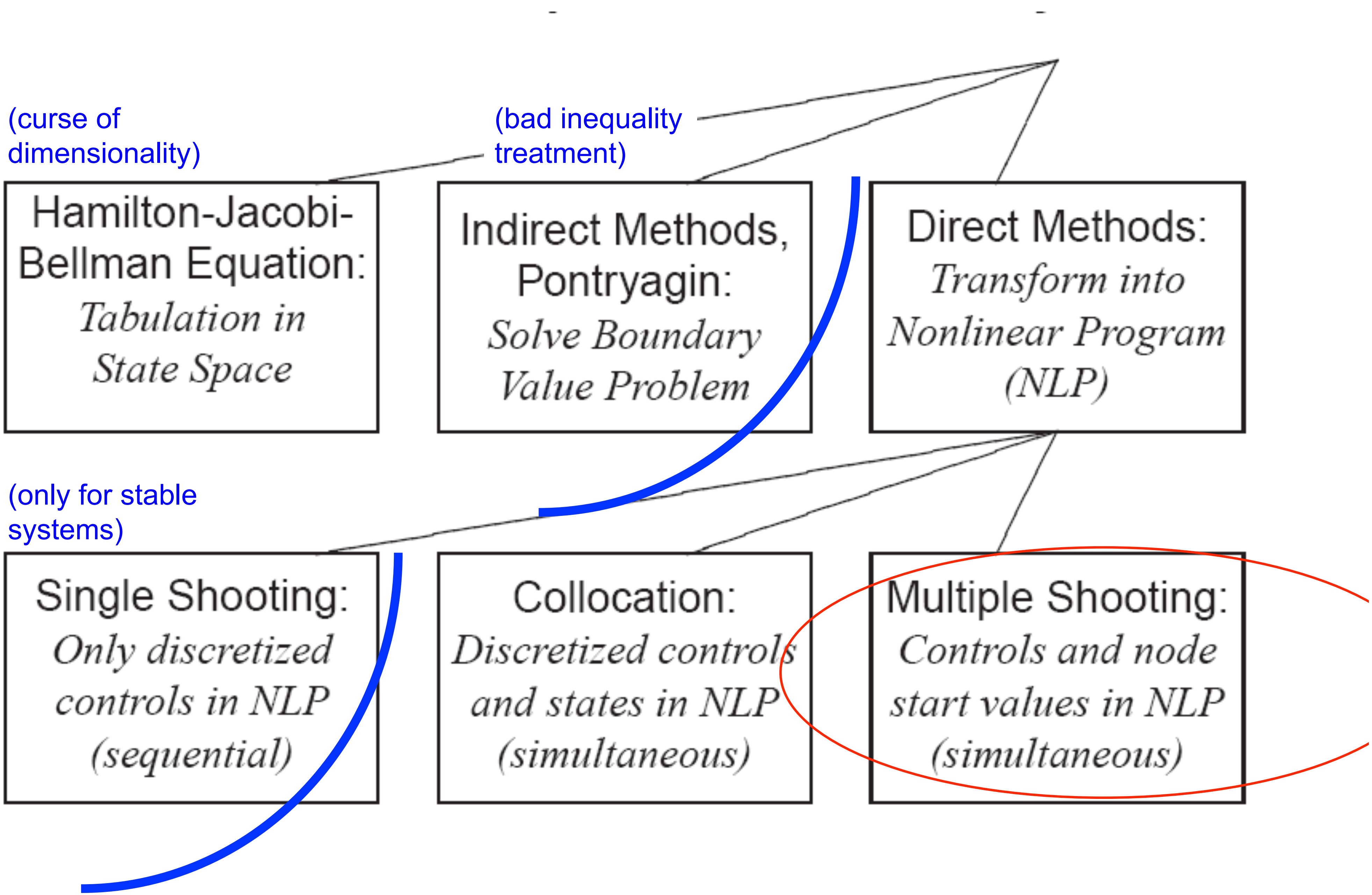


Solver qpOASES [PhD H.J. Ferreau, 2011], [Ferreau, Kirches, Potschka, Bock, D. , A parametric active-set algorithm for quadratic programming, Mathematical Programming Computation, 2014]

Overview

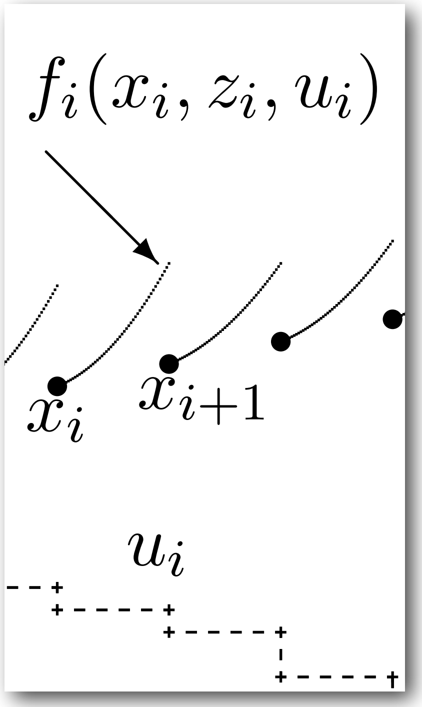
- Embedded Optimization and Model Predictive Control (MPC)
- **Real-Time Optimal Control Methods**
- Progress in Numerical Integration Methods
- Progress in Structured Quadratic Programming
- Some real-world NMPC applications

Optimal Control Solution Methods - Family Tree



Direct Multiple Shooting [Bock and Plitt, 1981] [Leineweber et al. 1999]

Hans Georg Bock



- Discretize controls piecewise on a coarse grid

$$u(t) = u_i \quad \text{for} \quad t \in [t_i, t_{i+1}]$$

- Solve relaxed DAE on each interval $[t_i, t_{i+1}]$ numerically, starting with artificial initial value x_i, z_i. Obtain trajectory pieces and state at end of interval, $f_i(x_i, z_i, u_i)$.

$$
\begin{array}{llcll}
\underset{x,\,z,\,u}{\text{minimize}} & \displaystyle\sum_{i=0}^{N-1} L_i(x_i, z_i, u_i) & + & E(x_N) & \\
\text{subject to} & x_0 - \bar{x}_0 & = & 0, & \\
& x_{i+1} - f_i(x_i, z_i, u_i) & = & 0, & i = 0, \ldots, N-1, \\
& g_i(x_i, z_i, u_i) & = & 0, & i = 0, \ldots, N-1, \\
& h_i(x_i, z_i, u_i) & \leq & 0, & i = 0, \ldots, N-1, \\
& r(x_N) & \leq & 0. &
\end{array}
$$

Dynamic Optimization Problem in NMPC

$$
\begin{array}{llcll}
\underset{x,\,z,\,u}{\text{minimize}} & \displaystyle\sum_{i=0}^{N-1} L_i(x_i, z_i, u_i) & + & E\,(x_N) & \\
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$$

Structured parametric Nonlinear Program

Initial Value $\bar{x}_0$ usually not known beforehand (“online data” in MPC)

Discrete time dynamics often come from ODE/DAE simulation, in direct multiple shooting method [Bock and Plitt, 1984]

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$$

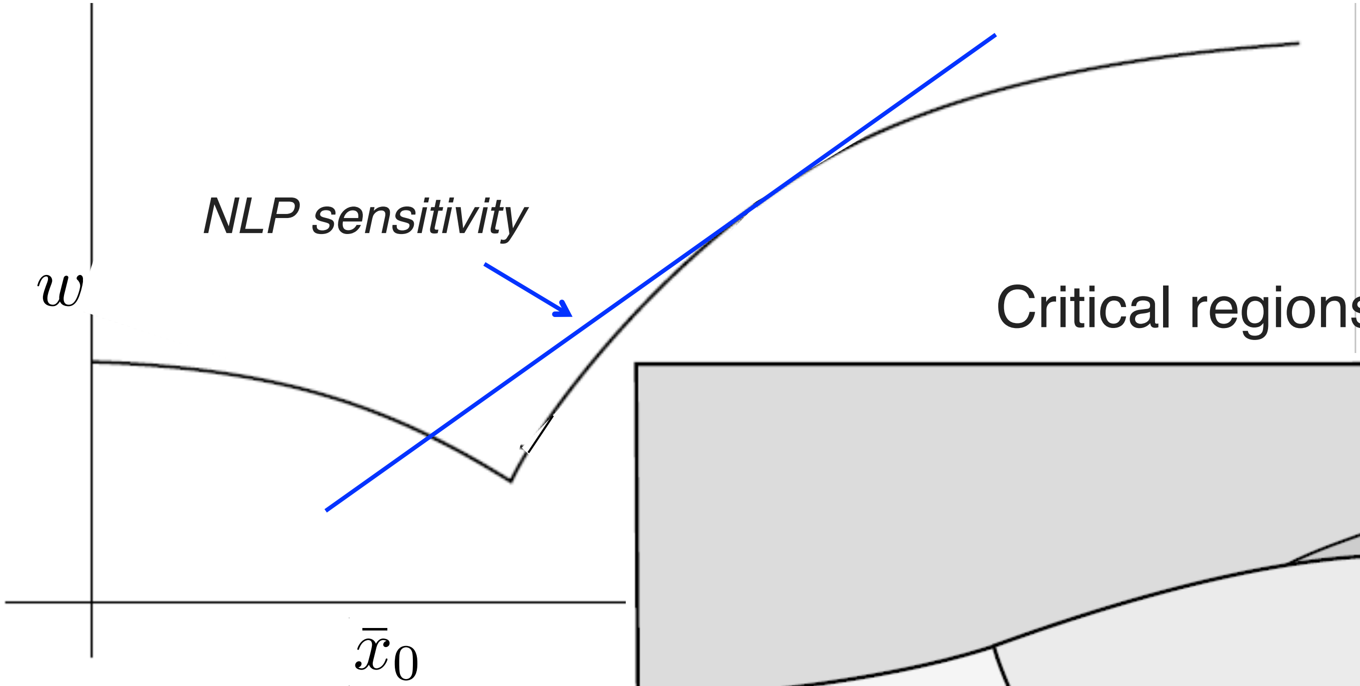
Summarize as

$$
\begin{array}{ll}
\underset{w \in \mathbb{R}^n}{\text{minimize}} & \phi(w) \\
\text{subject to} & g(w) + M\bar{x}_0 = 0 \\
& w \in \Omega
\end{array}
$$

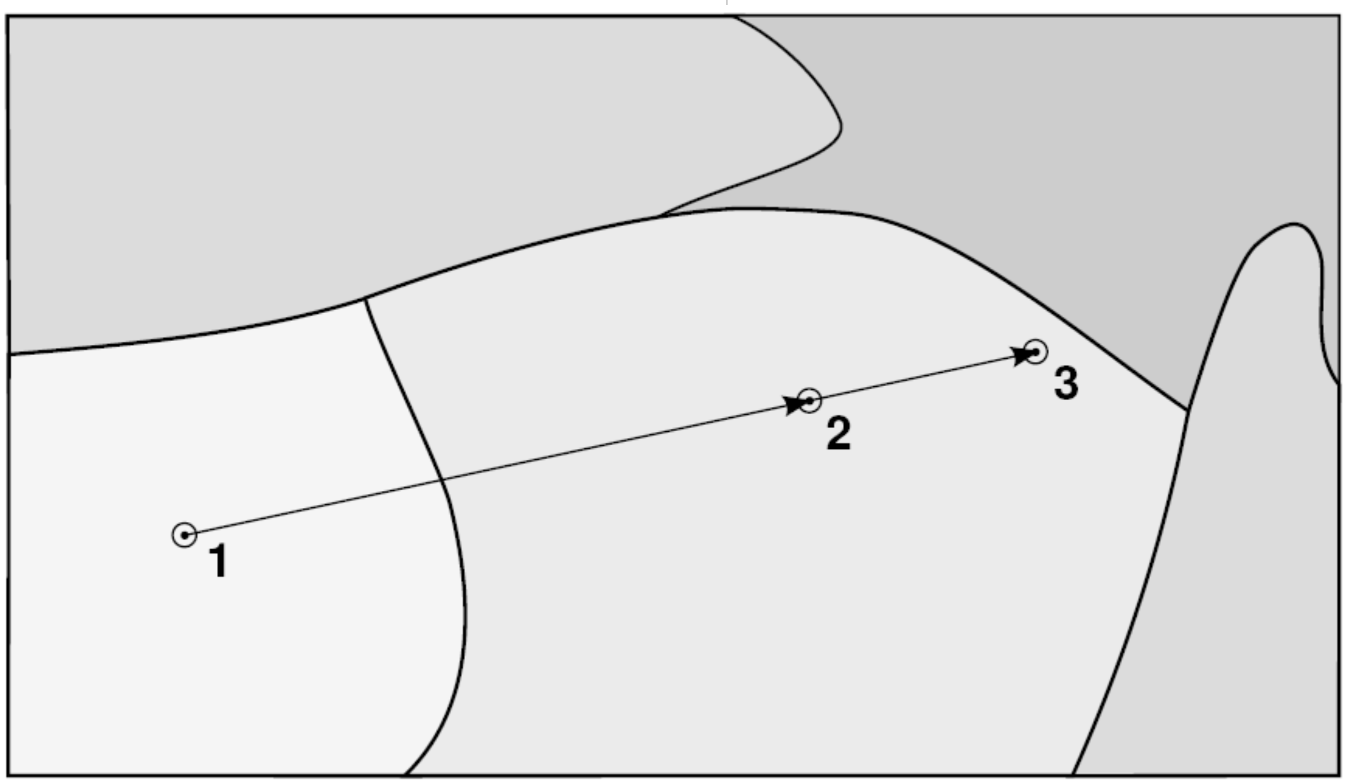
with convex ϕ and Ω

Nonlinear MPC = parametric Nonlinear Programming

Solution manifold is piecewise differentiable (kinks at active set changes)



Critical regions are non-polyhedral



How to deal with a sequence of large parameter changes?

NLP Pathfollowing

Sequential Convex Programming (SCP) and Real-Time Iteration (RTI)

$$\begin{array}{ll} \underset{w \in \mathbb{R}^n}{\text{minimize}} & \phi(w) \\ \text{subject to} & g(w) + M\bar{x}_0 = 0 \\ & w \in \Omega \end{array} \quad \rightarrow \quad \begin{array}{ll} \underset{w \in \mathbb{R}^n}{\text{minimize}} & \phi(w) \\ \text{subject to} & g(w_k) + g'(w_k)(w - w_k) + M \cdot (\bar{x}_0)_{k+1} = 0 \\ & w \in \Omega \end{array}$$

Repeat each sampling instant k two steps:

Step 1: Linearize nonlinear constraints at w_k to obtain convex problem (right).
(numerical integrations, nonlinear function and derivative evaluations)

Step 2: Obtain new value of parameter $(\bar{x}_0)_{k+1}$ and solve convex problem to obtain new iterate w_{k+1}. (quadratic program solution)

[D., Bock, Schloeder, Findeisen, Nagy, Allgower, JPC, 2002]

[Zavala, Anitescu, SICON, 2010]

[Tran Dinh, Savorgnan, D., SIOPT, 2013]

Sequential Convex Programming (SCP) and Real-Time Iteration (RTI)

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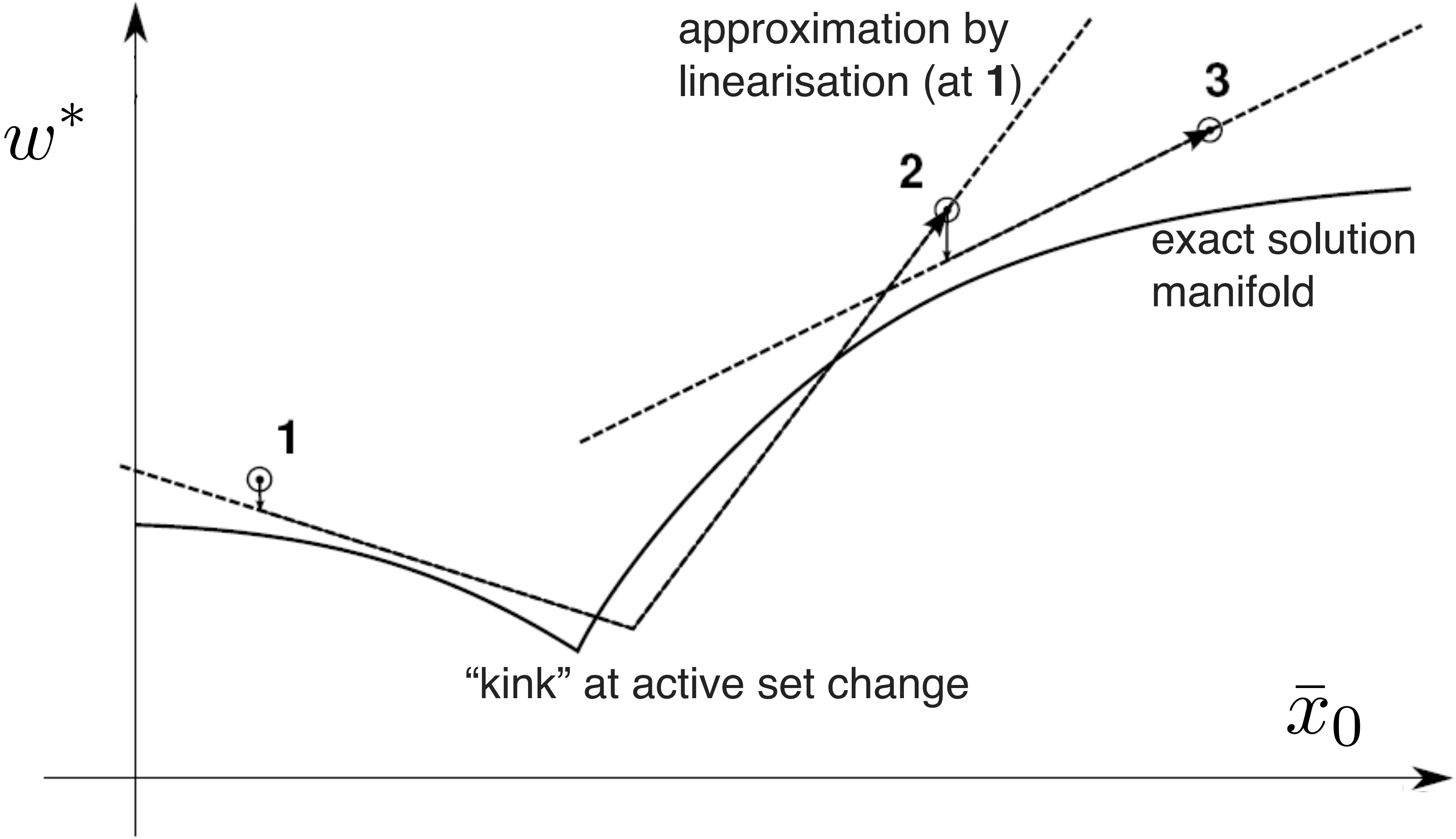
Real-Time Iteration

$$\begin{array}{ll} \underset{w \in \mathbb{R}^n}{\text{minimize}} & \phi(w) \\ \text{subject to} & g(w) + M\bar{x}_0 = 0 \\ & w \in \Omega \end{array}$$



Tangential prediction even across active set changes

Can divide computations in "preparation" and "feedback phase" [D. 2001]

Real-Time Iterations [PhD Diehl 2001, Heidelberg]

1) Keep states in problem - use direct multiple shooting [1]
2) Exploit convexity via Generalized Gauss-Newton [2]
3) Use tangential predictors for short feedback delay [3]
4) Iterate while problem changes (Real-Time Iterations) [4]
5) Auto-generate custom solvers in plain-C [5,6,7] (no malloc)

[1] Bock & Plitt, IFAC WC, 1984
[2] Bock 1983
[3] Bock, D. et al, 1999
[4] D. et al., 2002 / 2005
[5] Mattingley & Boyd, 2009
[6] Houska et al.: Automatica, 2011
[7] Verschueren et al.: Math.Prog. C, 2022

Computations in one Real-Time Iteration

NLP

$$
\begin{array}{lrcll}
\underset{x,\,z,\,u}{\text{minimize}} & \displaystyle\sum_{i=0}^{N-1} L_i(x_i, z_i, u_i) & + & E\,(x_N) & \\
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\end{array}
$$

1) Linearize constraints:
Integration & sensitivities

Sparse QP

$$
\begin{array}{lrcll}
\underset{x,\,u}{\text{minimize}} & \displaystyle\sum_{i=0}^{N-1} L_{\text{redQP},i}(x_i, u_i) & + & E_{\text{QP}}\,(x_N) & \\
\text{subject to} & x_0 - \bar{x}_0 & = & 0, & \\
& x_{i+1} - c_i - A_i x_i - B_i u_i & = & 0, & i = 0, \ldots, N-1, \\
& \bar{h}_i + \bar{H}_i^x x_i + \bar{H}_i^u u_i & \leq & 0, & i = 0, \ldots, N-1, \\
& r' + R x_N & \leq & 0. &
\end{array}
$$

2) Condense sparse QP

Condensed small QP

$$
\begin{array}{lrcl}
\underset{u}{\text{minimize}} & f_{\text{condQP},i}(\bar{x}_0, u) & & \\
\text{subject to} & \bar{r} + \bar{R}^{x_0} \bar{x}_0 + \bar{R}^u u & \leq & 0.
\end{array}
$$

3) Solve condensed QP

Computations in one Real-Time Iteration

NLP

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\begin{aligned}
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\end{aligned}
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\text{subject to} \quad & \bar{r} + \bar{R}^{x_0} \bar{x}_0 + \bar{R}^u u \leq 0.
\end{aligned}
$$

Can prepare without knowing $\bar{x}_0$
"Preparation phase"

1) Linearize constraints: Integration & sensitivities

2) Condense sparse QP

3) Solve condensed QP

Computations in one Real-Time Iteration

NLP

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\begin{array}{llrcll}
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\begin{array}{llrcll}
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\end{array}
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Condensed small QP

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\text{subject to} & \bar{r} + \bar{R}^{x_0} \bar{x}_0 + \bar{R}^u u & \leq & 0.
\end{array}
$$

"Feedback phase"

3) Solve condensed QP

Real-Time Iteration Contraction

$$\begin{array}{ll} \underset{w \in \mathbb{R}^n}{\text{minimize}} & \phi(w) \\ \text{subject to} & g(w) + M\bar{x}_0 = 0 \\ & w \in \Omega \end{array}$$

Regard primal dual iterates $z_k = (w_k, \lambda_k)$
and exact solutions z_k^* solving the full nonconvex problem for $(\bar{x}_0)_k$

Iterations are driven by parameter changes $\Delta\bar{x}_{0,k} := (\bar{x}_0)_{k+1} - (\bar{x}_0)_k$

We can establish a contraction estimate for the primal dual errors:

$$\boxed{\|z_{k+1} - z_{k+1}^*\| \leq (c_1 + c_2\|z_k - z_k^*\|)\ \|z_k - z_k^*\| + (c_3 + c_4\|\Delta\bar{x}_{0,k}\|)\ \|\Delta\bar{x}_{0,k}\|}$$

Contraction rate depends on bounds on nonlinearity, Jacobian error, and on strong regularity constant.

Contraction rate is independent of active set changes.

[Tran Dinh, Savorgnan, Diehl, SIOPT, 2013]

[Zanelli, Tran Dinh, Diehl, Automatica, 2021]

*** Advanced-Step Real-Time Iteration (AS-RTI)***

[Nurkanovic et al. 2020]

$$
\begin{aligned}
\underset{x,\,z,\,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} L_i(x_i, z_i, u_i) \quad + \quad E\left(x_N\right) \\
\text{subject to} \quad & x_0 - \bar{x}_0 = 0, \\
& x_{i+1} - f_i(x_i, z_i, u_i) = 0, \quad i = 0, \ldots, N-1, \\
& g_i(x_i, z_i, u_i) = 0, \quad i = 0, \ldots, N-1, \\
& h_i(x_i, z_i, u_i) \leq 0, \quad i = 0, \ldots, N-1, \\
& r\left(x_N\right) \leq 0.
\end{aligned}
$$

1) Linearize constraints: Integration & sensitivities

Sparse QP

$$
\begin{aligned}
\underset{x,\,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} L_{\text{redQP},i}(x_i, u_i) \quad + \quad E_{\text{QP}}\left(x_N\right) \\
\text{subject to} \quad & x_0 - \bar{x}_0 = 0, \\
& x_{i+1} - c_i - A_i x_i - B_i u_i = 0, \quad i = 0, \ldots, N-1, \\
& \bar{h}_i + \bar{H}_i^x x_i + \bar{H}_i^u u_i \leq 0, \quad i = 0, \ldots, N-1, \\
& r' + R x_N \leq 0.
\end{aligned}
$$

2) Condense sparse QP

Condensed small QP (feedback)

$$
\begin{aligned}
\underset{u}{\text{minimize}} \quad & f_{\text{condQP},i}(\bar{x}_0, u) \\
\text{subject to} \quad & \bar{r} + \bar{R}^{x_0} \bar{x}_0 + \bar{R}^u u \leq 0.
\end{aligned}
$$

3) Solve condensed QP for current state

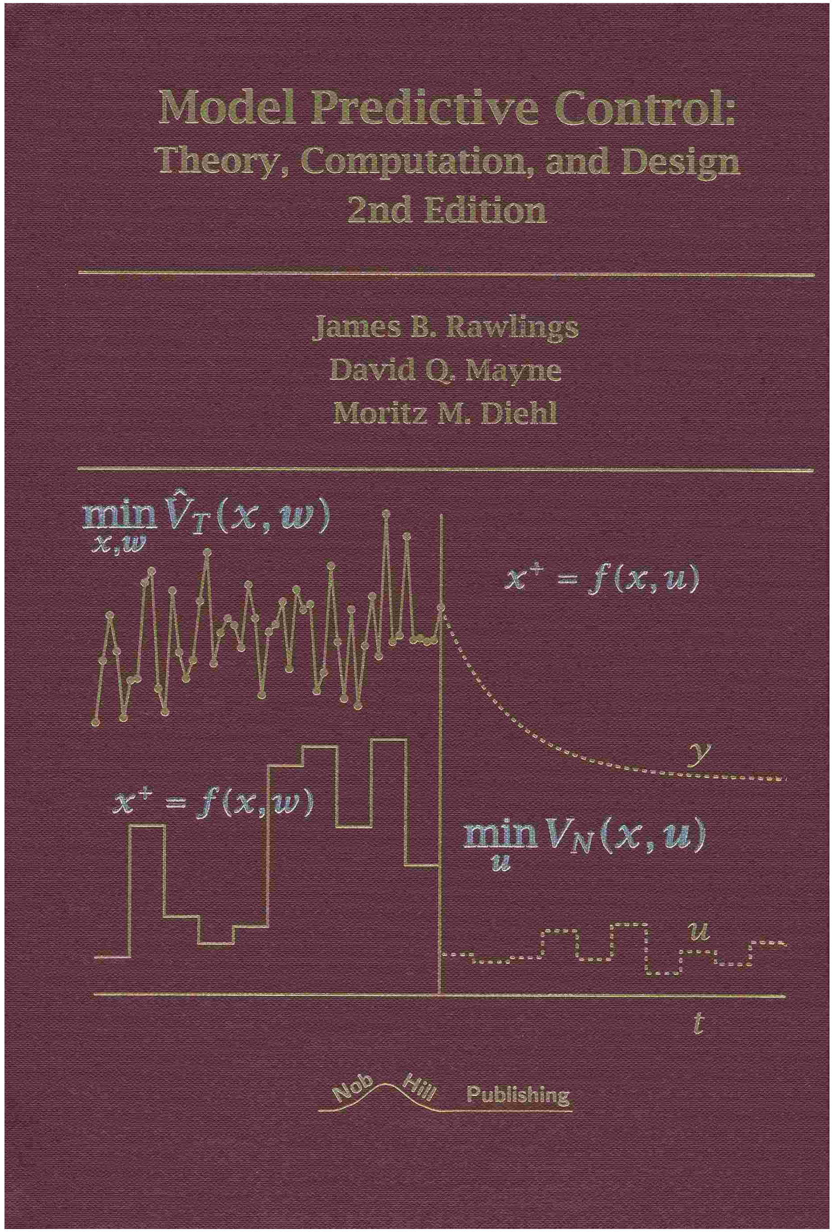
Condensed small QP (predicted state)

$$
\begin{aligned}
\underset{u}{\text{minimize}} \quad & f_{\text{condQP},i}(\bar{x}_0, u) \\
\text{subject to} \quad & \bar{r} + \bar{R}^{x_0} \bar{x}_0 + \bar{R}^u u \leq 0.
\end{aligned}
$$

4) Solve condensed QP again for predicted next state, for NLP initialisation

Also see MPC textbook, Ch. 8 on “Numerical Optimal Control”

(2nd edition, fourth printing, 618 pages, Nob Hill 2022)



Overview

- Embedded Optimization and Model Predictive Control (MPC)
- Real-Time Optimal Control Methods
- **Progress in Numerical Integration Methods**
- Progress in Structured Quadratic Programming
- Some real-world NMPC applications

Numerical Integrators for Embedded Optimization

General features:

- offline precomputation possible (model preprocessing, code generation)
- first and second order derivatives (Jacobian, Hessian) needed
- use implicit integration methods (with root-finding) for stiff and for DAE systems (e.g. Gauss-Legendre or Radau IIA collocation methods)

Some recent developments:

- ***Lifted implicit integrators*** perform only one root-finding iteration, combine advantages of direct collocation and direct multiple shooting
- ***Inexact Newton with Iterated Sensitivities (INIS)*** can work with inexact inverses of the implicit systems [SIAM J. Opt., 28(1), 74-95, 2018]
- ***General Nonlinear Static Feedback Structure (GNSF) Structure*** can be exploited [Frey et al. , ECC 2019]
- ***Casados-Integrators*** make acados numerics available in CasADi [Frey et al. , ECC 2023]
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Exploiting Linear Substructures in ODE/DAE Models

Generalized Nonlinear Static Feedback structure (GNSF)

$$E \begin{bmatrix} \dot{x}^{[1]} \\ z^{[1]} \end{bmatrix} = Ax^{[1]} + Bu + C\phi(\underbrace{L_{\dot{\mathrm{x}}}\dot{x}^{[1]} + L_{\mathrm{x}}x^{[1]} + L_{\mathrm{z}}z^{[1]}}_{=:y}, \underbrace{L_{\mathrm{u}}u}_{\hat{u}}) + c \tag{3a}$$

$$E^{\mathsf{LO}} \begin{bmatrix} \dot{x}^{[2]} \\ z^{[2]} \end{bmatrix} = A^{\mathsf{LO}}x^{[2]} + f_{\mathsf{LO}}(\dot{x}^{[1]}, x^{[1]}, z^{[1]}, u). \tag{3b}$$

Every ODE/DAE can be brought into this format. Gains are largest if output dimension of nonlinearity function ϕ is smallest

Implementation inside acados (successor of ACADO)

GNSF-IRK integrator implemented in acados using BLASFEO, including:

- Precomputation phase
- Simulation
- Efficient forward & adjoint sensitivity generation
- Output of algebraic variables z at start t_0
- and corresponding sensitivities

[Master thesis Jonathan Frey]

Numerical Experiments with Wind Turbine ODE Model

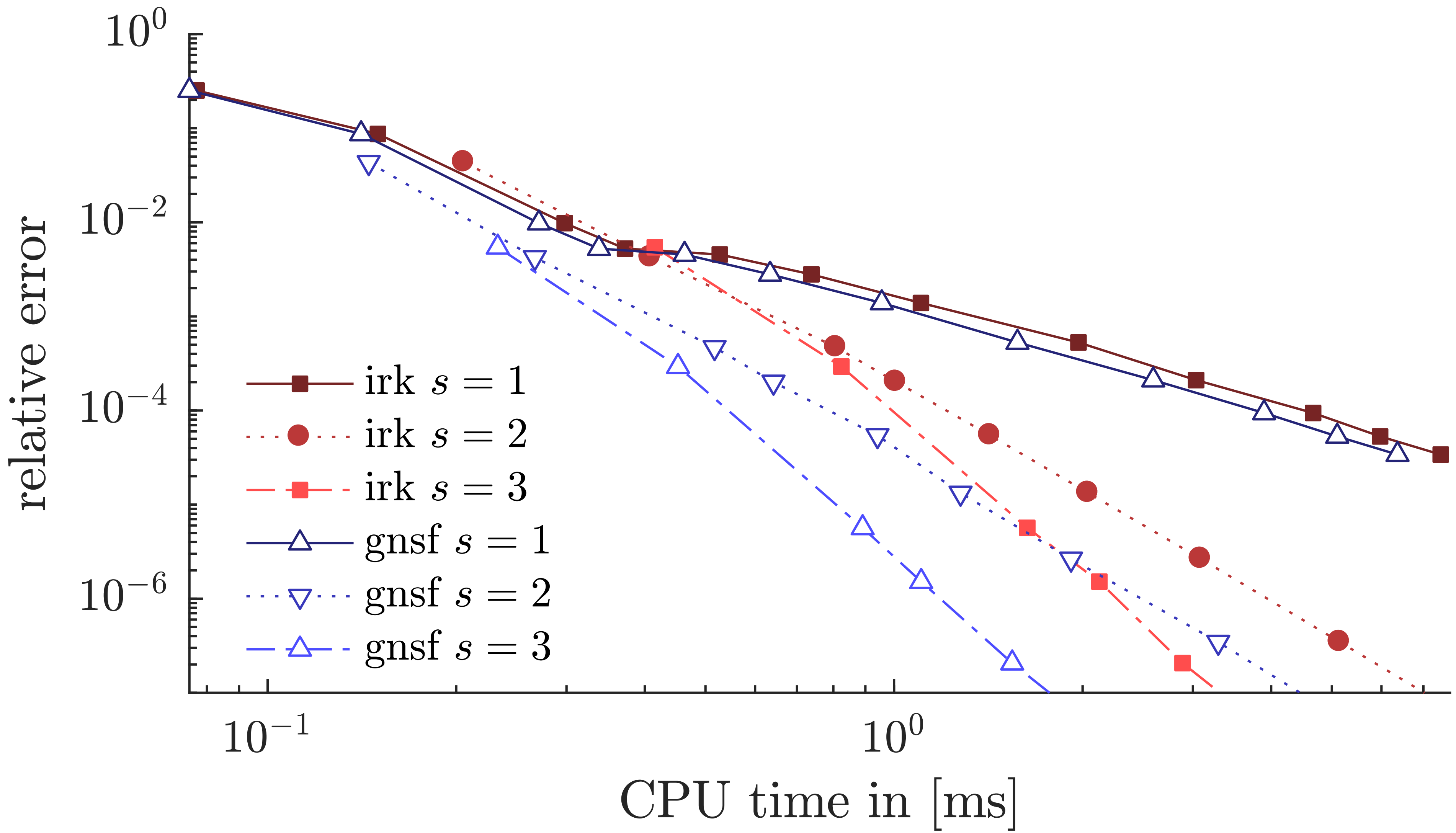
For numerical experiments, a wind turbine model was used:

- highly nonlinear
- polynomials to model aerodynamic coefficients
- used in the eco4wind project, provided by Senvion

Model dimensions				GNSF dimensions				
n_{x}	n_{u}	n_{z}	n_{param}	n_{x_1}	n_{z_1}	n_{out}	n_{y}	$n_{\hat{\mathrm{u}}}$
13	2	0	1	11	0	5	8	1

Numerical Comparisons with Different Stepsizes

CPU time includes forward derivatives, as needed in optimal control
Gauss-Legendre Collocation Integrators with s=1 to s=3 stages (2-6th order)



Numerical Comparisons with Different Stepsizes

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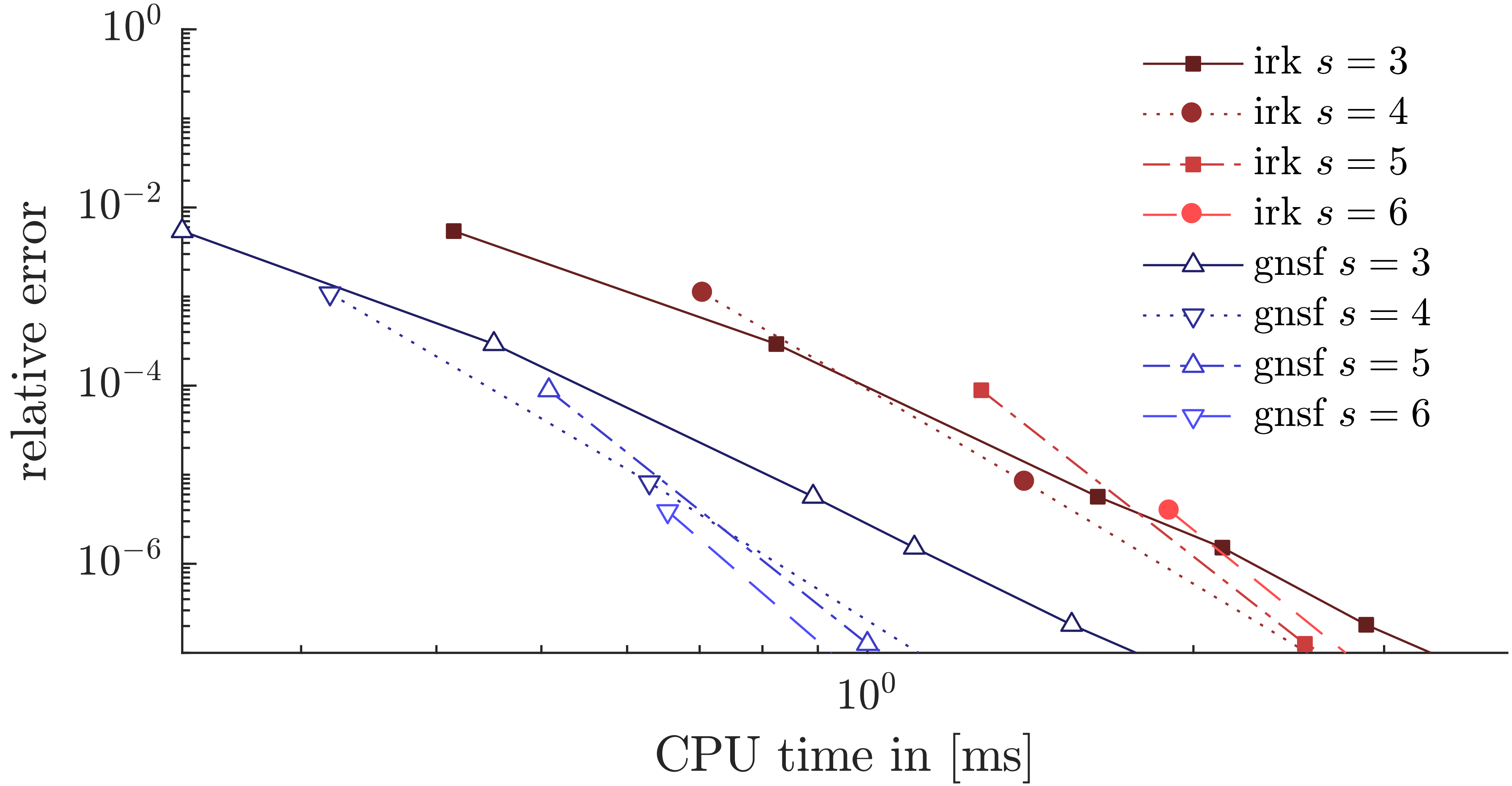
Gauss-Legendre Collocation Integrators with s=3 to s=6 stages (6th-12th order)



Speedup of GNSF integrators: factor 2-3 (depending on desired accuracy)

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Computations in one Real-Time Iteration (RTI)

NLP

$$
\begin{aligned}
\underset{x,\,z,\,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} L_i(x_i, z_i, u_i) \quad + \quad E\,(x_N) \\
\text{subject to} \quad & \begin{aligned}[t]
x_0 - \bar{x}_0 &= 0, \\
x_{i+1} - f_i(x_i, z_i, u_i) &= 0, \quad i = 0, \ldots, N-1, \\
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h_i(x_i, z_i, u_i) &\leq 0, \quad i = 0, \ldots, N-1, \\
r\,(x_N) &\leq 0.
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1) Linearize constraints:
Integration & sensitivities

Sparse Quadratic Program (QP)

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\begin{aligned}
\underset{x,\,u}{\text{minimize}} \quad & \sum_{i=0}^{N-1} L_{\mathrm{redQP},i}(x_i, u_i) \quad + \quad E_{\mathrm{QP}}\,(x_N) \\
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2) Solve sparse QP either

- via condensing, or
- via a Riccati type algorithm, or
- via a combination of both

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2a) **Condense** sparse QP

Condensed small QP

$$
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\text{subject to} & \bar{r} + \bar{R}^{x_0} \bar{x}_0 + \bar{R}^u u & \leq & 0.
\end{array}
$$

2b) Solve condensed QP
with small dense QP solver e.g.
qpOASES

Case Study: Quadratic Programming improvements 2012-2016 (all algorithms re-activated on same computer on 14.6.2017)

Andrea Zanelli

Dimitris Kouzoupis

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Comparison of different algorithmic QP solution approaches, using **ACADO Code Generation** on Linux Laptop, CPU i5 6200U with 2.7 GHz

Hanging Chain Optimal Control Benchmark

- 15 states, 3 controls, state and control constraints,
- vary MPC control horizon length from N=10 to N=100 intervals
- direct multiple shooting leads to sparse NLP with N*(15+3) variables, N*3 state constraints, N*6 input bounds (1800 variables, 300 state constraints, 600 input bounds for N=100)

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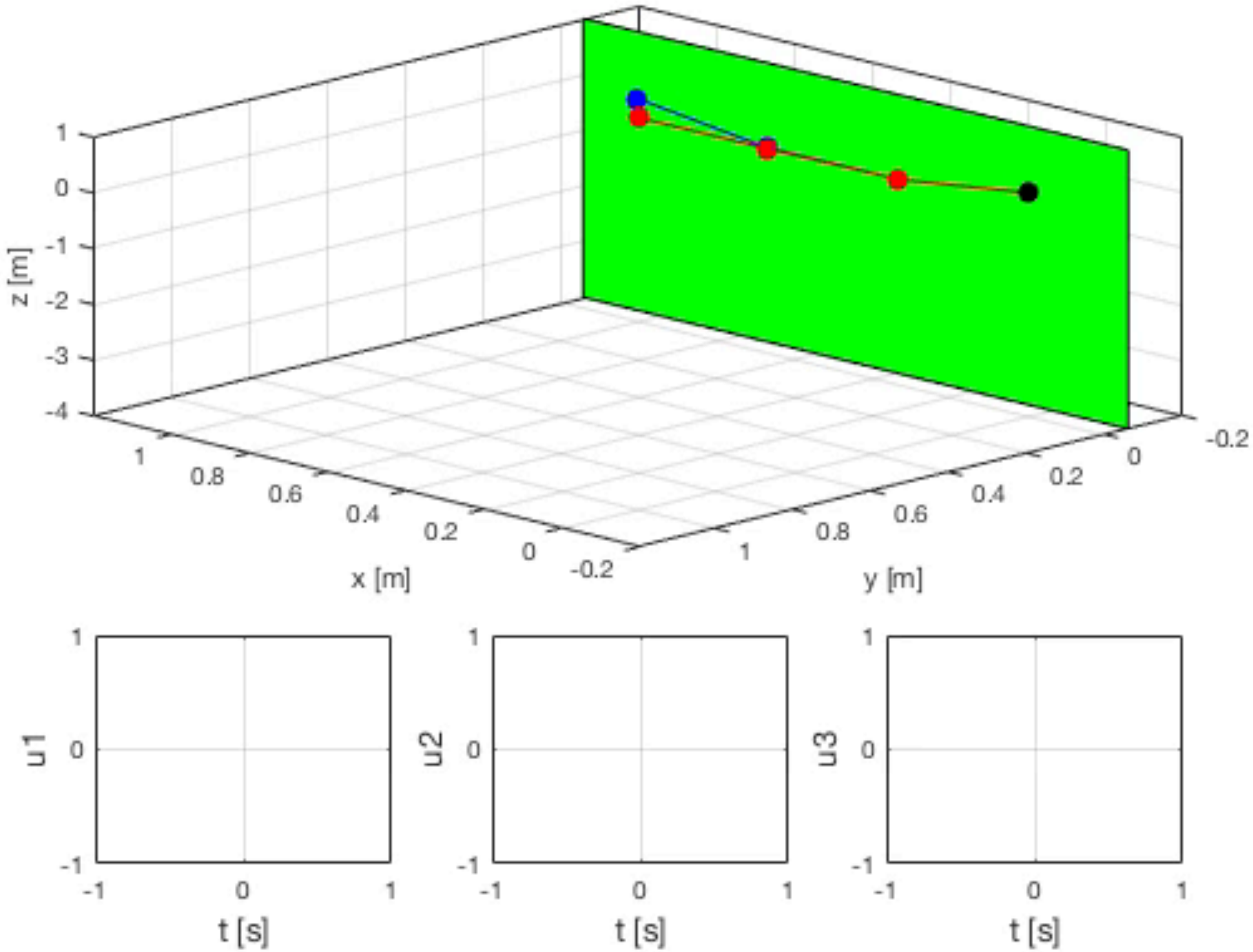
- always use: Numerical integration with code generated Implicit Runge Kutta (IRK-GL2) method [Quirynen 2012], two integration steps per interval of 100 ms.

Hanging Chain Benchmark

Andrea Zanelli

Dimitris Kouzoupis

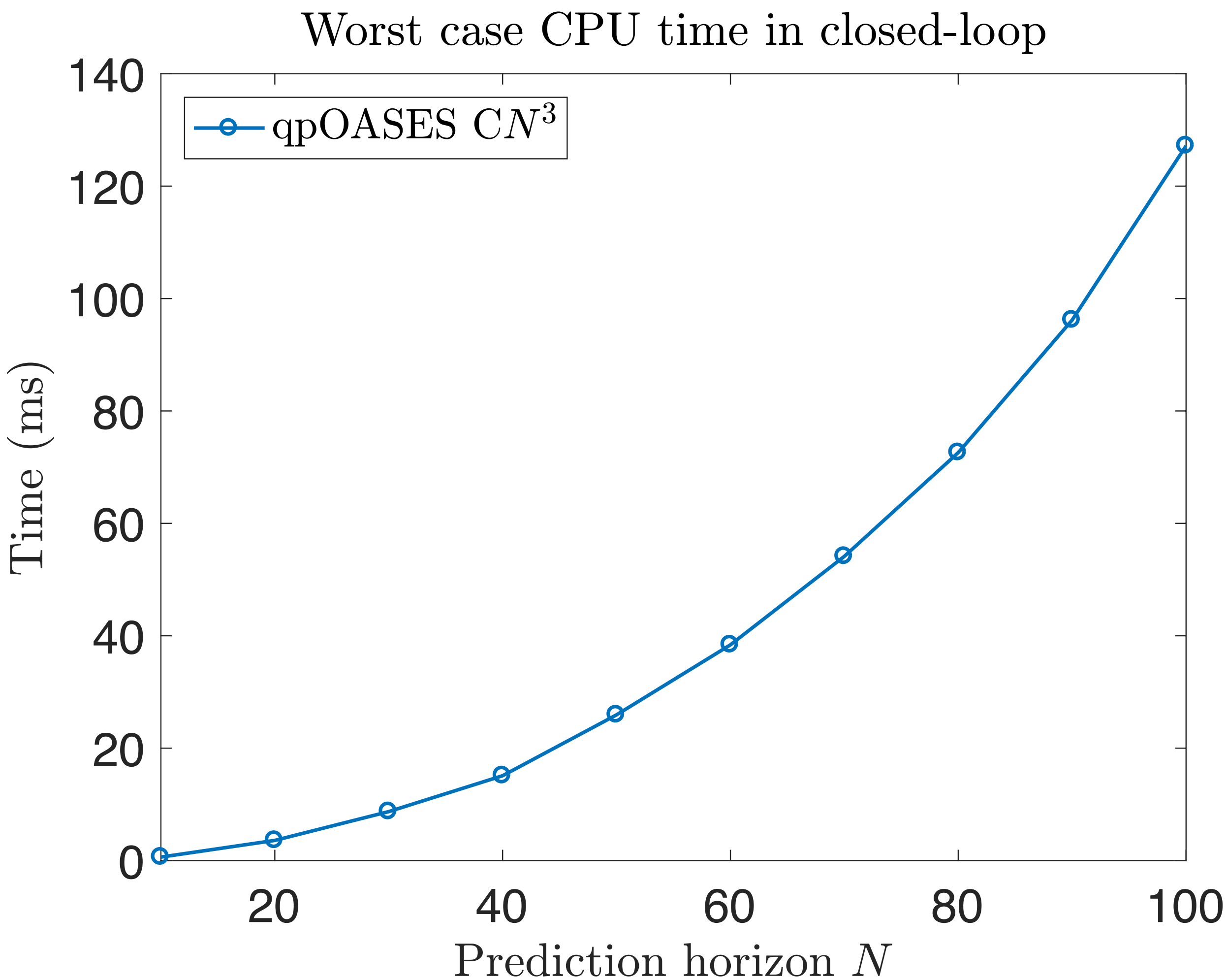


2012: ACADO Code Generation with **Condensing**

- efficient block sparse condensing with **O(N³) complexity**
- qpOASES to solve "condensed" QPs (with 3*N variables)

Hans Joachim Ferreau

Milan Vukov



2013: Code generated sparse QP solver **FORCES**

- use interior point method with sparse linear algebra (cf. Steinbach 1995)
- code generate Riccati solvers with **O(N) complexity**
- include FORCES as QP solver in ACADO [Vukov, Domahidi et al., CDC, 2013]

Alexander Domahidi
[PhD ETH 2013]

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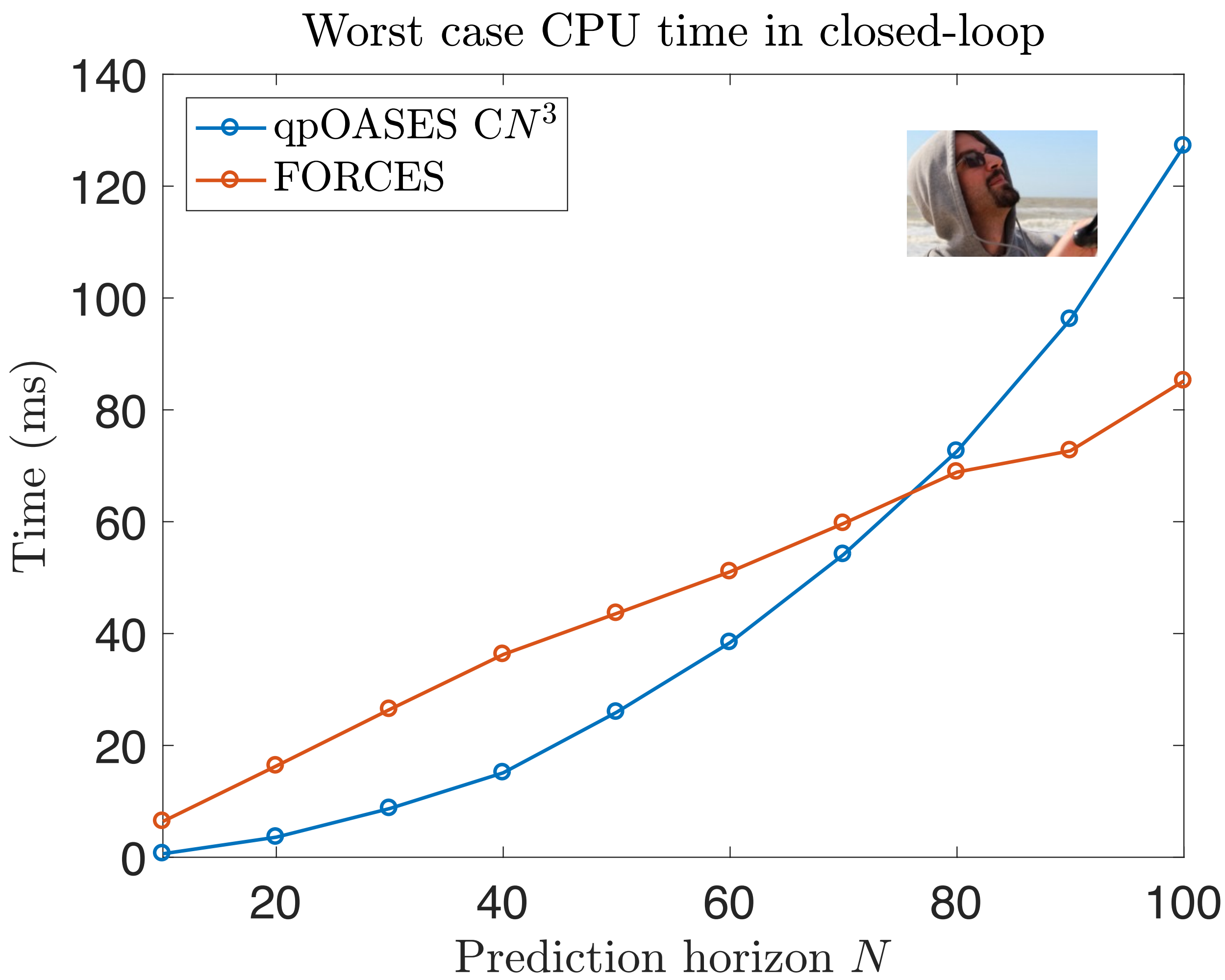

2013: A Surprising Improvement in **Condensing**

- reorder block matrix multiplications, reduce **$O(N^3)$ to $O(N^2)$ complexity!**
- independently discovered by G. Frison and J. Andersson.
  Implemented efficiently by M. Vukov in ACADO.

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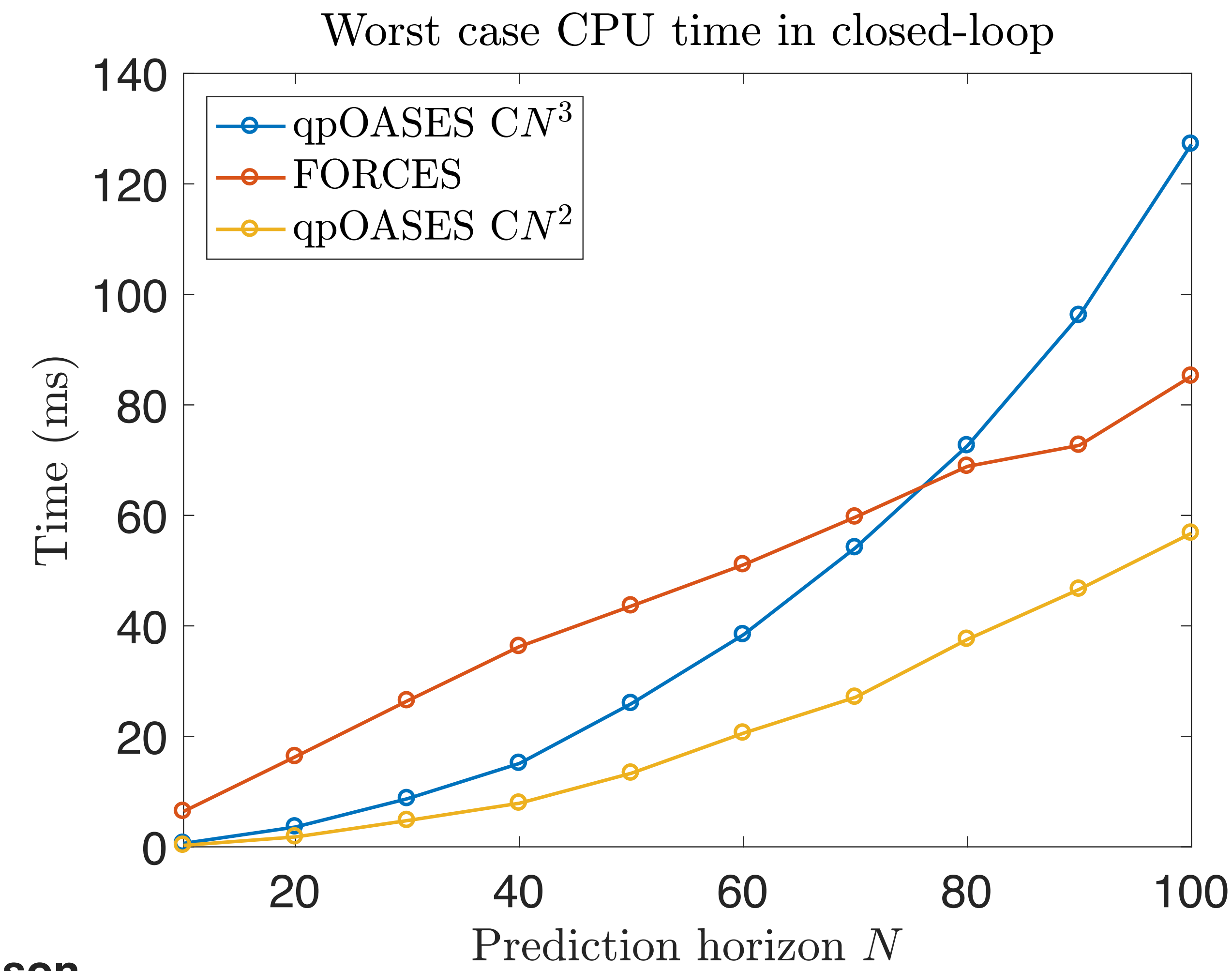

2014: The dual Newton strategy **qpDUNES**

Janick Frasch

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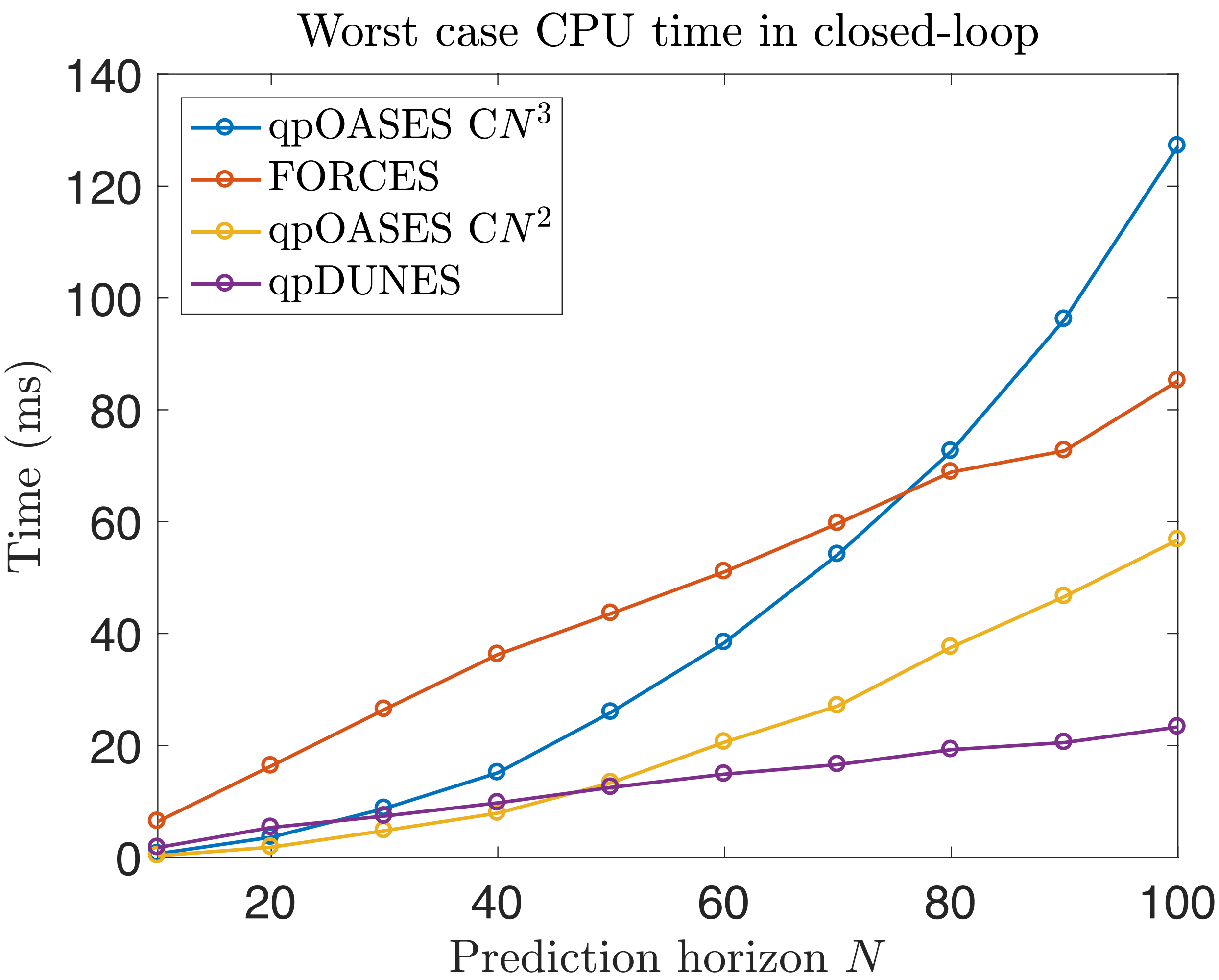

2015: Efficient Register Management in **HPMPC**

- use interior point method with Riccati solver of **O(N) complexity**
- use linear algebra routines tailored to embedded optimization (now in BLASFEO) obtaining near peak CPU performance **(Gianluca Frison)**
- include in ACADO (M. Vukov, A. Zanelli, G. Frison)

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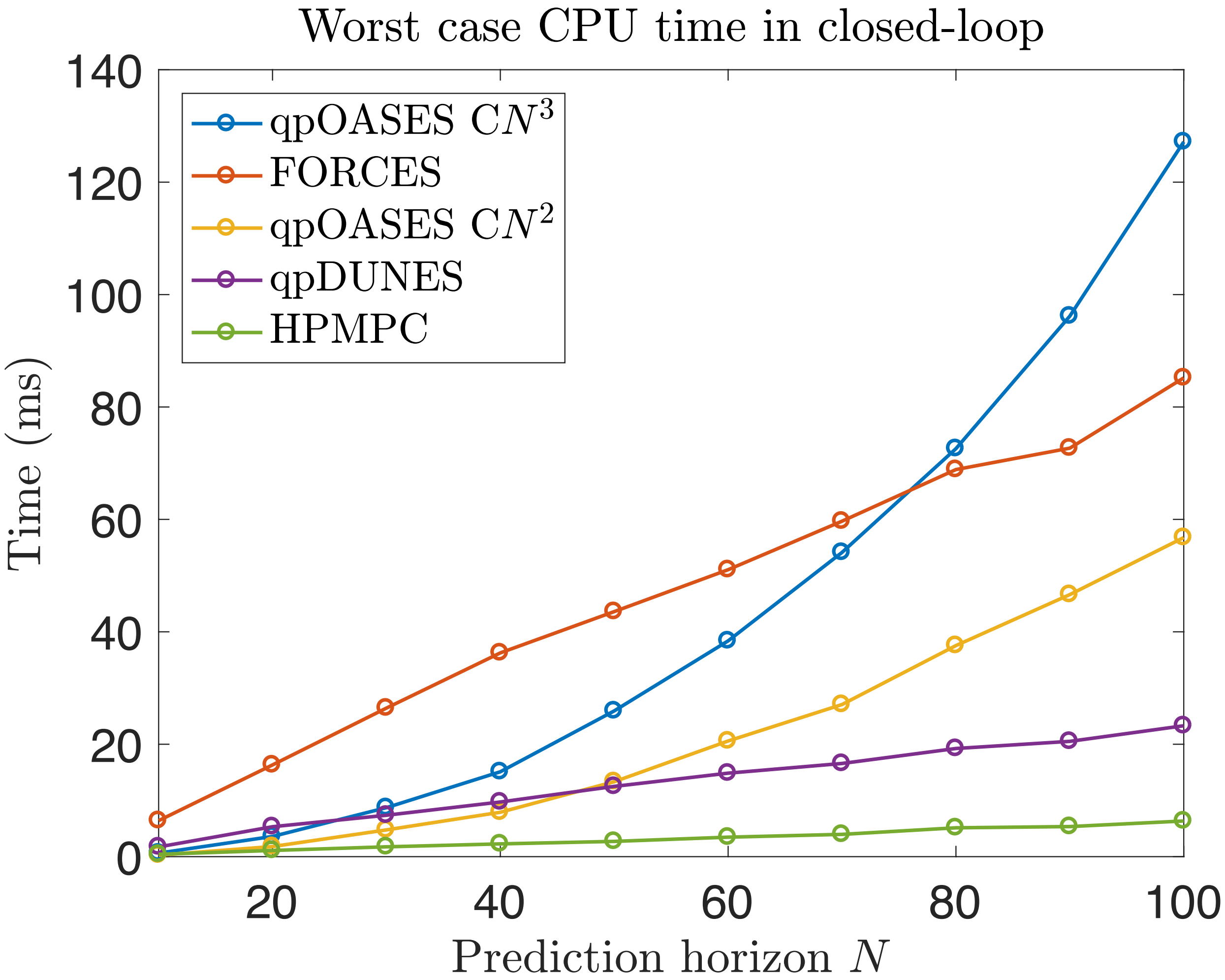

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- **partial condensing** (proposed by Daniel Axehill, Linköping) combines advantages of condensing and Riccati recursion and further boost performance of HPMPC (by G. Frison, DTU/Freiburg)

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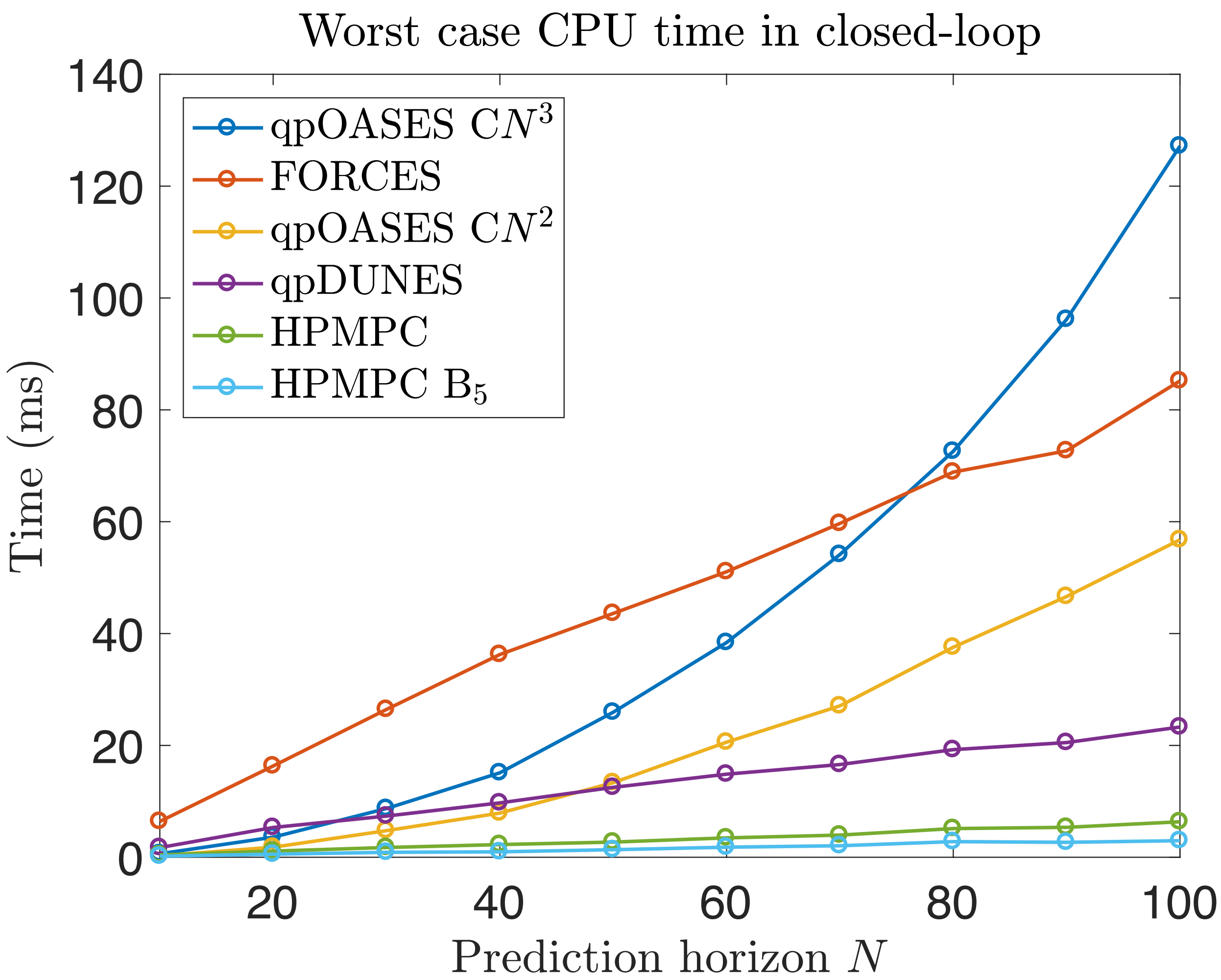

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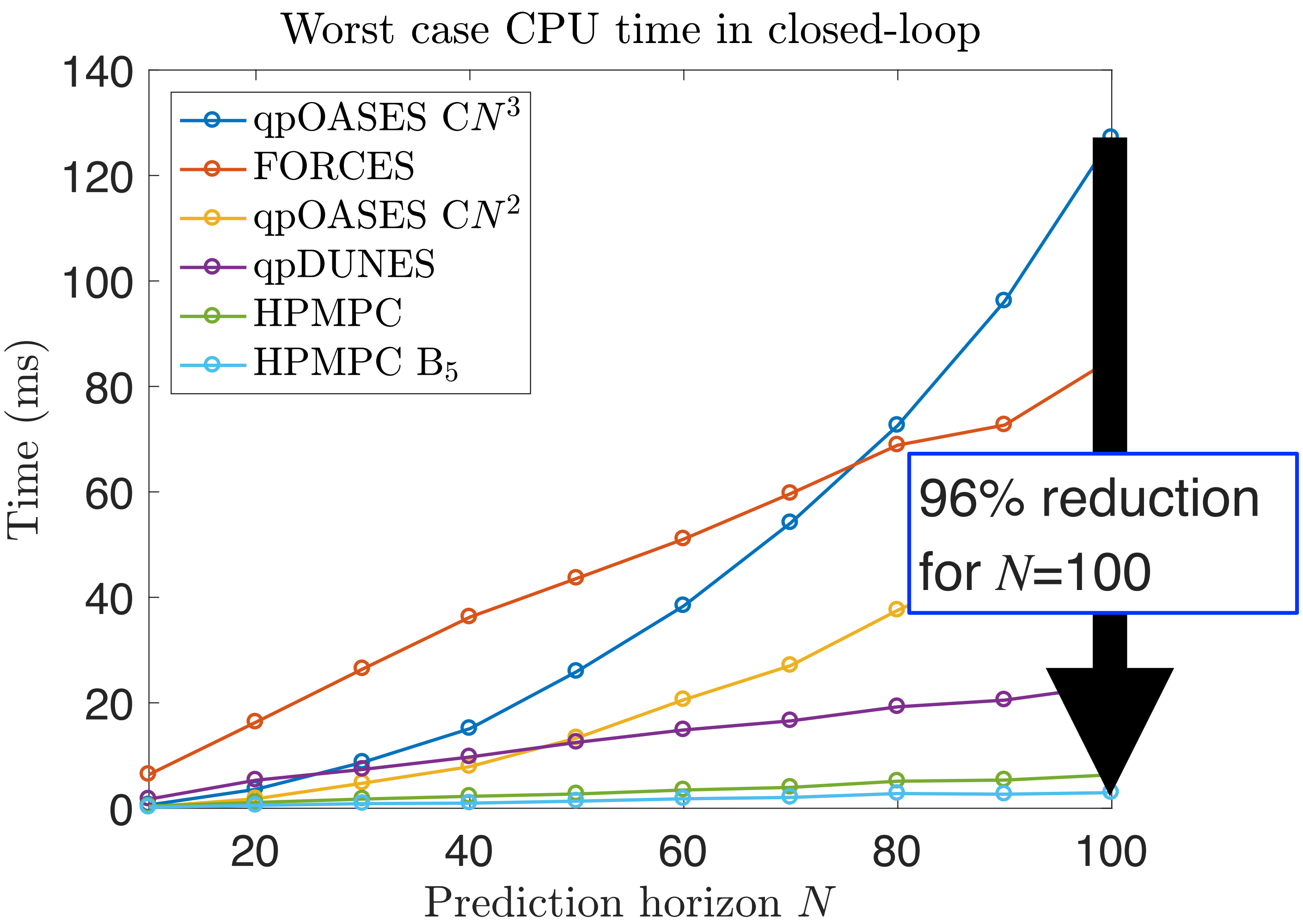

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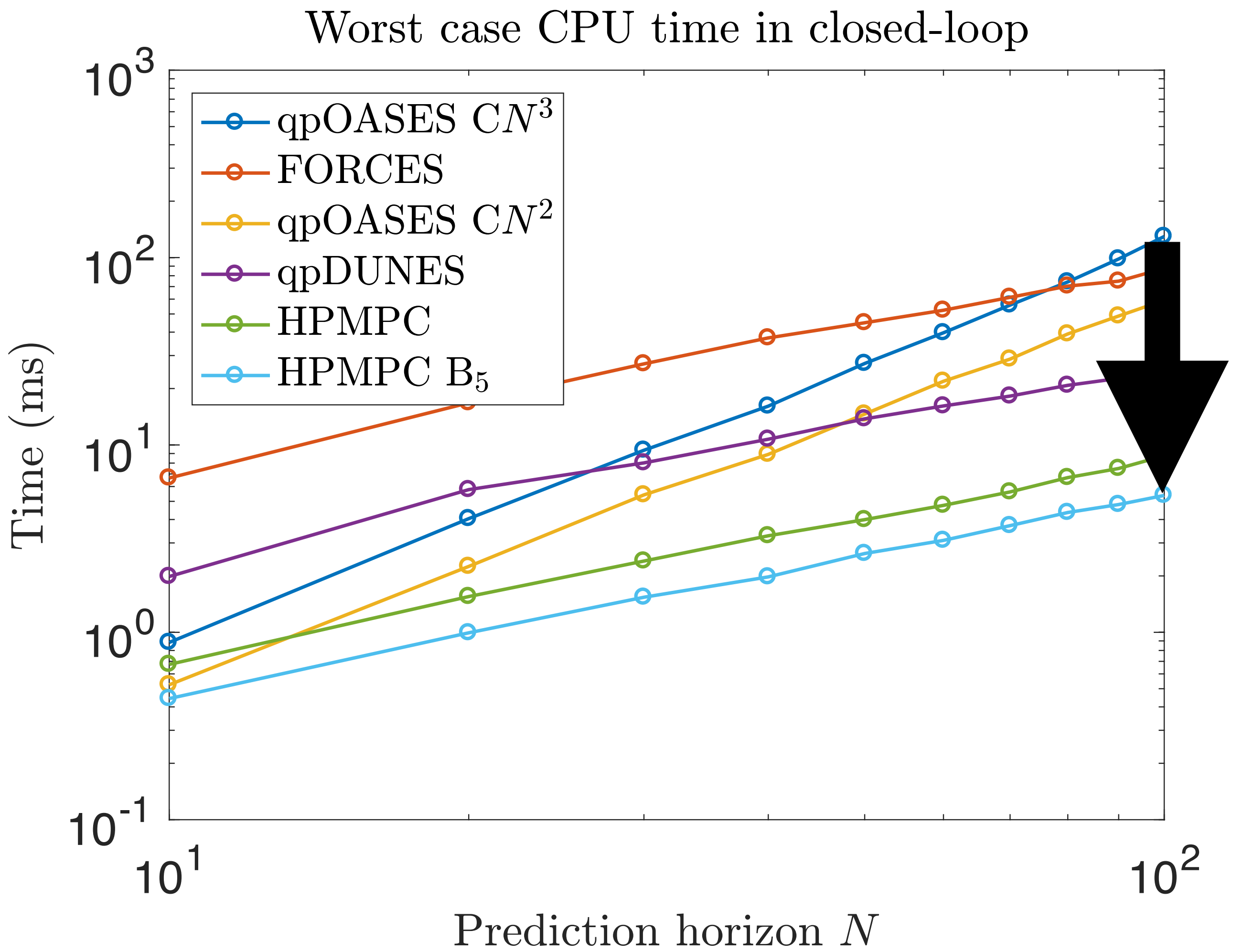

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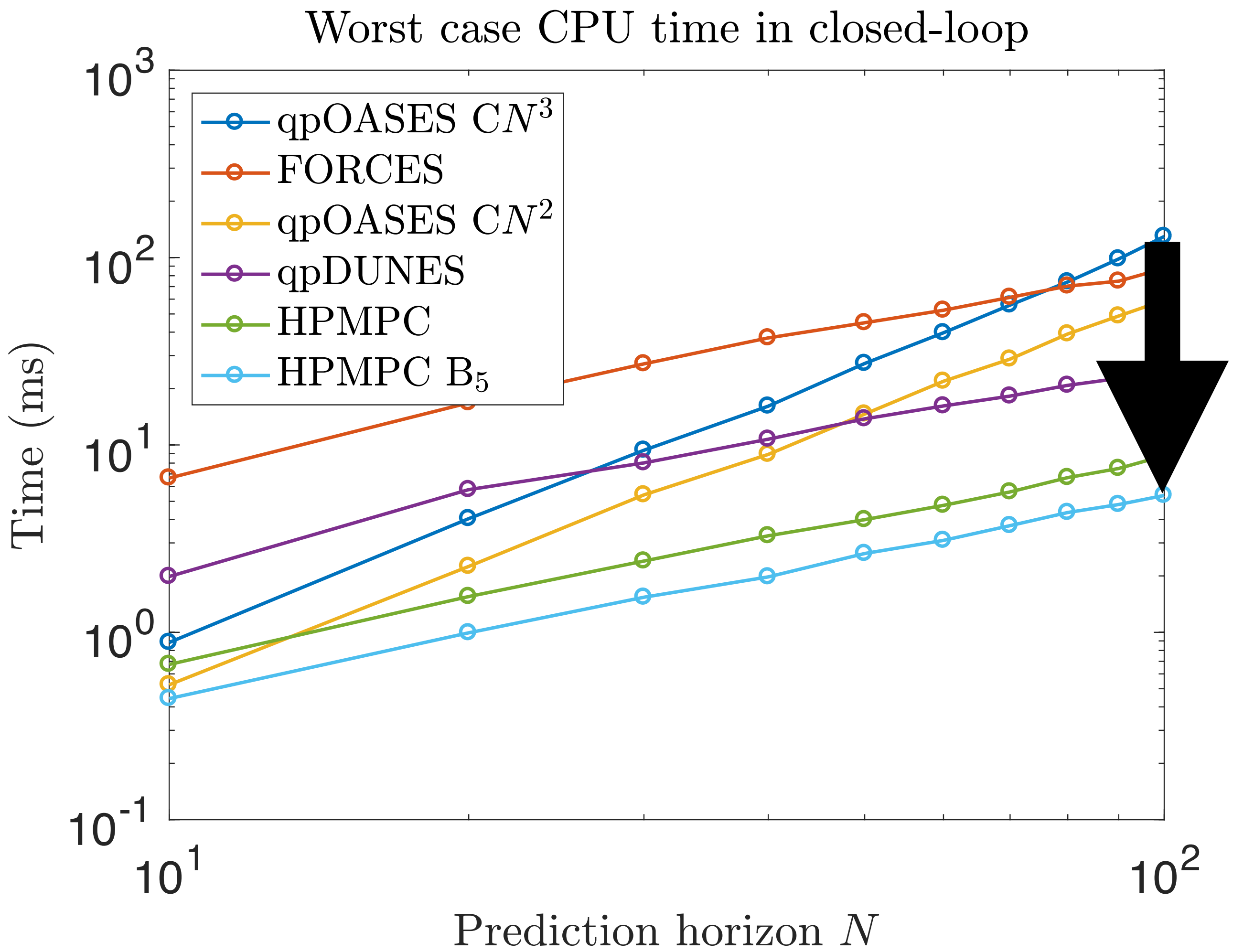
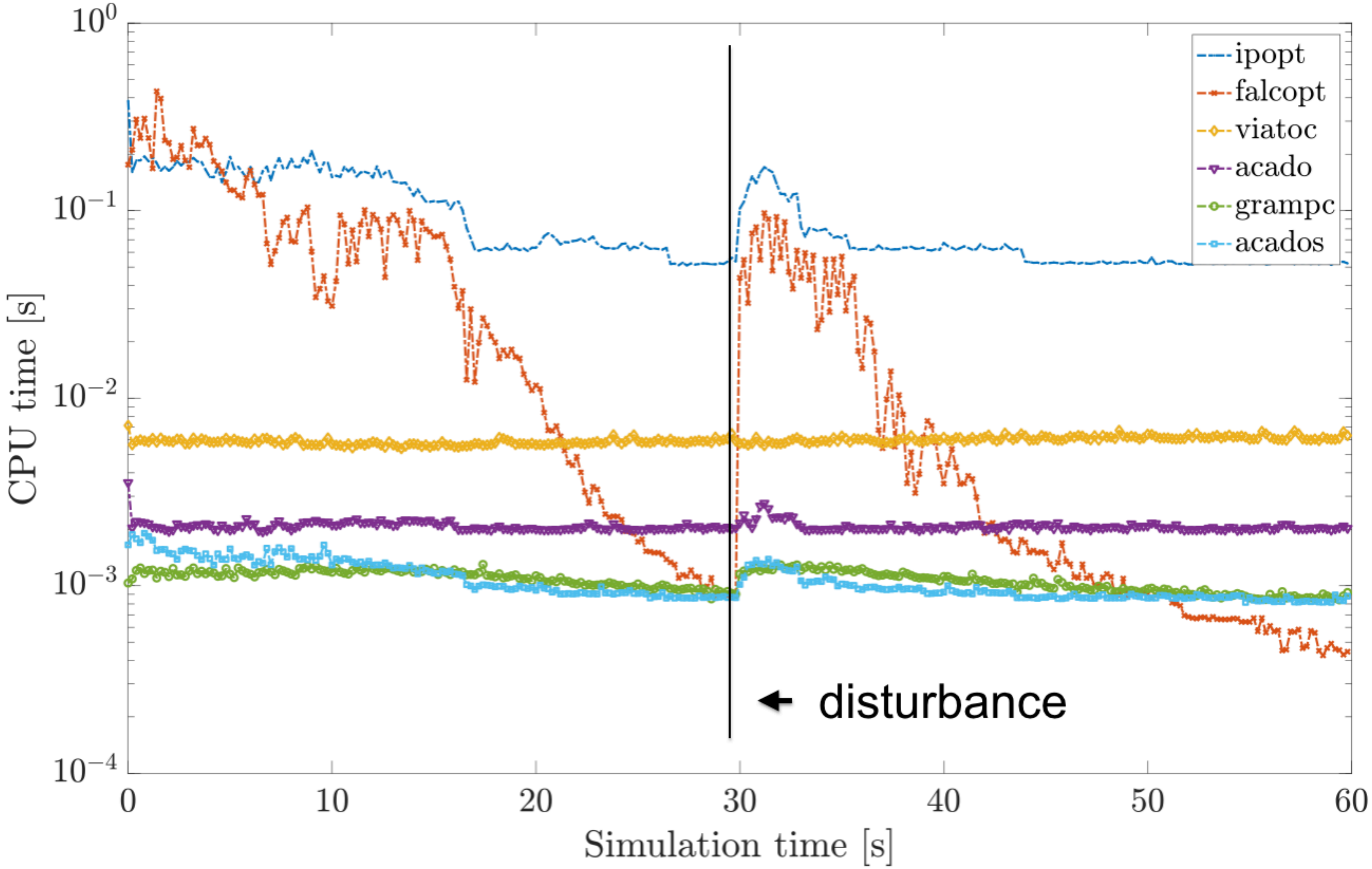
"Basic Linear Algebra Subroutines for Embedded Optimization (BLASFEO)"



Current development: acados - plain C-code library

benchmark - chain of masses[2]: 33 states, 3 controls, horizon length 40



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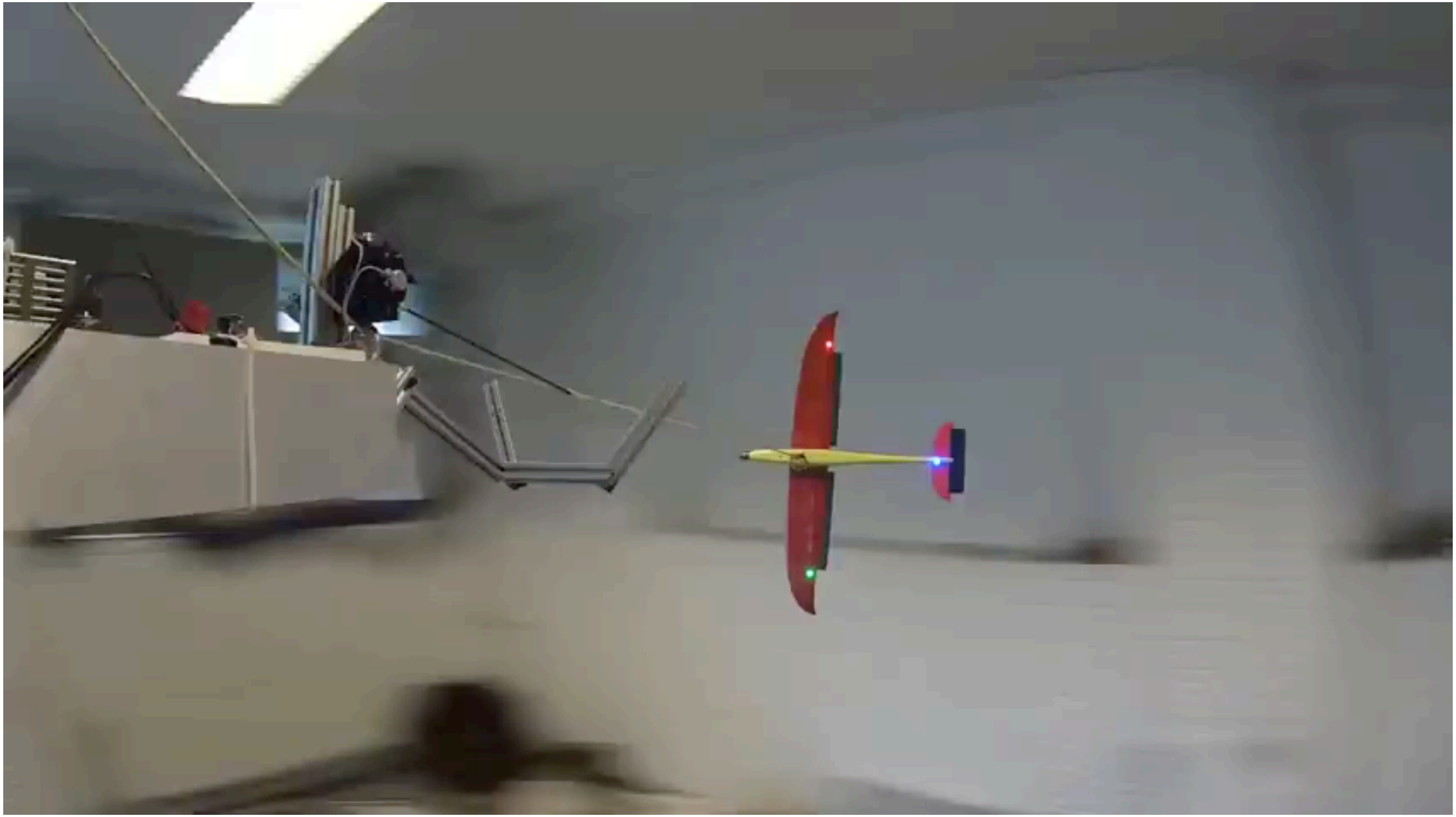
- **Some real-world NMPC applications**

1) Flight Carousel for Tethered Airplanes

Experiments within the ERC Project HIGHWIND Leuven/Freiburg



Milan Vukov

Moving Horizon Estimation and Nonlinear Model Predictive Control on the Flight Carousel

(sampling time 50 Hz, using ACADO Code Generation)

Closed loop experiments with NMPC & NMHE

KU LEUVEN

UNI FREIBURG

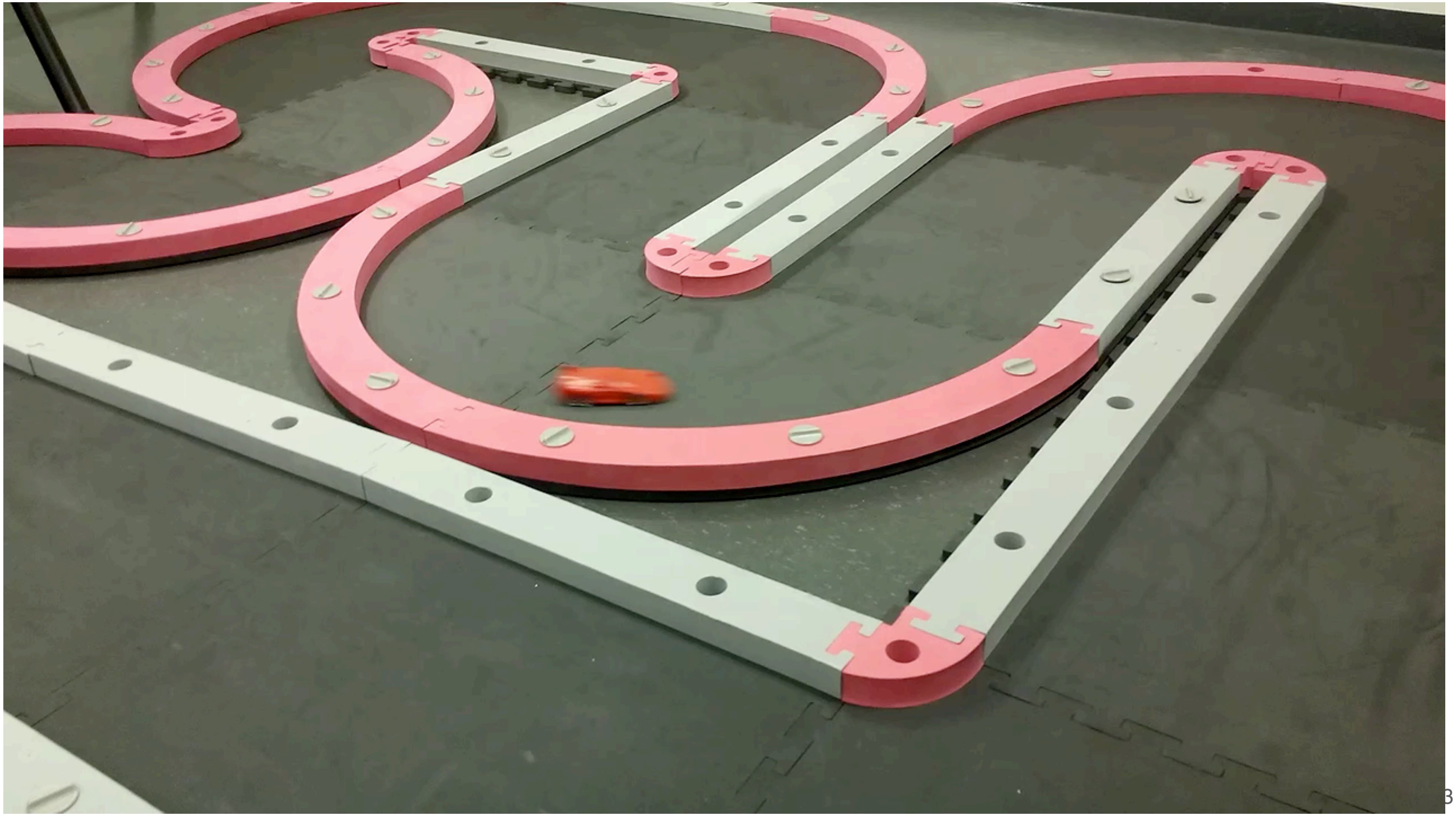
ERC HIGHWIND

2) Nonlinear MPC Example: time-optimal “racing” of model cars

Robin Verschueren

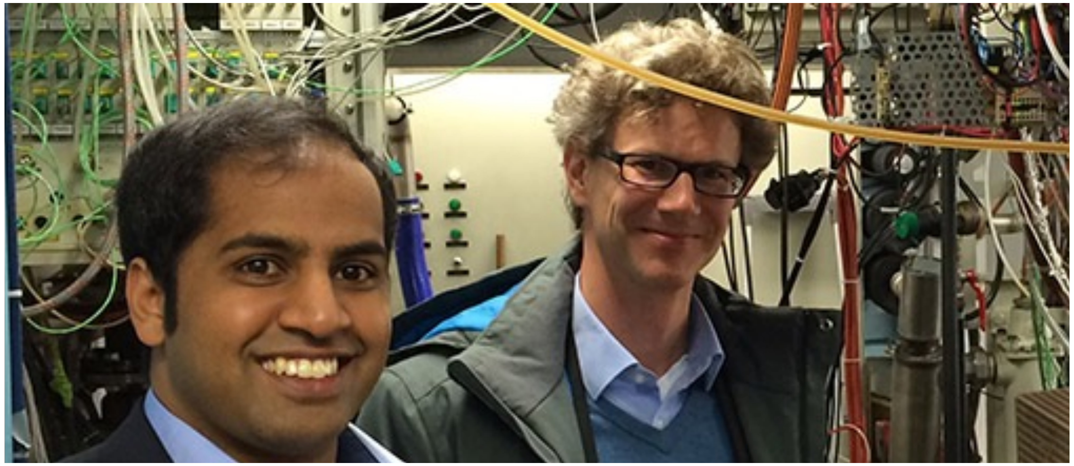
Freiburg/Leuven/ETH/Siemens-PLM. 100 Hz sampling time using ACADO [Verschueren, De Bruyne, Zanon, Frasch, D. CDC 2014]

(Nonlinear MPC video from 22.6.2016 in Freiburg)



3) Nonlinear MPC of Two-Stage Turbocharger with ACADO

Cooperation with Dr. Thiva Albin (RWTH Aachen) and **Rien Quirynen**



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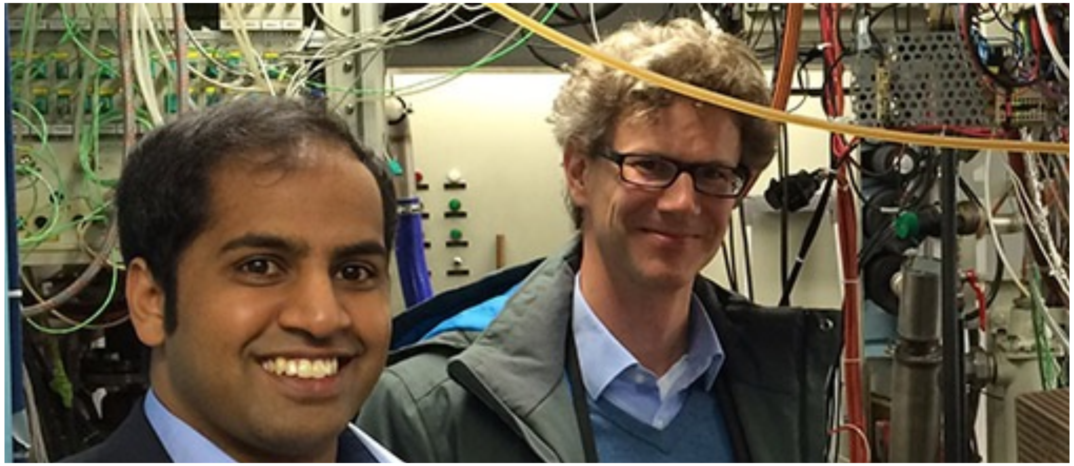
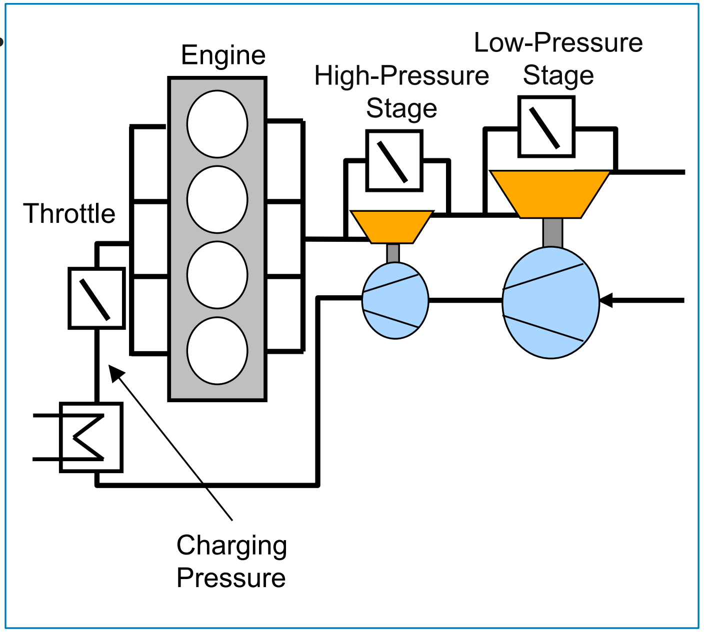




Actuated Variables

- Wastegate- High Pressure
- Wastegate- Low Pressure

Controlled Variable / Sensors

- Charging pressure
- No sensors in the exhaust gas path

test car of RWTH Aachen

3) Nonlinear MPC of Two-Stage Turbocharger with ACADO

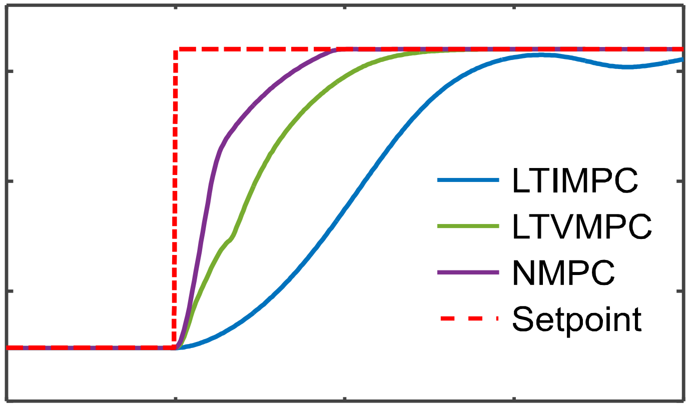
Cooperation with Dr. Thiva Albin (RWTH Aachen) and **Rien Quirynen**

- use nonlinear DAE model with 4 states, 2 controls
- use ACADO Code Generation from MATLAB
- export C-code into Simulink
- deploy on dSPACE Autobox

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}J_{tc,hp}n_{tc,hp}^2\right) &= P_{t,hp} - P_{c,hp} \\
\frac{d}{dt}\left(\frac{1}{2}J_{tc,lp}n_{tc,lp}^2\right) &= P_{t,lp} - P_{c,lp} \\
P_{c,hp} &= \dot{m}_{c,hp}c_pT_{uc,hp}\frac{1}{\eta_{s,c,hp}}\left(\Pi_{c,hp}^{\frac{\kappa-1}{\kappa}} - 1\right) \\
P_{c,lp} &= \dot{m}_{c,lp}c_pT_{amb}\frac{1}{\eta_{s,c,lp}}\left(\Pi_{c,lp}^{\frac{\kappa-1}{\kappa}} - 1\right) \\
P_{t,hp} &= \dot{m}_{t,hp}c_pT_{ut,hp}\eta_{s,t,hp}\left(1 - \Pi_{t,hp}^{\frac{1-\kappa}{\kappa}}\right) \\
P_{t,lp} &= \dot{m}_{t,lp}c_pT_{ut,lp}\eta_{s,t,lp}\left(1 - \Pi_{t,lp}^{\frac{1-\kappa}{\kappa}}\right)
\end{aligned}
$$

3) Nonlinear MPC of Two-Stage Turbocharger with ACADO

Nonlinear MPC superior to Linear MPC in simulations:



Implemented in test car of RWTH Aachen and tested on a test drive and the road.

[driving a happy M.D. to Aachen Hbf on 2.11.2015]

4) Electrical Compressor Control at ABB (Norway)

- work of Dr. Joachim Ferreau and Dr. Thomas Besselmann, ABB
- nonlinear MPC with qpOASES and ACADO, 1ms sampling time
- first tests at 48 MW Drive
- currently, 15% of Norwegian Gas Exports are controlled by Nonlinear MPC

4) Electrical Compressor Control at ABB (Norway)

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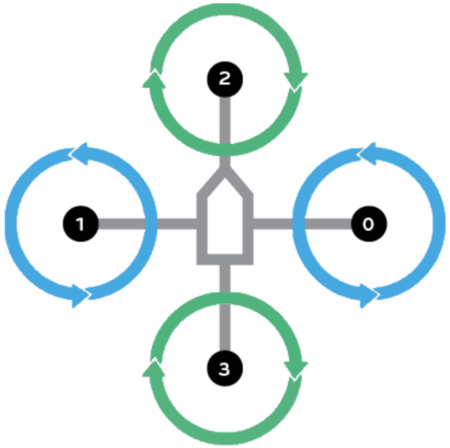
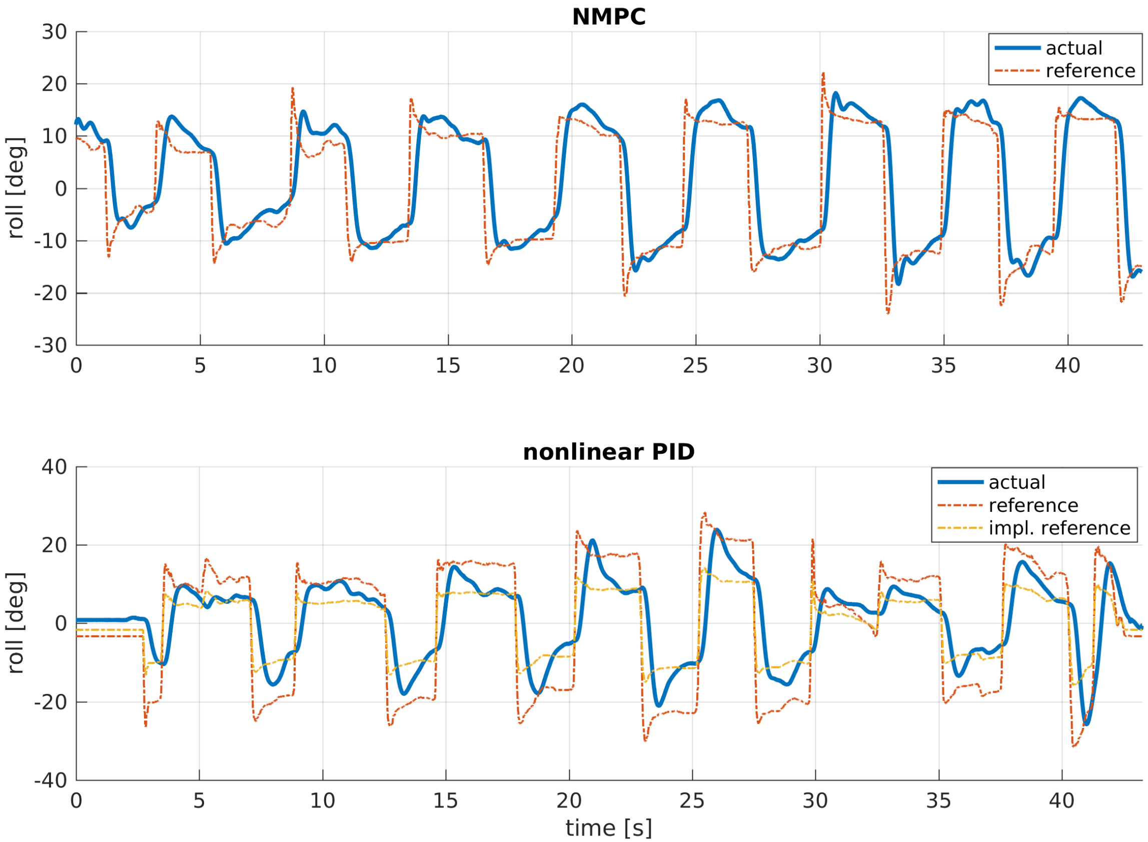
```
Joachim Ferreau (email from 7.3.2016):

The NMPC installations in Norway
(actually 5 compressors at two
different sites) are doing fine since
last autumn – roughly 80 billion NMPC
instances solved by now. In addition,
they have proven to work as expected
when handling external voltage dips.
```

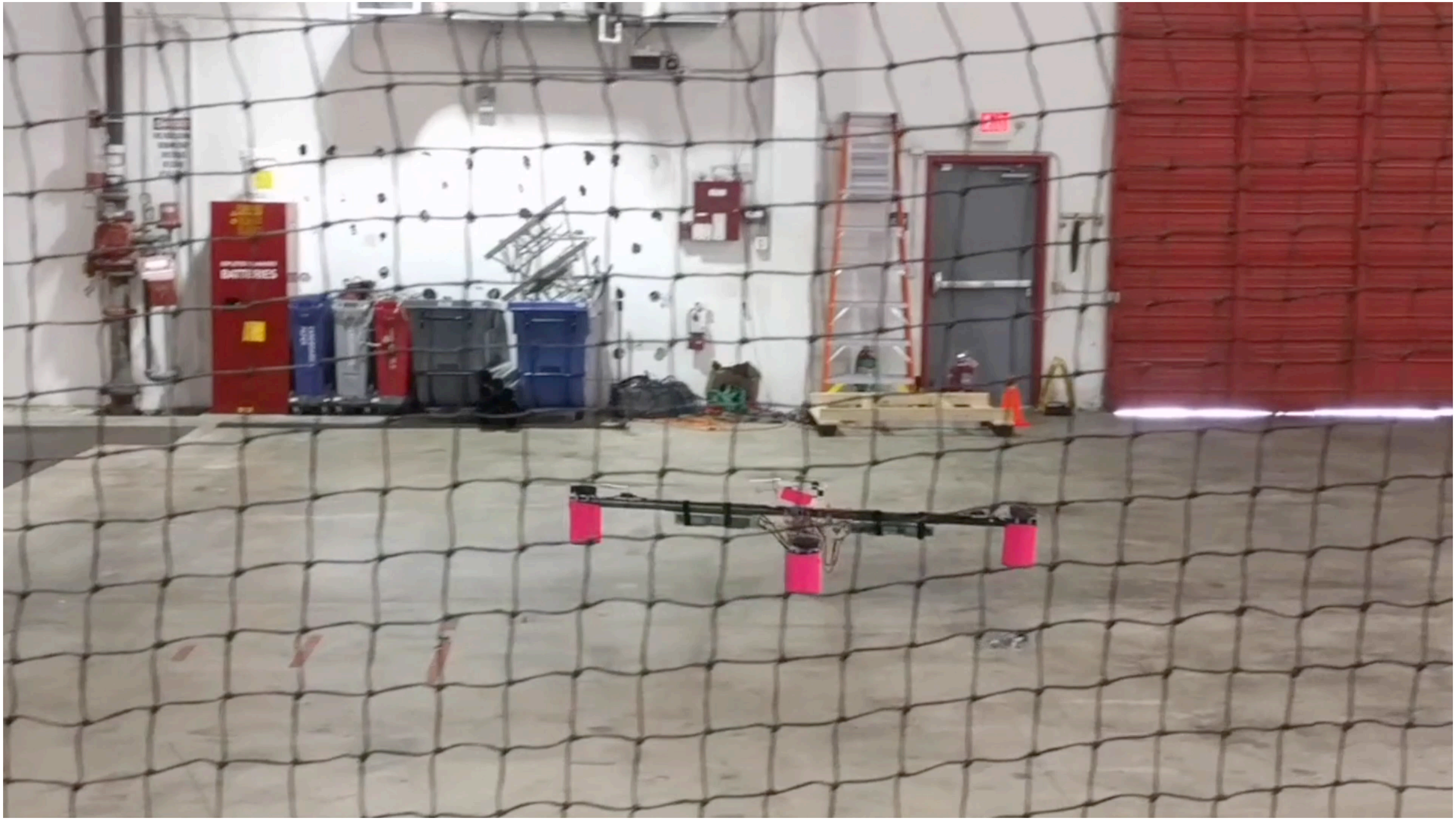
5) Human sized quadcopter control



- work by Greg Horn (Kittyhawk, California) and Andrea Zanelli (Freiburg) in April 2017
- aim is to track roll angle commands better than a custom PID
- nonlinear MPC with 11 states, 4 controls, 10 intervals
- use **HPMPC**, custom linear algebra **BLASFEO**, and “partial tightening” NMPC scheme within **acados** environment
- achieve 2 ms per optimisation on ARM cortex A9 @900 MHz



5) Human sized quadcopter control (NMPC) at Kittyhawk



Conclusions and outlook

Model Predictive Control (MPC) uses more CPU resources than standard techniques, but allows the development of **more powerful** controllers with **wider range of validity**

good numerical methods can solve nonlinear optimal control problems at **milli- and microsecond sampling times** on embedded systems

open source software (CasADi, qpOASES, ACADO, HPIPM, BLASFEO, acados) well-tested in dozens of embedded MPC applications: cranes, wafer steppers, model race cars, combustion engines, electrical drives, tethered airplanes, power converters,…

Latest open source developments in the Freiburg team regard

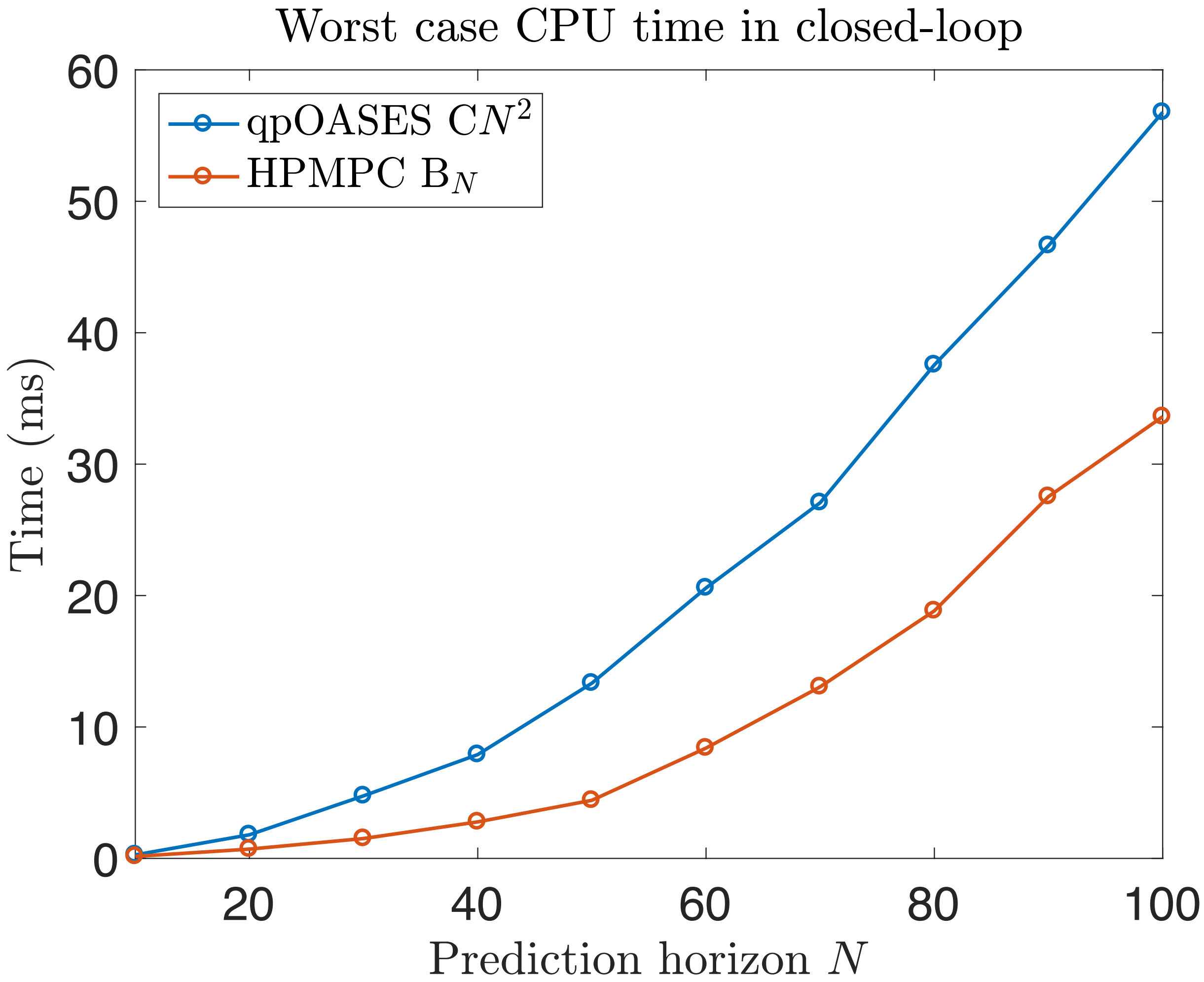
- mixed-integer nonlinear optimal control algorithms
- nonsmooth optimal control algorithms (see previous summer school)
- continuous least-squares integration and penalty-barrier methods

Appendix

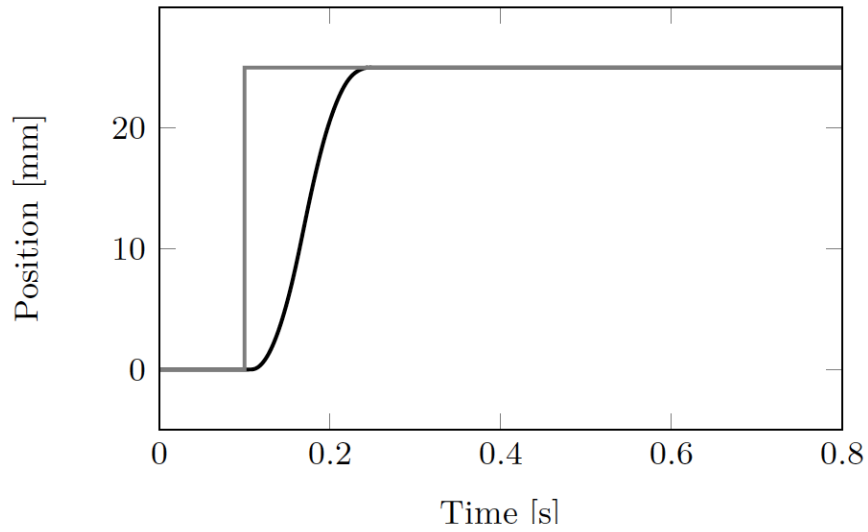
2017: Full condensing with BLASFEO and HPMPC

- block size in HPMPC with partial condensing equal to prediction horizon.

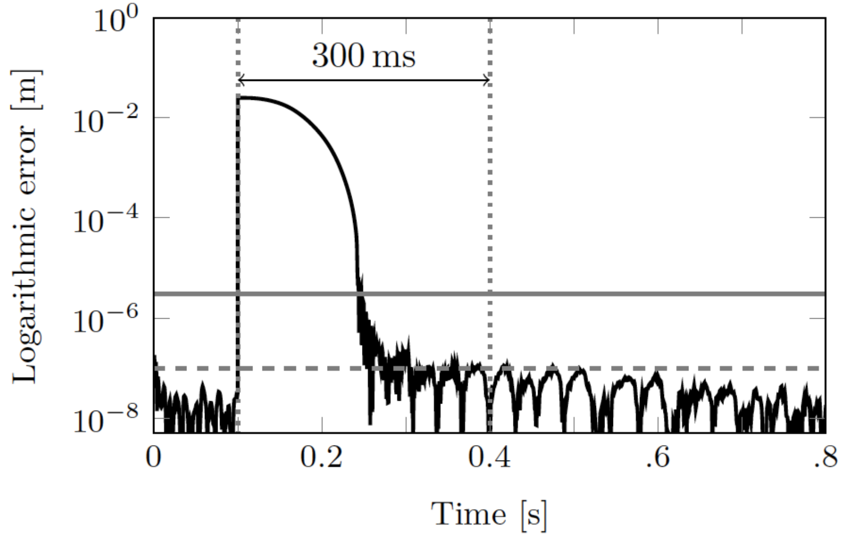
Gianluca Frison



Time Optimal MPC at ETEL (CH): 25cm step, 100nm accuracy



TOMPC at 250 Hz (+PID with 12 kHz)



Lieboud's results after 1 week at ETEL:

- 25 cm step in 300 ms
- 100 nm accuracy

equivalent to: „fly 2,5 km with MACH15,
stop with 1 mm position accuracy"