Lecture 8: Time-freezing II: Rigid bodies with friction and inelastic impacts

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Systems Control and Optimization Laboratory (syscop)

Summer School on Direct Methods for Optimal Control of Nonsmooth Systems
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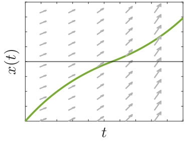
universität freiburg

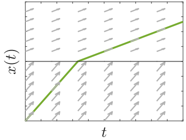
Outline of the lecture

Nonsmooth Dynamics (NSD) - a classification

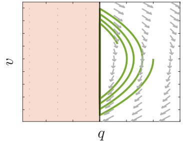
Regard an ordinary differential equation (ODE) with a **nonsmooth** right-hand side (RHS).
Distinguish three cases:



NSD1: non-differentiable RHS, e.g., $\dot{x} = 1 + |x|$



NSD2: state dependent switch of RHS, e.g., $\dot{x} = 2 - \mathrm{sign}(x)$



NSD3: **state dependent jump**, e.g., bouncing ball, $v(t_+) = -0.9\, v(t_-)$

Controlled complementarity Lagrangian systems

Controlled CLS

$$\dot{q} = v$$

$$M(q)\dot{v} = f_{\mathrm{v}}(q, v) + B_u(q)u$$

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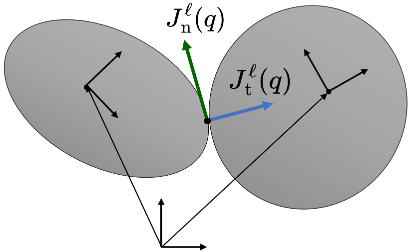
$$M(q)\dot{v} = f_{\mathrm{v}}(q,v) + B_u(q)u + \sum_{\ell=1}^{n_{\mathrm{c}}} (J_{\mathrm{n}}^{\ell}(q)\lambda_{\mathrm{n}}^{\ell} \qquad)$$

$$0 \leq \lambda_{\mathrm{n}}^{\ell} \perp f_c^{\ell}(q) \geq 0, \quad \forall \ell \in \mathcal{C}$$



- $\mathcal{C} = \{1, \dots, n_c\}$ - number of contact, $J_{\mathrm{n}}^{\ell}(q)$ - contact normal, $\epsilon_{\mathrm{r}}^{\ell}$ - coeff. of restitution
- blue terms: impact model $f_c^{\ell}(q) = 0$ becomes active, triggers state jump

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Controlled CLS

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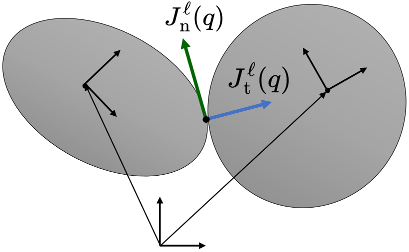
$$\text{if } (f_c^{\ell}(q(t_{\mathrm{s}})) \leq 0 \text{ then}$$

$$J_{\mathrm{n}}^{\ell}(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^+) \geq -\epsilon_{\mathrm{r}}^{\ell} J_{\mathrm{n}}^{\ell}(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^-), \quad \forall \ell \in \mathcal{C}$$



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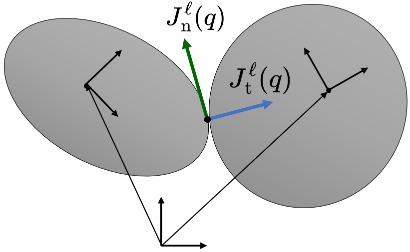

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- green terms: Coulomb's friction model (maximum dissipation principle)
- $J_{\mathrm{t}}^{\ell}(q) \in \mathbb{R}^{n_q \times n_{\mathrm{t}}}$, $n_{\mathrm{t}} \in \{1, 2\}$ spans the tangent plane

Controlled complementarity Lagrangian systems

Controlled CLS

$$\dot{q} = v$$

$$M(q)\dot{v} = f_{\mathrm{v}}(q,v) + B_u(q)u + \sum_{\ell=1}^{n_{\mathrm{c}}} (J_{\mathrm{n}}^{\ell}(q)\lambda_{\mathrm{n}}^{\ell} + J_{\mathrm{t}}^{\ell}(q)\lambda_{\mathrm{t}}^{\ell})$$

$$0 \leq \lambda_{\mathrm{n}}^{\ell} \perp f_c^{\ell}(q) \geq 0, \quad \forall \ell \in \mathcal{C}$$

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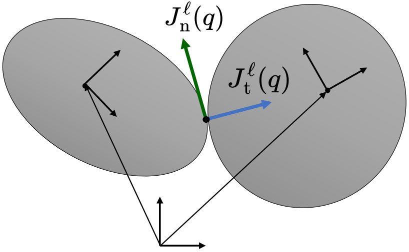
$$\lambda_{\mathrm{t}}^{\ell} \in \arg \min_{\tilde{\lambda}_{\mathrm{t}}^{\ell} \in \mathbb{R}^{n_{\mathrm{t}}}} \quad -v^{\top} J_{\mathrm{t}}^{\ell}(q)\tilde{\lambda}_{\mathrm{t}}^{\ell}$$

$$\text{s.t.} \quad \|\tilde{\lambda}_{\mathrm{t}}^{\ell}\|_2 \leq \mu^{\ell}\lambda_{\mathrm{n}}^{\ell}, \quad \forall \ell \in \mathcal{C}$$



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Controlled complementarity Lagrangian systems - single contact

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\begin{aligned}
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&J_{\mathrm{n}}(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^+) \geq 0 \\
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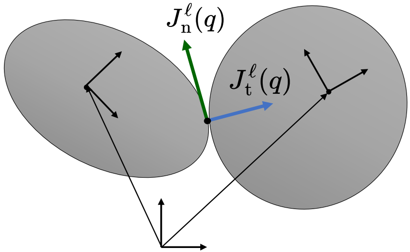

- $J_{\mathrm{n}}(q)$ - contact normal
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The friction cone

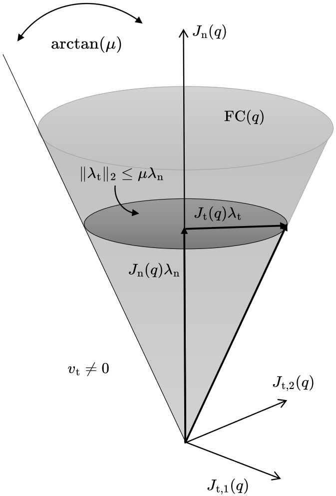
$$M(q)\dot{v} = f_{\mathrm{v}}(q,v) + B_u(q)u + J_{\mathrm{n}}(q)\lambda_{\mathrm{n}} + J_{\mathrm{t}}(q)\lambda_{\mathrm{t}}$$

Solution map of friction model

$$\lambda_{\mathrm{t}} \in \begin{cases} \{-\mu\lambda_{\mathrm{n}} \frac{v_{\mathrm{t}}}{\|v_{\mathrm{t}}\|_2}\}, & \text{if } \|v_{\mathrm{t}}\|_2 > 0 \\ \end{cases}$$

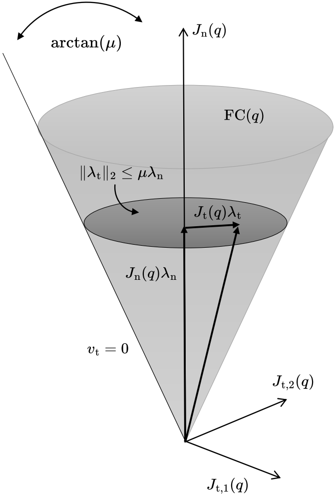
- Tangential velocity defined as $v_{\mathrm{t}} := J_{\mathrm{t}}(q)^\top v$
- In 2D solution map reduces to $\lambda_{\mathrm{t}} \in -\mu\lambda_{\mathrm{n}}\mathrm{sign}(v_{\mathrm{t}})$
- Set of all possible contact forces

  $$\mathrm{FC}(q) = \{J_{\mathrm{n}}(q)\lambda_{\mathrm{n}} + J_{\mathrm{t}}(q)\lambda_{\mathrm{t}} \mid \lambda_{\mathrm{n}} \geq 0,\ \|\lambda_{\mathrm{t}}\|_2 \leq \mu\lambda_{\mathrm{n}}\}$$



The friction cone

$$\boxed{M(q)\dot{v} = f_{\mathrm{v}}(q,v) + B_u(q)u + J_{\mathrm{n}}(q)\lambda_{\mathrm{n}} + J_{\mathrm{t}}(q)\lambda_{\mathrm{t}}}$$



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Controlled complementarity Lagrangian systems - single contact

Controlled CLS

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0 &\leq \lambda_{\mathrm{n}} \perp f_c(q) \geq 0 \\
&\text{if } f_c(q(t_{\mathrm{s}})) \leq 0 \text{ then } J_{\mathrm{n}}(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^+) \geq 0 \\
&\lambda_{\mathrm{t}} \in \arg\min_{\tilde{\lambda}_{\mathrm{t}} \in \mathbb{R}^{n_{\mathrm{t}}}} \quad -v^\top J_{\mathrm{t}}(q)\tilde{\lambda}_{\mathrm{t}} \\
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$$

- $J_{\mathrm{n}}(q)$ - contact normal
- blue terms: impact model $f_c(q) = 0$ becomes active, triggers state jump
- For a moment let us study the CLS without friction (no green terms)
- We consider the two modes when $f_c(q) > 0$ (free flight) and $f_c(q) = 0$ (active contact)

CLS modes and the contact LCP

When should an active constraint become inactive?

Unconstrained ODE mode (free flight)

$$\begin{aligned} \dot{q} &= v \\ M(q)\dot{v} &= f_{\mathrm{v}}(q,v) + B_u(q)u \end{aligned}$$

Contact mode - DAE of index 3

$$\begin{aligned} \dot{q} &= v \\ M(q)\dot{v} &= f_{\mathrm{v}}(q,v) + B_u(q)u + J_{\mathrm{n}}(q)\lambda_{\mathrm{n}} \\ 0 &= f_c(q) \end{aligned}$$

The *contact LCP* tells us if the system will stay in contact mode or switch to the ODE mode:

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The *contact LCP* tells us if the system will stay in contact mode or switch to the ODE mode:

$$\begin{aligned} &0 \leq \frac{\mathrm{d}^2}{\mathrm{d}t^2} f_c(q(t)) \perp \lambda_{\mathrm{n}}(t) \geq 0 \\ &0 \leq D(q)\lambda_{\mathrm{n}} + \varphi(x) \perp \lambda_{\mathrm{n}} \geq 0, \\ &\lambda_{\mathrm{n}} = \max(0, -D(q)^{-1}\varphi(x)) \end{aligned}$$

where $D(q)$ is the Delassus' matrix (scalar in single contact case) and

$$D(q) := \nabla_q f_c(q)^\top M(q)^{-1} \nabla_q f_c(q) \succ 0, \quad \varphi(x) := \nabla_q f_c(q)^\top f_{\mathrm{v}}(q,v,u) + \nabla_q(\nabla_q f_c(q)^\top v)^\top v.$$

CLS modes and the contact LCP

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$$\begin{aligned}0 \leq \frac{\mathrm{d}^2}{\mathrm{d}t^2} f_c(q(t)) &\perp \lambda_{\mathrm{n}}(t) \geq 0\\ 0 \leq D(q)\lambda_{\mathrm{n}} + \varphi(x) &\perp \lambda_{\mathrm{n}} \geq 0,\\ \lambda_{\mathrm{n}} &= \max(0, -D(q)^{-1}\varphi(x))\end{aligned}$$

- If $\varphi(x) < 0$ contact stays closed with $\lambda_{\mathrm{n}} > 0$
- If $\varphi(x) > 0$ contact becomes inactive with $\lambda_{\mathrm{n}} = 0$

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$$D(q) := \nabla_q f_c(q)^\top M(q)^{-1} \nabla_q f_c(q) \succ 0, \quad \varphi(x) := \nabla_q f_c(q)^\top f_{\mathrm{v}}(q,v,u) + \nabla_q(\nabla_q f_c(q)^\top v)^\top v.$$

Summary od CLS modes and switches

- Summarized state $x = (q, v)$
- The *free flight ODE*:

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f_{\mathrm{ODE}}(x,u) := \begin{bmatrix} v \\ M(q)^{-1}\hat{f}_{\mathrm{v}}(q,v,u) \end{bmatrix}, \quad \hat{f}_{\mathrm{v}}(q,v,u) := f_{\mathrm{v}}(q,v) + B_u(q)u$$

- The ODE during persistent contact obtained after index reduction:

$$\frac{\mathrm{d}}{\mathrm{d}t}x = f_{\mathrm{DAE}}(x,u) := \begin{bmatrix} v \\ M(q)^{-1}(\hat{f}_{\mathrm{v}}(q,v,u) - J_{\mathrm{n}}(q)D(q)^{-1}\varphi(x)) \end{bmatrix}$$

The possible transitions are:

1. From ODE to DAE - with a state jump in the normal contact velocity
2. From DAE to ODE - solution continious, conditions given by contact LCP

Outline of the lecture

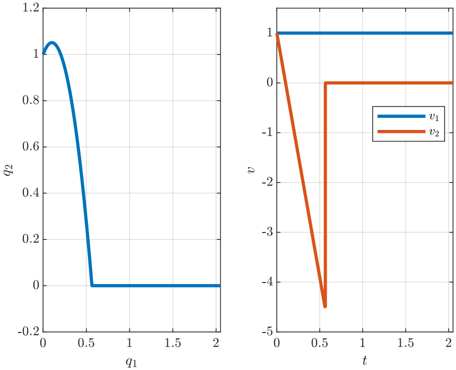
Warm up example

A 2D particle without friction

2D frictionless particle with an inelastic impact

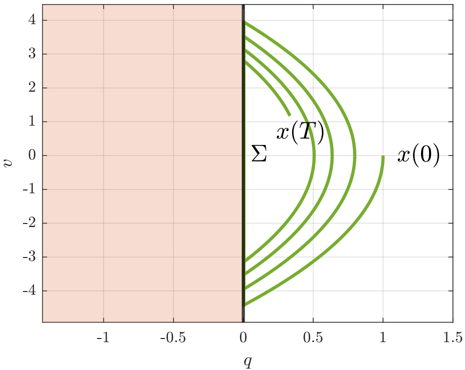
$$
\begin{aligned}
\dot{q} &= v,\\
m\dot{v} &= \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \lambda_{\mathrm{n}} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\\
0 &\leq \lambda_{\mathrm{n}} \perp q_2 \geq 0,\\
v_2(t_{\mathrm{s}}^{+}) &= 0, \text{ if } q_2(t_{\mathrm{s}}) = 0 \text{ and } v_2(t_{\mathrm{s}}^{-}) < 0.
\end{aligned}
$$

Trajectory with $u(t) = 0$:

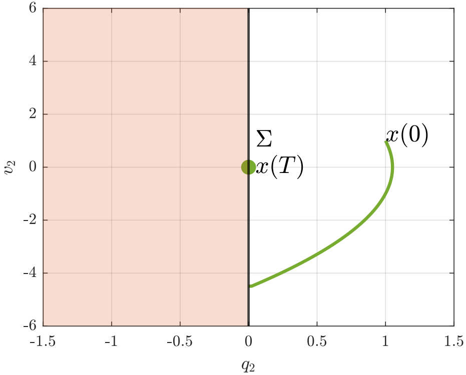


Warm up example

Phase plots: elastic vs. inelastic impact



elastic impact



inelastic impact

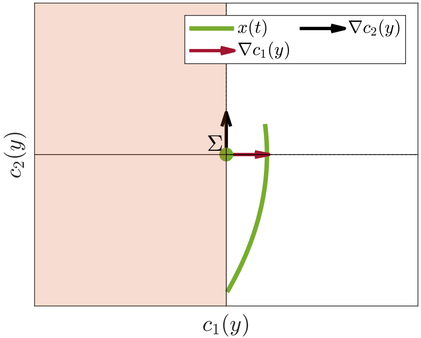
- State space in numerical time τ: $y = (q, v, t) \in \mathbb{R}^{n_y}$, $n_y = n_x + 1$ and $x = (q, v)$

Switching functions

$$c_1(y) := f_c(q)$$

$$c_2(y) := \nabla_q f_c(q)^\top v \quad \left(= \frac{\mathrm{d} f_c}{\mathrm{d}t}(q) \right)$$

Regions



- R_1 - unconstrained dynamics
- R_2 - auxiliary dynamics
- **After impact**: $c_1(y) = c_2(y) = 0$
- sliding mode on $\Sigma = \{y \mid c_1(y) = 0, c_2(y) = 0\}$

Time-freezing for inelastic impacts

Back to the more general setting

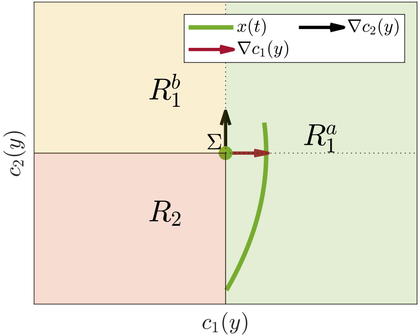
- State space in numerical time τ: $y = (q, v, t) \in \mathbb{R}^{n_y}$, $n_y = n_x + 1$ and $x = (q, v)$

Switching functions

$$c_1(y) := f_c(q)$$
$$c_2(y) := \nabla_q f_c(q)^\top v \quad \left(= \frac{\mathrm{d} f_c}{\mathrm{d}t}(q)\right)$$

Regions

$$R_1^a = \{y \in \mathbb{R}^{n_y} \mid c_1(y) > 0\}$$
$$R_1^b = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) > 0\}$$
$$R_1 = R_1^a \cup R_2{}^b$$
$$R_2 = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$



- R_1 - unconstrained dynamics
- R_2 - auxiliary dynamics
- **After impact**: $c_1(y) = c_2(y) = 0$
- sliding mode on $\Sigma = \{y \mid c_1(y) = 0, c_2(y) = 0\}$

Unconstrained and auxiliary dynamics

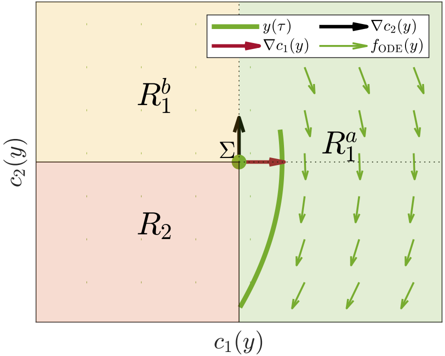
Unconstrained free-flight ODE in R_1

$$y' = f_{\mathrm{ODE}}(y,u) := \begin{bmatrix} v \\ \hat{f}_{\mathrm{v}}(q,v,u) \\ 1 \end{bmatrix}$$

Auxiliary ODE in R_2

$$y'(\tau) = f_{\mathrm{aux,n}}(y) := \begin{bmatrix} \mathbf{0} \\ M(q)^{-1} J_{\mathrm{n}}(q) a_{\mathrm{n}} \\ 0 \end{bmatrix}$$

with $a_{\mathrm{n}} > 0$.



- $f_{\mathrm{ODE}}(y,u)$ stops $y(\tau)$ on Σ!
- dynamics on Σ is $y' \in \overline{\mathrm{conv}}\{f_{\mathrm{ODE}}(y) f_{\mathrm{aux,n}}(y)\}$

Unconstrained and auxiliary dynamics

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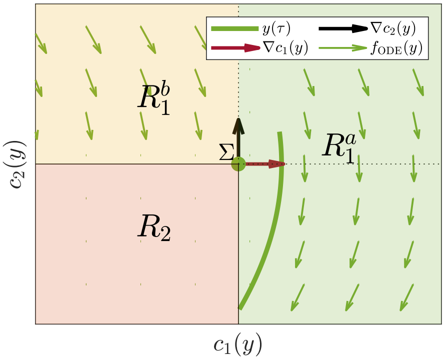

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Unconstrained and auxiliary dynamics

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Auxiliary ODE in R_2

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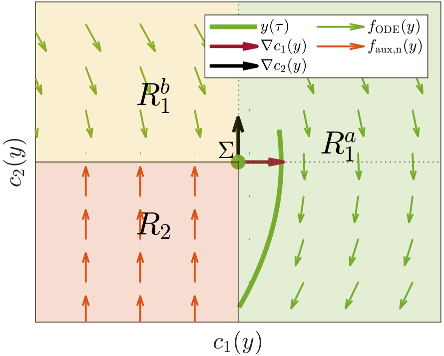
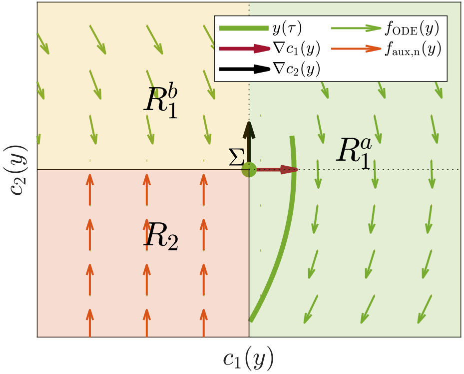


- $f_{\mathrm{ODE}}(y,u)$ stops $y(\tau)$ on Σ!
- dynamics on Σ is $y' \in \overline{\mathrm{conv}}\{f_{\mathrm{ODE}}(y) f_{\mathrm{aux,n}}(y)\}$

Contact breaking

The contact LCP function $\varphi(x)$ tells us about the vector field in R_1

- $\varphi(x)$ determines stability of Σ (remember the contact LCP)
- staying in sliding mode (persistent contact) or leaving sliding mode (contact breaking) is possible

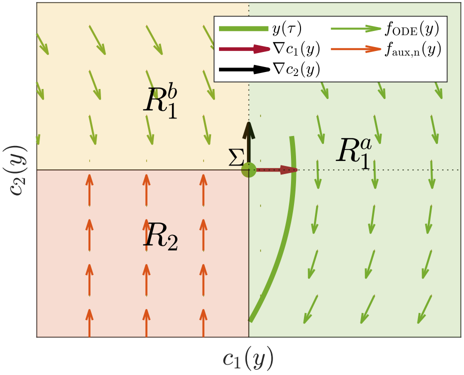


Sliding mode if $\varphi(x) \leq 0$

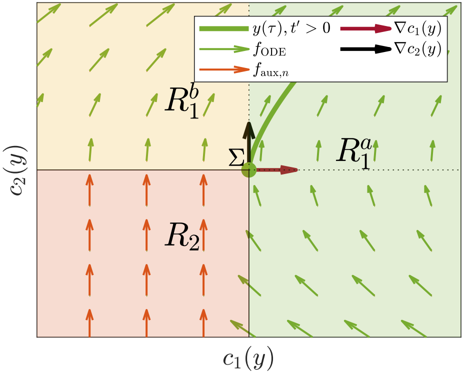
Contact breaking

The contact LCP function $\varphi(x)$ tells us about the vector field in R_1

- $\varphi(x)$ determines stability of Σ (remember the contact LCP)
- staying in sliding mode (persistent contact) or leaving sliding mode (contact breaking) is possible



Sliding mode if $\varphi(x) \leq 0$



Breaking contact if $\varphi(x) > 0$

Warm up example: a linearly increasing vertical force beats gravity

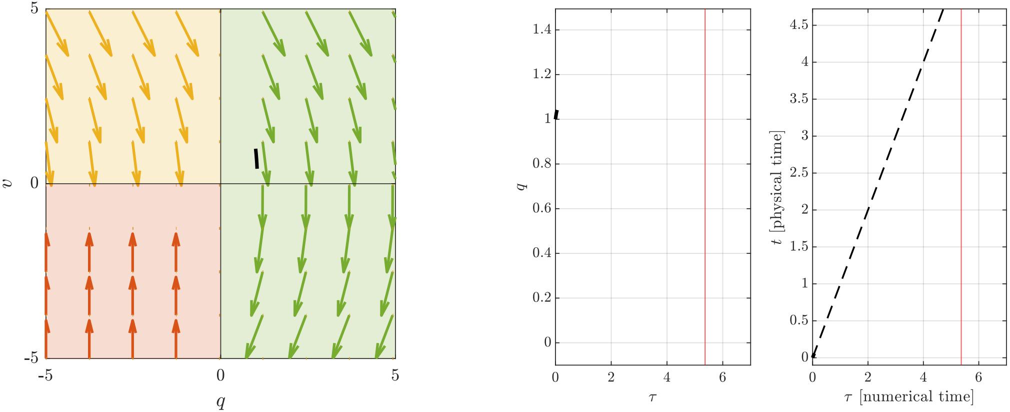


Illustration of leaving a sliding mode - contact breaking

Warm up example: a linearly increasing vertical force beats gravity

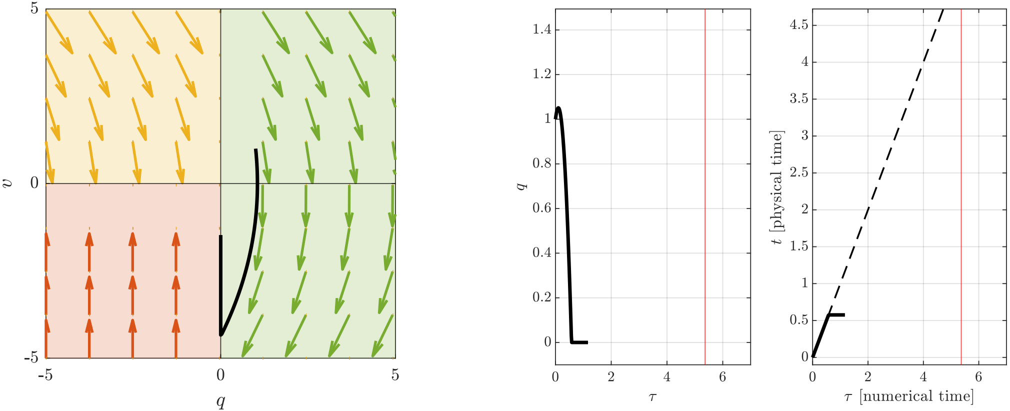


Illustration of leaving a sliding mode - contact breaking

Warm up example: a linearly increasing vertical force beats gravity

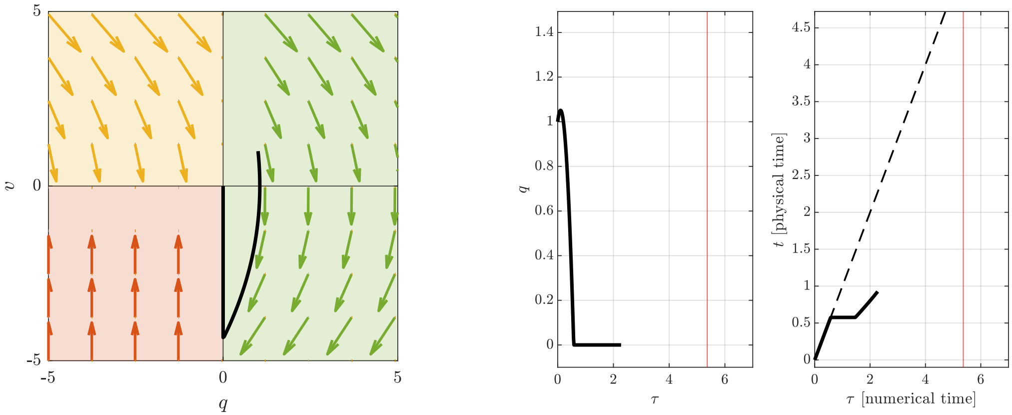
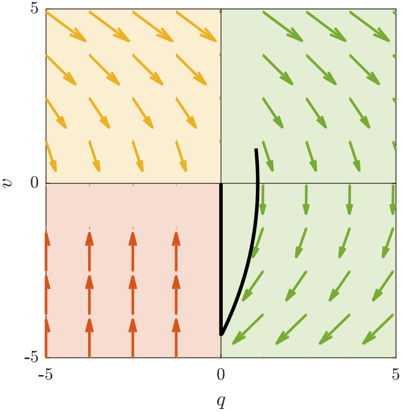
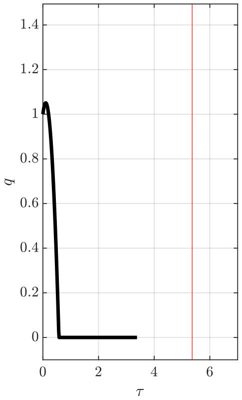


Illustration of leaving a sliding mode - contact breaking

Warm up example: a linearly increasing vertical force beats gravity





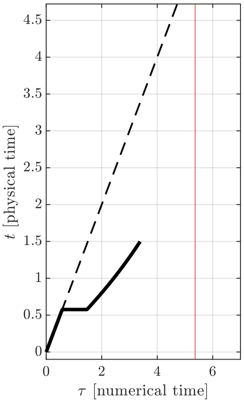
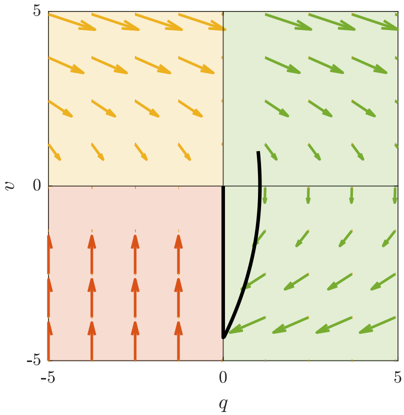
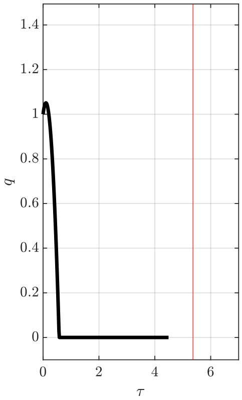
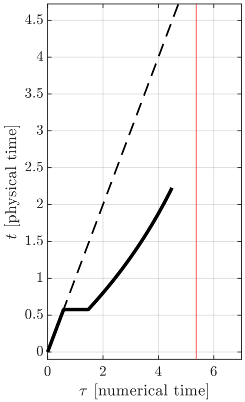


Illustration of leaving a sliding mode - contact breaking

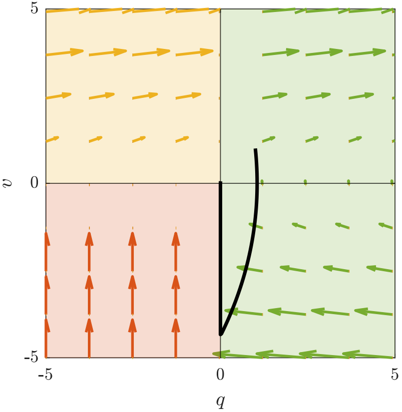
Warm up example: a linearly increasing vertical force beats gravity

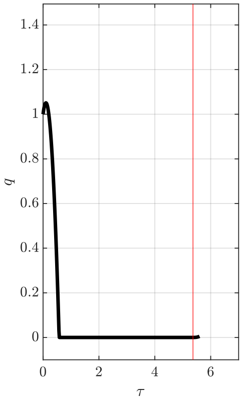


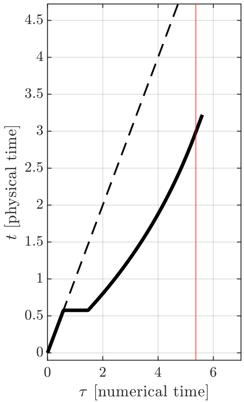




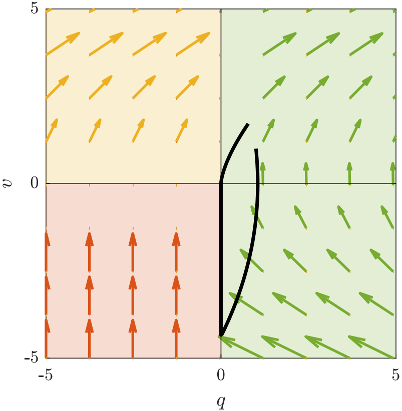
Warm up example: a linearly increasing vertical force beats gravity

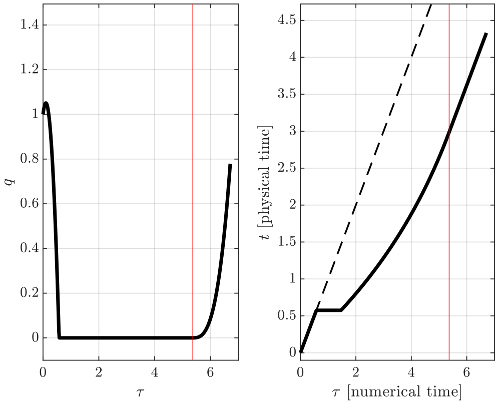






Warm up example: a linearly increasing vertical force beats gravity



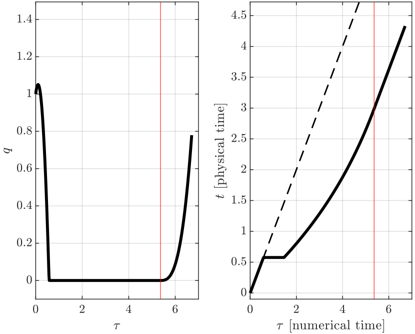


Why is the time slowed-down?

Time-freezing system

$$y' \in F_{\mathrm{TF}}(y,u) = \{\theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_{\mathrm{aux},n}(y) \mid \theta^\top e = 1,\ \theta \geq 0\}$$

- fractional $\theta_1, \theta_2 \in (0,1)$ ensures sliding on Σ
- speed of time $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \theta_1 \cdot 1 + \theta_2 \cdot 0 < 1$ - slow down
- resulting dynamics equal to *reduced* DAE index 3 dynamics $f_{\mathrm{DAE}}(x,u)$ (contact mode)
- auxiliary dynamics plays role of contact force (keeps $v = 0$ and avoids penetration)



The sliding mode is unique

Time-freezing system

$$y' \in F_{\mathrm{TF}}(y,u) = \{\theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_{\mathrm{aux},n}(y) \mid \theta^\top e = 1,\ \theta \geq 0\} \tag{1}$$

Theorem

Let $y(\tau)$ be a solution of the dyn. system (1) with $y(0) \in \Sigma = \{y \in \mathbb{R}^{n_y} \mid c_1(y) = 0, c_2(y) = 0\}$ and $\tau \in [0, \tau_{\mathrm{f}}]$. Suppose that $\varphi(x(\tau), u(\tau)) \leq 0$ for all $\tau \in [0, \tau_{\mathrm{f}}]$ (persistent contact), then the following statements are true:

(i) *the convex multipliers $\theta_1, \theta_2 \geq 0$ are unique,*

(ii) *the dynamics of the sliding mode are given by $y' = \beta(x,u) \begin{bmatrix} f_{\mathrm{DAE}}(x,u) \\ 1 \end{bmatrix}$, where*

$\beta(x,u) \in (0,1]$ is a time-rescaling factor given by

$$\beta(x,u) := \frac{D(q)a_{\mathrm{n}}}{D(q)a_{\mathrm{n}} - \varphi(x,u)}. \tag{2}$$

Obtaining a Filippov system in Stewart's or Step form

Time-freezing system

$$y' \in F_{\mathrm{TF}}(y,u) = \{\theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_{\mathrm{aux,n}}(y) \mid \theta_1 + \theta_2 = 1, \theta \geq 0\}$$

Regions

$$R_1^a = \{y \in \mathbb{R}^{n_y} \mid c_1(y) > 0\}$$
$$R_1^b = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) > 0\}$$
$$R_1 = R_1^a \cup {R_2}^b$$
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Obtaining a Filippov system in Stewart's or Step form

Time-freezing system

$$y' \in F_{\mathrm{TF}}(y,u) = \{\theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_{\mathrm{aux,n}}(y) \mid \theta_1 + \theta_2 = 1, \theta \geq 0\}$$

Switching functions and sign matrix

$$c(y) = \begin{bmatrix} f_c(q) \\ \nabla_q f_c(q)^\top v \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \quad \begin{matrix} R_1 \\ R_1 \\ R_1 \\ R_2 \end{matrix}$$

$$g(y) = -S^\top c(y)$$

Regions

$$R_1^a = \{y \in \mathbb{R}^{n_y} \mid c_1(y) > 0\}$$

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$$R_1 = R_1^a \cup R_2{}^b$$
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Step representation

$$y' = \theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_{\mathrm{aux,n}}(y)$$
$$\theta_1 = \alpha_1 + (1-\alpha_1)\alpha_2$$
$$\theta_2 = (1-\alpha_1)(1-\alpha_2)$$
$$\alpha_1 \in \gamma(c_1(y)),\ \alpha_2 \in \gamma(c_2(y))$$

Obtaining a Filippov system in Stewart's or Step form

Time-freezing system

$$y' \in F_{\mathrm{TF}}(y,u) = \{\theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_{\mathrm{aux,n}}(y) \mid \theta_1 + \theta_2 = 1, \theta \geq 0\}$$

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Switching functions and sign matrix

$$c(y) = \begin{bmatrix} f_c(q) \\ \nabla_q f_c(q)^\top v \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \quad \begin{matrix} R_1 \\ R_1 \\ R_1 \\ R_2 \end{matrix}$$

$$g(y) = -S^\top c(y)$$

Stewart's representation

$$y' = (\theta_1 + \theta_2 + \theta_3) f_{\mathrm{ODE}}(y,u) + \theta_4 f_{\mathrm{aux,n}}(y)$$
$$\theta = \arg\min_{\tilde{\theta} \in \mathbb{R}^4} \; g(y)^\top \tilde{\theta}$$
$$\text{s.t} \quad \tilde{\theta} \geq 0, \; e^\top \tilde{\theta} = 1$$

Outline of the lecture

Complementarity Lagrangian systems with impact and friction

Controlled CLS with friction

$$
\begin{aligned}
\dot{q} &= v \\
M(q)\dot{v} &= f_{\mathrm{v}}(q,v) + B_u(q)u + J_{\mathrm{n}}(q)\lambda_{\mathrm{n}} + J_{\mathrm{t}}(q)\lambda_{\mathrm{t}} \\
0 &\leq \lambda_{\mathrm{n}} \perp f_c(q) \geq 0 \\
&\text{if } f_c(q(t_{\mathrm{s}})) \leq 0 \text{ then } J_{\mathrm{n}}(q(t_{\mathrm{s}}))^\top v(t_{\mathrm{s}}^+) \geq 0 \\
&\lambda_{\mathrm{t}} \in \arg \min_{\tilde{\lambda}_{\mathrm{t}} \in \mathbb{R}^{n_{\mathrm{t}}}} \quad -v^\top J_{\mathrm{t}}(q)\tilde{\lambda}_{\mathrm{t}} \\
&\qquad\qquad \text{s.t.} \quad \|\tilde{\lambda}_{\mathrm{t}}\|_2 \leq \mu\lambda_{\mathrm{n}}
\end{aligned}
$$

- we regard $f_c(x) \in \mathbb{R}$ (single unilateral constraint)
- $J_{\mathrm{t}}(q) \in \mathbb{R}^{n_q \times n_{\mathrm{t}}}$ spans the tangent plane at contact points $C(q) := \{q \in \mathbb{R}^{n_q} \mid f_c(q) = 0\}$, $n_{\mathrm{t}} \in \{1, 2\}$, tang. velocity $v_{\mathrm{t}} = J_{\mathrm{t}}(q)^\top v$
- We derive time-freezing for the friction terms

Coulomb's friction

Solution map for a given λ_{n}

Coulomb's friction law

$$\lambda_{\mathrm{t}} \in \arg \min_{\tilde{\lambda}_{\mathrm{t}} \in \mathbb{R}^{n_{\mathrm{t}}}} \quad -v_{\mathrm{t}}^\top \tilde{\lambda}_{\mathrm{t}}$$
$$\text{s.t.} \quad \|\tilde{\lambda}_{\mathrm{t}}\|_2 \leq \mu \lambda_{\mathrm{n}}.$$

Friction solution map

$$\lambda_{\mathrm{t}} \in \begin{cases} \{-\mu \lambda_{\mathrm{n}} \frac{v_{\mathrm{t}}}{\|v_{\mathrm{t}}\|_2}\}, & \text{if } \|v_{\mathrm{t}}\|_2 > 0, \\ \{\tilde{\lambda}_{\mathrm{t}} \mid \|\tilde{\lambda}_{\mathrm{t}}\|_2 \leq \mu \lambda_{\mathrm{n}}\}, & \text{if } \|v_{\mathrm{t}}\|_2 = 0. \end{cases}$$

Coulomb's friction

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- reduces to $\lambda_{\mathrm{t}} \in -\mu \lambda_{\mathrm{n}} \mathrm{sign}(v_{\mathrm{t}})$ in planar case
- the normal impulse is $a_{\mathrm{n}} \tau_{\mathrm{jump}} \implies$ the tangential impulse should be $-\mu a_{\mathrm{n}} \tau_{\mathrm{jump}} \mathrm{sign}(v_{\mathrm{t}})$
- tangential state jumps happens simultaneously with normal impulse

Coulomb's friction

Solution map for a given λ_{n}

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- tangential state jumps happens simultaneously with normal impulse
- **Conclusion**: define aux. dyn. in tangential directions $J_{\mathrm{t}}(q)$ "proportional" to $f_{\mathrm{aux,n}}$ and let them evolve simultaneously

Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_t

New additional switching function $c_3(y) = v_\mathrm{t}$

The "old R_2" where the jumps were happening split into two regions to account for sign of v_t

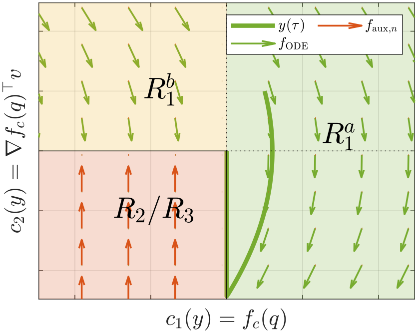
Regions

$$Q = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$

$$R_1 = R_1^a \cup R_1^b$$

$$R_2 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) > 0\}$$

$$R_3 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) < 0\}$$



Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\} \quad (3)$$

Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_t

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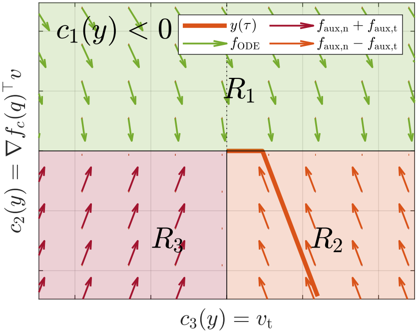

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$$y' \in F_{\mathrm{TF}}(y,u) = \Big\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \Big\} \tag{3}$$

Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_t

New additional switching function $c_3(y) = v_\mathrm{t}$

The "old R_2" where the jumps were happening split into two regions to account for sign of v_t

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$$Q = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$

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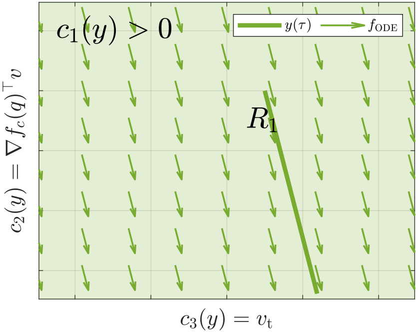

Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\} \tag{3}$$

Regions with tangential auxiliary dynamics

Refine the definitions for $c_1(y) < 0$ and $c_2(y) < 0$ to account for the sign of v_t

New additional switching function $c_3(y) = v_\mathrm{t}$

The "old R_2" where the jumps were happening split into two regions to account for sign of v_t

Regions

$$Q = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$

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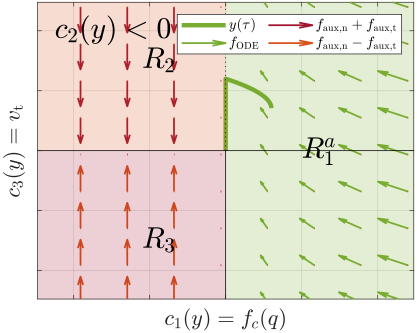


Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\} \tag{3}$$

Time-freezing with friction in the planar case

PSS modes

$$
\begin{aligned}
f_1(y,u) &= (f_{\mathrm{ODE}}(x,u), 1) \\
f_2(y) &= f_{\mathrm{aux,n}}(y) - f_{\mathrm{aux,t}}(y) \\
f_3(y) &= f_{\mathrm{aux,n}}(y) + f_{\mathrm{aux,t}}(y)
\end{aligned}
$$

Auxiliary ODE for tangential directions

$$
f_{\mathrm{aux,t}}(y) := \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1} J_{\mathrm{t}}(q)\mu\, a_{\mathrm{n}} \\ 0 \end{bmatrix}
$$

$$
f_{\mathrm{aux,n}}(y) := \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1} J_{\mathrm{n}}(q) a_{\mathrm{n}} \\ 0 \end{bmatrix}
$$

- Simply sum the auxiliary dynamics in normal and tangential directions (recall that $J_{\mathrm{t}}(q) \in \mathbb{R}^{n_q \times 1}$ and $J_{\mathrm{n}}(q) \perp J_{\mathrm{t}}(q)$)
- State jump is over when $J_{\mathrm{n}}(q)^\top v = 0$
- With $v_{\mathrm{t}} = 0$ sliding mode on $\Gamma = \{y \mid c_1(y) = 0, c_2(y) = 0, c_3(y) = 0\}$

Time-freezing with friction - sliding mode dynamics

- $\dot{x} = f_{\mathrm{Slip}}(x,u)$ reduced DAE in slip mode, $v_{\mathrm{t}} \neq 0$
- $\dot{x} = f_{\mathrm{Stick}}(x,u)$ reduced DAE in stick mode, $v_{\mathrm{t}} = 0$

Theorem (Slip-stick sliding mode)

Let $y(\tau)$ be a solution of time freezing system (3) with $y(0) \in \Sigma$ and $\tau \in [0, \tau_{\mathrm{f}}]$. Let $J_{\mathrm{n}}(q)^\top M(q)^{-1} J_{\mathrm{t}}(q) = 0$ (orthogonality in kinetic metric). Suppose that $\varphi(x(\tau), u(\tau)) \leq 0$ for all $\tau \in [0, \tau_{\mathrm{f}}]$ (persistent contact), then the following statements are true:

(i) *If $v_{\mathrm{t}} \neq 0$ (slip motion), then the sliding mode dynamics are given by*

$$y' = \beta(x,u) \begin{bmatrix} f_{\mathrm{Slip}}(x,u) \\ 1 \end{bmatrix}$$

(ii) *If $v_{\mathrm{t}} = 0$ (stick motion), then the sliding mode dynamics are given by*

$$y' = \beta(x,u) \begin{bmatrix} f_{\mathrm{Stick}}(x,u) \\ 1 \end{bmatrix}$$

where $\beta(x,u) \in (0,1]$ is a time-rescaling factor defined in Eq. (2).

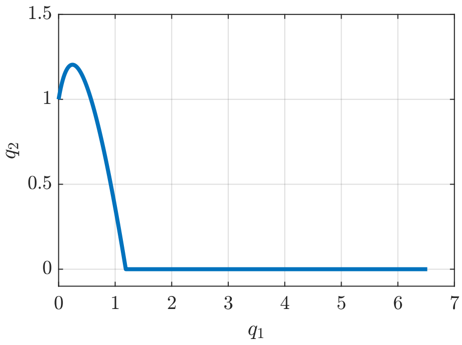
Simulation example - slip/stick

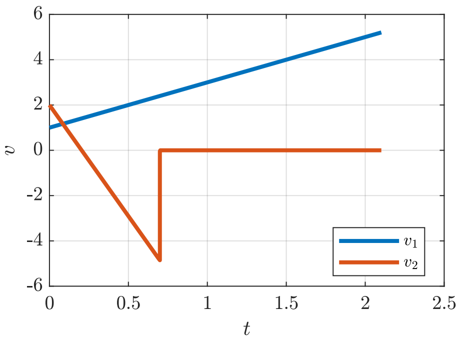
Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0$

No friction





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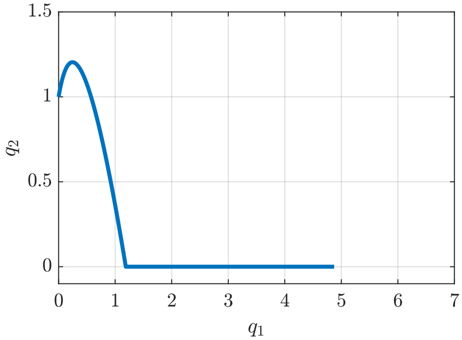
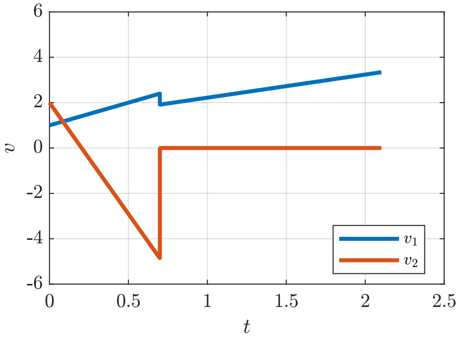
Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.1$

External force stronger than friction





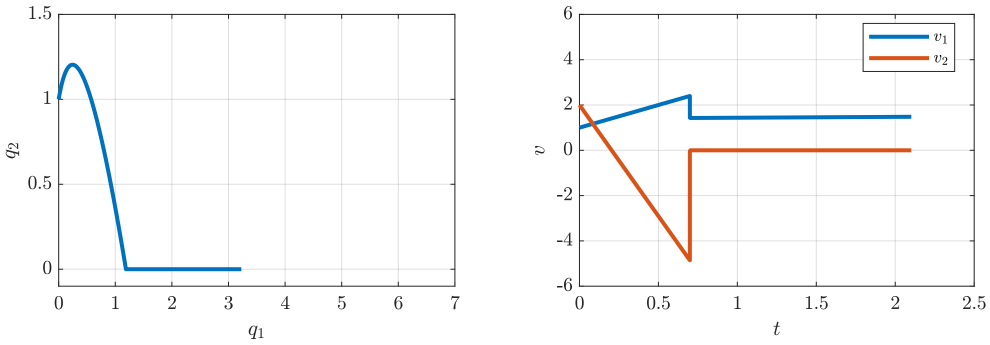
Simulation example - slip/stick

Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.2$

External force equal to friction



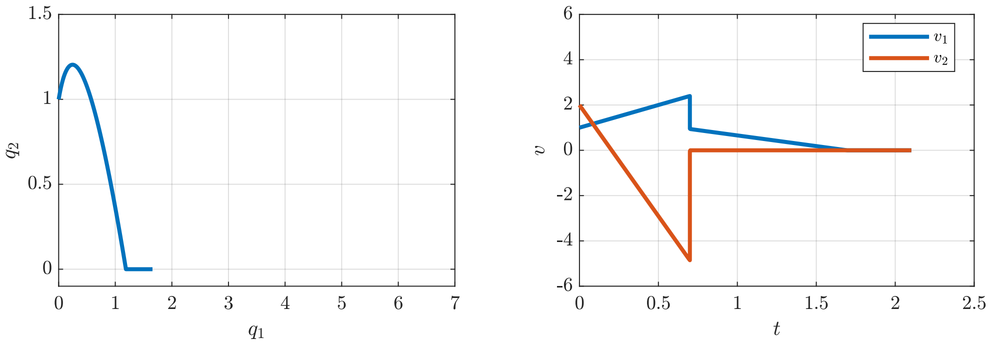
Simulation example - slip/stick

Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.3$

External force weaker than friction



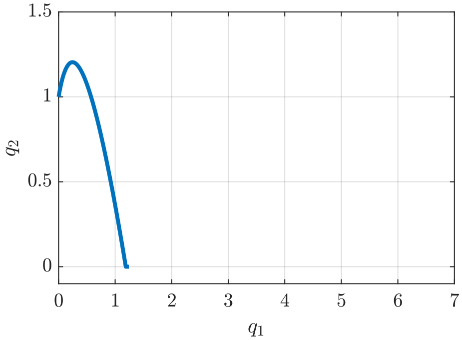
Simulation example - slip/stick

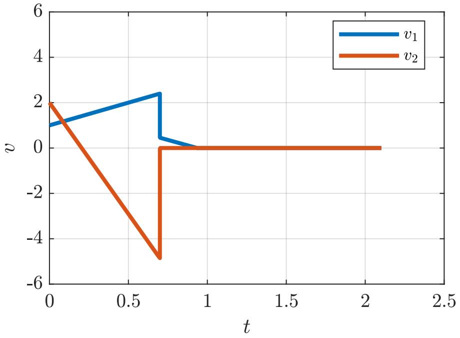
Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.4$

External force weaker than friction





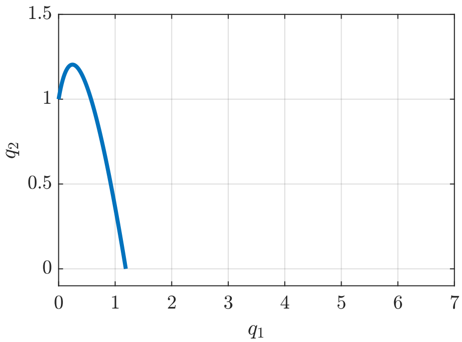
Simulation example - slip/stick

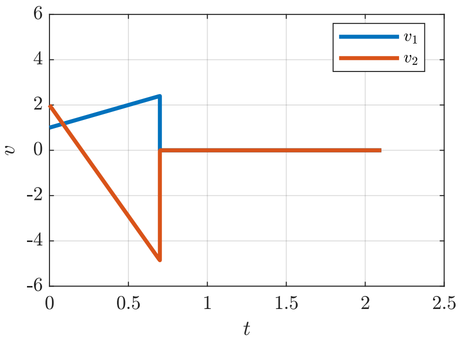
Increasing $\mu = 0$ to $\mu = 0.5$ with $\Delta\mu = 0.1$.

External force $u_x = 2$

$\mu = 0.5$

Tangential velocity zero after impact



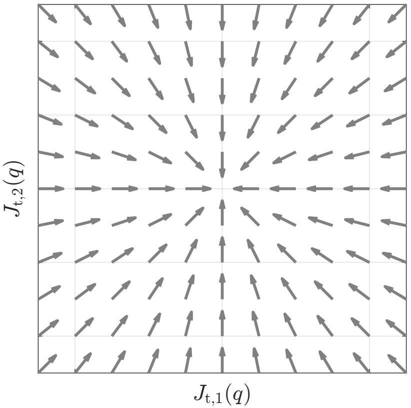


Friction for 3D contacts

Friction solution map

$$\lambda_t \in \begin{cases} \{-\mu\lambda_n \frac{v_t}{\|v_t\|_2}\}, & \text{if } \|v_t\|_2 > 0, \\ \{\tilde{\lambda}_t \mid \|\tilde{\lambda}_t\|_2 \leq \mu\lambda_n\}, & \text{if } \|v_t\|_2 = 0. \end{cases}$$

- The set $\{v_t \mid v_t = 0\}$ has an empty interior
- Problematic for defining Filippov system via θ multipliers
- Problem not present with polyhedral approximations

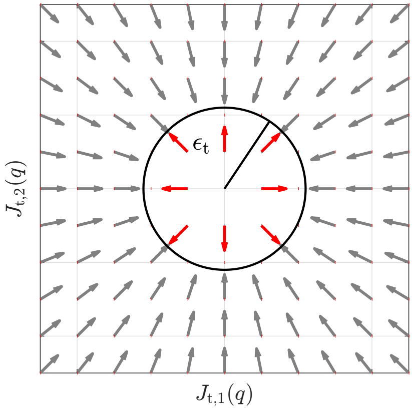


Friction for 3D contacts - relaxed solution

Relaxed riction solution map

$$\lambda_t = \begin{cases} -\mu \lambda_n \frac{v_t}{\|v_t\|_2}, & \text{if } \|v_t\|_2 > \epsilon_t, \\ v_t, & \text{if } \|v_t\|_2 < \epsilon_t, \end{cases}$$

- $\epsilon_t > 0$ can be arbitrarily small
- Obtain set with nonempty interior
- Slip mode: approximation is exact
- Stick mode: sliding mode on $\|v_t\|_2 = \epsilon_t$
- Approximation can be made arbitrarily accurate



Friction for 3D contacts - the time-freezing system

Time-freezing system with friction

$$y' \in F_{\mathrm{TF}}(y,u) = \left\{ \sum_{i=1}^{3} f_i(y,u) \mid \theta \geq 0,\ e^\top \theta = 1 \right\}$$

PSS modes

$$\begin{aligned} f_1(y,u) &= (f_{\mathrm{ODE}}(x,u), 1) \\ f_2(y) &= f_{\mathrm{aux,n}}(y) - f_{\mathrm{aux,t,2}}(y) \\ f_3(y) &= f_{\mathrm{aux,n}}(y) + f_{\mathrm{aux,t,3}}(y) \end{aligned}$$

- Use same definition of regions R_1, R_2 and R_3
- Switching function $c_3(y) = \|v_\mathrm{t}\|_2 - \epsilon_\mathrm{t}$

Auxiliary ODEs for 3D friction

$$f_{\mathrm{aux,t,2}}(y) = \begin{bmatrix} \mathbf{0}_{n_q,1} \\ -M(q)^{-1} J_\mathrm{t}(q) \mu a_\mathrm{n} \frac{v_\mathrm{t}}{\|v_\mathrm{t}\|} \\ 0 \end{bmatrix}$$

$$f_{\mathrm{aux,t,3}}(y) = \begin{bmatrix} \mathbf{0}_{n_q,1} \\ M(q)^{-1} J_\mathrm{t}(q) v_\mathrm{t} \\ 0 \end{bmatrix}$$

Obtaining a Filippov system in Stewart's or Step form

$$y' \in \Big\{ \sum_{i=1}^{3} \theta_i f_i(y,u) \mid e^\top \theta = 1, \theta \geq 0 \Big\}$$

Switching functions and sign matrix

$$c(y) = \begin{bmatrix} f_c(q) \\ J_\mathrm{n}(q)^\top v \\ J_\mathrm{t}(q)^\top v \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{matrix} R_1 \\ R_1 \\ R_1 \\ R_1 \\ R_1 \\ R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$g(y) = -S^\top c(y)$$

Regions

$$Q = \{y \in \mathbb{R}^{n_y} \mid c_1(y) < 0, c_2(y) < 0\}$$

$$R_1 = R_1^a \cup R_1^b$$

$$R_2 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) > 0\}$$

$$R_3 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) < 0\}$$

Step representation

$$y' = \theta_1 f_{\mathrm{ODE}}(y,u) + \theta_2 f_2(y) + \theta_3 f_3(y)$$

$$\theta_1 = \alpha_1 + (1-\alpha_1)\alpha_2$$

$$\theta_2 = (1-\alpha_1)(1-\alpha_2)\alpha_3$$

$$\theta_3 = (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)$$

$$\alpha_1 \in \gamma(c_1(y)),\ \alpha_2 \in \gamma(c_2(y)),\ \alpha_3 \in \gamma(c_3(y))$$

Obtaining a Filippov system in Stewart's or Step form

$$y' \in \Big\{ \sum_{i=1}^{3} \theta_i f_i(y,u) \mid e^\top \theta = 1, \theta \geq 0 \Big\}$$

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$$R_3 = Q \cap \{y \in \mathbb{R}^{n_y} \mid c_3(y) < 0\}$$

Stewart's representation

$$y' = \sum_{i=1}^{6} \theta_i f_{\mathrm{ODE}}(y,u) + \theta_7 f_{\mathrm{aux},1}(y) + \theta_8 f_{\mathrm{aux},2}(y)$$

$$\theta = \arg\min_{\tilde\theta \in \mathbb{R}^8} \; g(y)^\top \tilde\theta \quad \text{s.t} \quad \tilde\theta \geq 0,\; e^\top \tilde\theta = 1$$

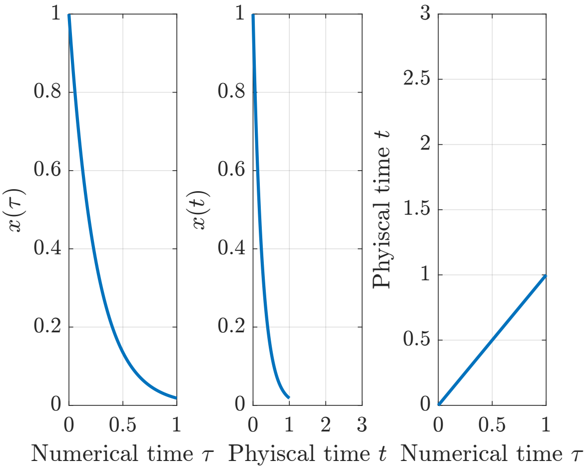
Outline of the lecture

Time-transformations for ODEs

- ODE in physical time

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t)), t \in [0, 1]$$
$$x(0) = x_0$$

- Introduce time scaling $t = s\tau$



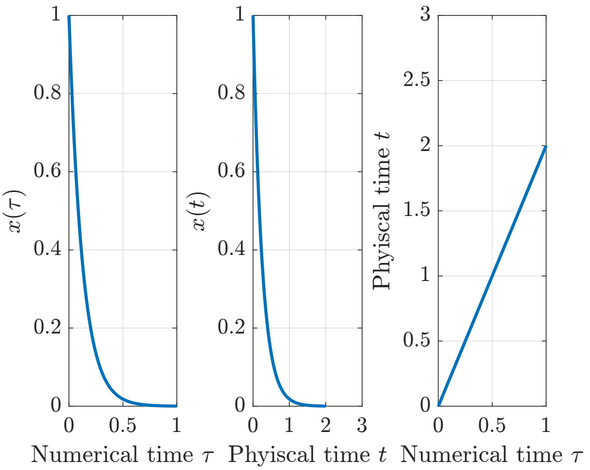
Time-transformations for ODEs

- ODE in physical time

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t)), t \in [0,1]$$
$$x(0) = x_0$$

- Introduce time scaling $t = s\tau$
- Rescaled dynamics in numerical time:

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = s\, f(x)$$
$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = s$$
$$x(0) = x_0,\ t(0) = 0$$



Time-transformations for ODEs

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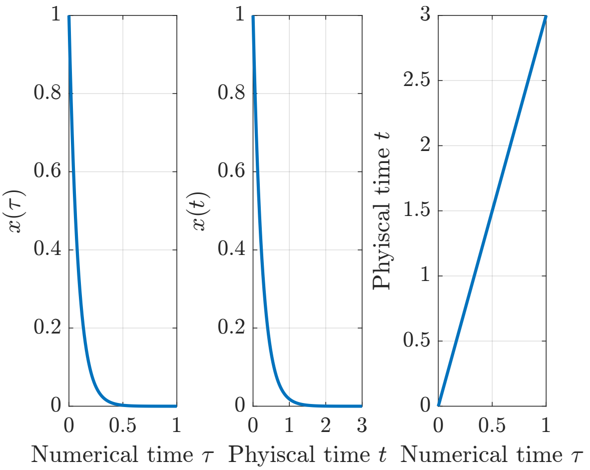
$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = s\, f(x)$$
$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = s$$
$$x(0) = x_0,\; t(0) = 0$$

- s can be an optimization variable, e.g., in time optimal control



Optimal control with time-freezing

OCP with CLS

$$\min_{x(\cdot),u(\cdot),\lambda(\cdot)} \int_0^T L(x,u)\mathrm{d}t + E(x(T))$$

$$\text{s.t.} \quad x(0) = \bar{x}_0$$

CLS

$$0 \geq h(x(t),u(t)),\ t \in [0,T]$$

$$0 \geq r(x(T))$$

Optimal control with time-freezing

OCP with qudrature state

$$\min_{x(\cdot),u(\cdot),\lambda(\cdot)} \quad \ell(T) + E(x(T)) =: \Phi(x(T))$$

$$\text{s.t.} \quad x(0) = \bar{x}_0, \;\; \ell(0) = 0$$

CLS

$$\dot{\ell}(t) = L(x(t), u(t))$$

$$0 \geq h(x(t), u(t)), \;\; t \in [0, T]$$

$$0 \geq r(x(T))$$

Integrate stage costs together with dynamics.

Optimal control with time-freezing

OCP with qudrature state

$$\min_{x(\cdot),u(\cdot),\lambda(\cdot)} \ \ell(T) + E(x(T)) =: \Phi(x(T))$$

$$\text{s.t.} \quad x(0) = \bar{x}_0, \ \ell(0) = 0$$

CLS

$$\dot{\ell}(t) = L(x(t), u(t))$$

$$0 \geq h(x(t), u(t)), \ t \in [0, T]$$

$$0 \geq r(x(T))$$

Integrate stage costs together with dynamics.

- In time-freezing OCP redefine quadrature state

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ell(\tau) = \begin{cases} L(x(\tau), u(\tau)), & \text{if } y \in R_1, \\ 0, & \text{otherwise.} \end{cases}$$

- On which time domains are the problems defined?
  - initial OCP on $t \in [0, T]$
  - time-freezing OCP $\tau \in [0, \tilde{T}]$
- If time freezes, then $T \neq t(\tilde{T})$
- Need time transformation to catch up

Optimal control with time-freezing

OCP with qudrature state

$$\begin{aligned}
\min_{x(\cdot),u(\cdot),\lambda(\cdot)} \quad & \ell(T) + E(x(T)) =: \Phi(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0, \; \ell(0) = 0 \\
& \text{CLS} \\
& \dot{\ell}(t) = L(x(t), u(t)) \\
& 0 \geq h(x(t), u(t)), \; t \in [0, T] \\
& 0 \geq r(x(T))
\end{aligned}$$

Integrate stage costs together with dynamics.

Time-freezing OCP with step reformulation

$$\begin{aligned}
\min_{y(\cdot),z(\cdot),u(\cdot),s(\cdot)} \quad & \Psi(x(\tilde{T})) \\
\text{s.t.} \quad & x(0) = \bar{x}_0, \; t(0) = 0, \\
& y'(\tau) = s(\tau) F(y(\tau), u(\tau)) \theta(\tau) \\
& 0 = g_{\text{Step}}(\theta(\tau), \alpha(\tau)) \\
& 0 = c(y(\tau)) - \lambda^{\text{p}}(\tau) + \lambda^{\text{n}}(\tau) \\
& 0 \leq \alpha(\tau) \perp \lambda^{\text{n}}(\tau) \geq 0 \\
& 0 \leq e - \alpha(\tau) \perp \lambda^{\text{p}}(\tau) \geq 0 \\
& 0 \leq h(x(\tau), u(\tau)), \; \tau \in [0, \tilde{T}] \\
& 0 \leq r(x(\tilde{T})) \\
& t(\tilde{T}) = T
\end{aligned}$$

A guiding example of solving an OCP

Example

A 2D ball with friction and impacts

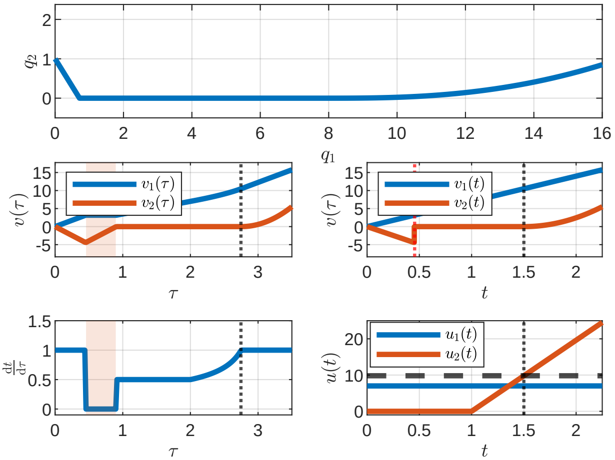
$$
\begin{aligned}
\min_{x(\cdot),z(\cdot),u(\cdot)} \quad & \int_0^T u(t)^\top u(t) \, \mathrm{d}t \\
\text{s.t.} \quad & x(0) = (0,1,0,0) \\
& \dot{q} = v, && t \in [0,T] \\
& m\dot{v} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \lambda_{\mathrm{n}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \lambda_{\mathrm{t}} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, && t \in [0,T] \\
& 0 \leq \lambda_{\mathrm{n}} \perp q_2 \geq 0, && t \in [0,T] \\
& v_2(t_{\mathrm{s}}^+) = 0, \ \text{if } q_2(t_{\mathrm{s}}) = 0 \text{ and } v_2(t_{\mathrm{s}}^-) < 0 \\
& \lambda_{\mathrm{t}} \in -\mu \lambda_{\mathrm{n}} \mathrm{sign}(v_1), && t \in [0,T] \\
& u_{\min} \leq u(t) \leq u_{\max}, && t \in [0,T] \\
& x(T) = (3,0,0,0)
\end{aligned}
$$

Control input

$$u_1(t) = 7$$

$$u_2(t) = \begin{cases} 0, & \text{if } t < 1 \\ 2g(t-1), & \text{if } t \geq 1 \end{cases}$$

- state jumps only in vertical direction (v_2)
- decreased speed of time in contact phases
- lift off when u_2 beats gravity g



Understanding the dynamics of time-freezing systems with state jumps

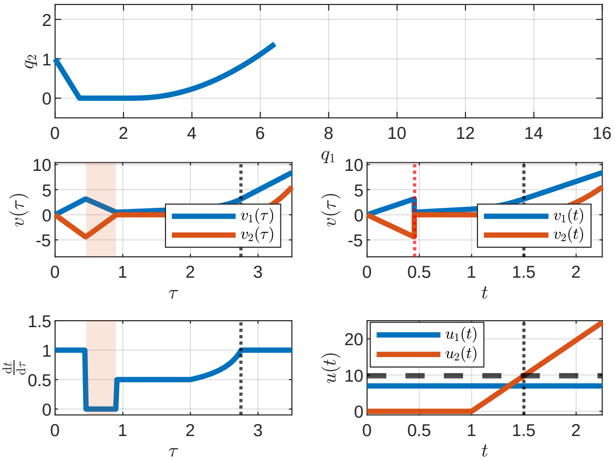
A simulation problem with fixed control and with friction $(\mu = 0.6)$

Control input

$$u_1(t) = 7$$

$$u_2(t) = \begin{cases} 0, & \text{if } t < 1 \\ 2g(t-1), & \text{if } t \geq 1 \end{cases}$$

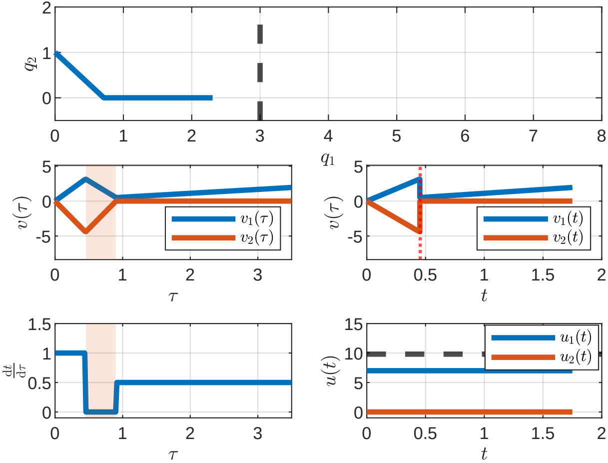
- state jumps horizontal (v_1) and in vertical direction (v_2)
- decreased speed of time in contact phases
- lift off when u_2 beats gravity g



Control input

$$u_1(t) = 7$$
$$u_2(t) = 0$$

- state jumps horizontal (v_1) and in vertical direction (v_2)
- decreased speed of time in contact phases
- we miss the goal



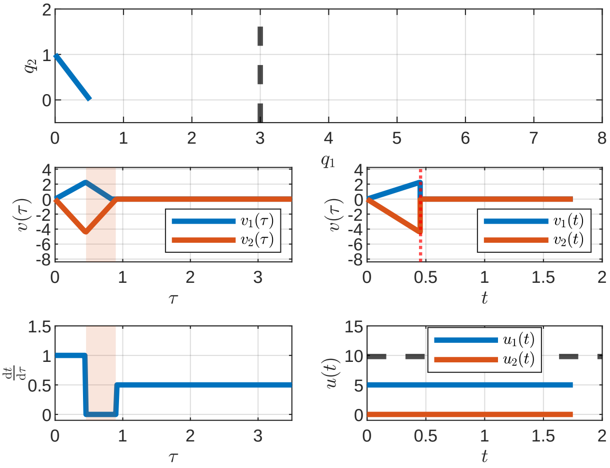
Understanding the dynamics of time-freezing systems with state jumps

How to reach the goal? Decrease the thrust force?

Control input

$$u_1(t) = 5$$
$$u_2(t) = 0$$

- state jumps horizontal (v_1) and in vertical direction (v_2)
- decreased speed of time in contact phases
- we miss the goal

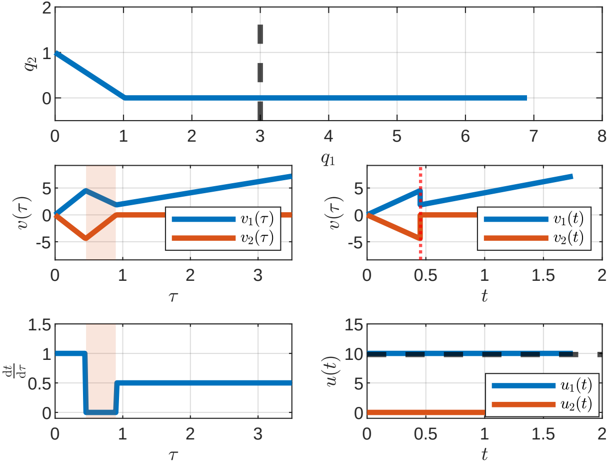


How to reach the goal? Increase the thrust force?

Control input

$$u_1(t) = 10$$
$$u_2(t) = 0$$

- state jumps horizontal (v_1) and in vertical direction (v_2)
- decreased speed of time in contact phases
- we miss the goal

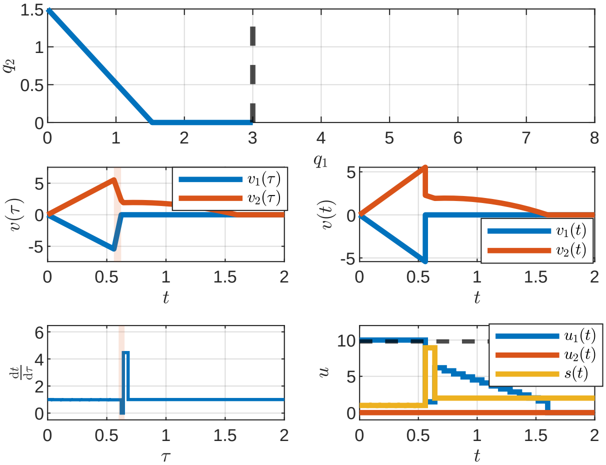


How to reach the goal? Solve an optimal control problem!

Control input

$$u_1(t) = u_1^*(t)$$
$$u_2(t) = 0$$

- state jumps horizontal (v_1) and in vertical direction (v_2)
- speed of time control variable $s(t)$ compensates slow downs
- the goal is reached!



Conclusions and outlook

Conclusions

- Optimal control problems with state jumps are *very* difficult.
- Time-freezing allows us to transform systems with state jumps of level NSD3 to the easier level NSD2.
- Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for switched systems of level NSD2.
- The time-freezing Filippov system can be treated both in Stewart's and the Heaviside step form.
- Alternative: FESD for NSD3 system = FESD-J, but time-freezing + FESD seems to converge better.

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Outlook

- Time-freezing for multiple and simultaneous impacts with friction (preprint in preparation)
- Time-freezing for more general hybrid automaton
- Do generic time-freezing principles, easily applicable to *any* system with state jumps, exist?

References

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  Mathew Halm and Michael Posa, arXiv Preprint, 2023.
- Numerical Methods for Optimal Control of Nonsmooth Dynamical Systems, A. Nurkanović., PhD Thesis, *to appear in 2023*