Lecture 6: Finite Elements with Switch Detection for Filippov Systems

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Systems Control and Optimization Laboratory (syscop)
Summer School on Direct Methods for Optimal Control of Nonsmooth Systems
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universität freiburg

Outline of the lecture

How to discretize optimal control problems subject to Filippov systems?

In direct optimal control, we first discretize, and then solve a finite-dimensional nonlinear program.

Original optimal control problem in continuous time

$$\min_{x(\cdot),u(\cdot)} \int_0^T L(x,u)\mathrm{d}t + E(x(T))$$
$$\text{s.t.} \quad x(0) = \bar{x}_0$$
$$\dot{x}(t) \in F_{\mathrm{F}}(x(t),u(t))$$
$$0 \geq h(x(t),u(t)),\ t \in [0,T]$$
$$0 \geq r(x(T))$$

Assume smooth (convex) L, E, h, r
Nonsmooth dynamics make problem nonconvex.

How to discretize optimal control problems subject to Filippov systems?

In direct optimal control, we first discretize, and then solve a finite-dimensional nonlinear program.

Optimal control problem with Stewart's formulation

$$\begin{aligned}
\min_{\substack{x(\cdot),u(\cdot),\\ \theta(\cdot),\lambda(\cdot),\mu(\cdot)}} \quad & \int_0^T L(x,u)\mathrm{d}t + E(x(T)) \\
\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = F(x(t),u(t))\,\theta(t) \\
& 0 = G_{\mathrm{LP}}(x(t),\theta(t),\lambda(t),\mu(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}$$

Assume smooth (convex) L, E, h, r
Nonsmooth dynamics make problem nonconvex.

How to discretize optimal control problems subject to Filippov systems?

In direct optimal control, we first discretize, and then solve a finite-dimensional nonlinear program.

Optimal control problem with Stewart's formulation

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\text{s.t.} \quad & x(0) = \bar{x}_0 \\
& \dot{x}(t) = F(x(t),u(t))\,\theta(t) \\
& 0 = G_{\mathrm{LP}}(x(t),\theta(t),\lambda(t),\mu(t)) \\
& 0 \geq h(x(t),u(t)),\ t \in [0,T] \\
& 0 \geq r(x(T))
\end{aligned}$$

Assume smooth (convex) L, E, h, r
Nonsmooth dynamics make problem nonconvex.

Goal: discretized optimal control problem (an NLP)

$$\begin{aligned}
\min_{s,z,u} \quad & \sum_{k=0}^{N-1} \Phi_L(s_k,z_k,u_k) + E(s_N) \\
\text{s.t.} \quad & s_0 = \bar{x}_0 \\
& s_{k+1} = \Phi_f(s_k,z_k,u_k) \\
& 0 = \Phi_{\mathrm{int}}(s_k,z_k,u_k) \\
& 0 \geq h(s_k,u_k),\ k = 0,\ldots,N-1 \\
& 0 \geq r(s_N)
\end{aligned}$$

Variables $s = (s_0,\ldots)$, $z = (z_0,\ldots)$ and $u = (u_0,\ldots,u_{N-1})$
Nonsmooth Φ_{int}

What happens if we use time stepping methods in direct optimal control?

Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]

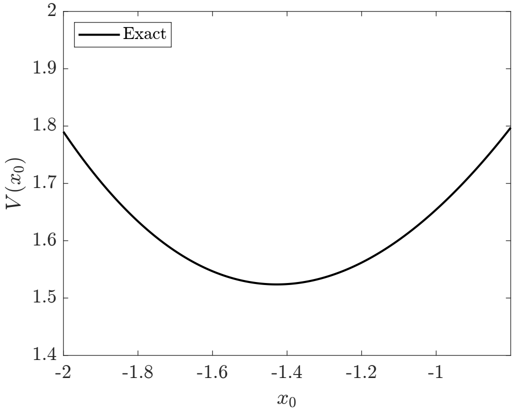
Continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^0([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) = 2 - \mathrm{sign}(x(t)), \quad t \in [0,2]$$

Free initial value $x(0)$ is the effective degree of freedom.

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V(x_0)$$



- Denote by $V(x_0)$ the nonsmooth objective value for the unique feasible trajectory starting at $x(0) = x_0$.

Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]

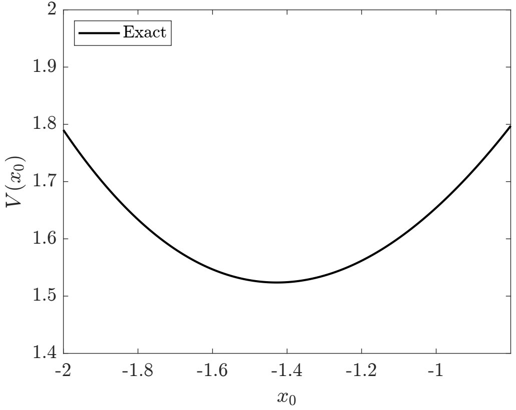
Continuous-time OCP

$$\min_{x(\cdot),\lambda(\cdot),s(\cdot)} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) = 2 - s(t)$$
$$0 \leq \lambda(t) - x(t) \perp 1 + s(t) \geq 0$$
$$0 \leq \lambda(t) \perp 1 - s(t) \geq 0, \ t \in [0, 2]$$

Free initial value $x(0)$ is the effective degree of freedom.

Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V(x_0)$$



- Denote by $V(x_0)$ the nonsmooth objective value for the unique feasible trajectory starting at $x(0) = x_0$.

Direct optimal control with a time stepping IRK discretization

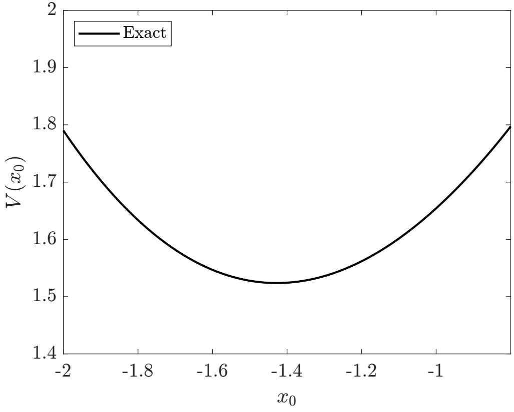
Tutorial example inspired by [Stewart & Anitescu, 2010]

Continuous-time OCP

$$\min_{x(\cdot),\lambda(\cdot),s(\cdot)} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

$$\begin{aligned}\text{s.t.} \quad & \dot{x}(t) = 2 - s(t) \\ & 0 \leq \lambda(t) - x(t) \perp 1 + s(t) \geq 0 \\ & 0 \leq \lambda(t) \perp 1 - s(t) \geq 0, \ t \in [0, 2]\end{aligned}$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\mathrm{s}} = 1$, accuracy $O(h^2)$



Locally quadratic objective.

Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]

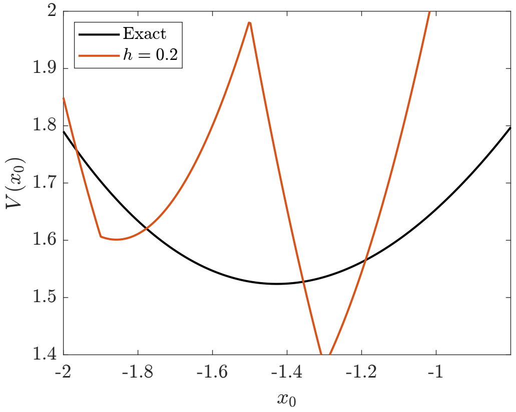
Discrete-time OCP

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2$$

$$\text{s.t.} \quad x_{n+1} = \phi_f(x_n, z_n)$$

$$0 = \phi_{\text{int}}(x_n, z_n), \;\; n = 0, \dots N-1$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\text{s}} = 1$, accuracy $O(h^2)$
- step size $h = 0.2$, i.e., $N = 10$ integration steps



Many artificial local minima and wrong derivatives.

Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]

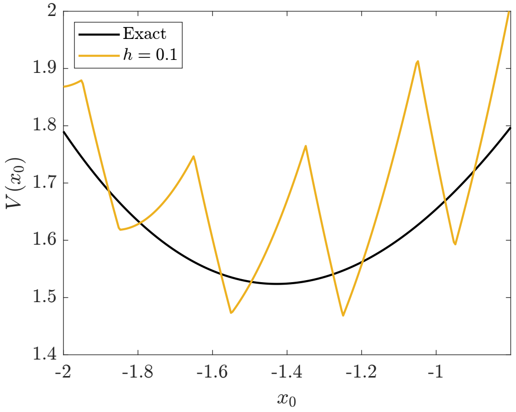
Discrete-time OCP

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2$$

$$\text{s.t.} \quad x_{n+1} = \phi_f(x_n, z_n)$$

$$0 = \phi_{\text{int}}(x_n, z_n), \;\; n = 0, \ldots N-1$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\text{s}} = 1$, accuracy $O(h^2)$
- step size $h = 0.1$, i.e., $N = 20$ integration steps



Many artificial local minima and wrong derivatives.

Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]

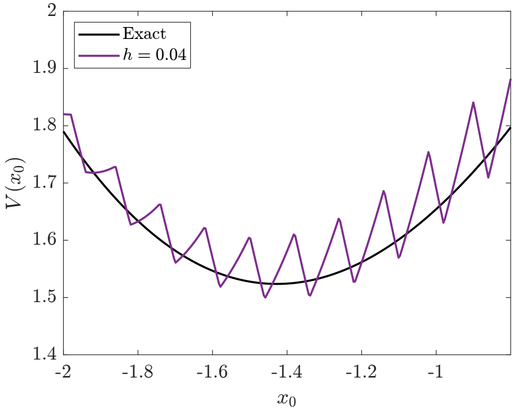
Discrete-time OCP

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2$$

$$\text{s.t.} \quad x_{n+1} = \phi_f(x_n, z_n)$$

$$0 = \phi_{\text{int}}(x_n, z_n), \; n = 0, \dots N-1$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\text{s}} = 1$, accuracy $O(h^2)$
- step size $h = 0.04$, i.e., $N = 50$ integration steps



Many artificial local minima and wrong derivatives.

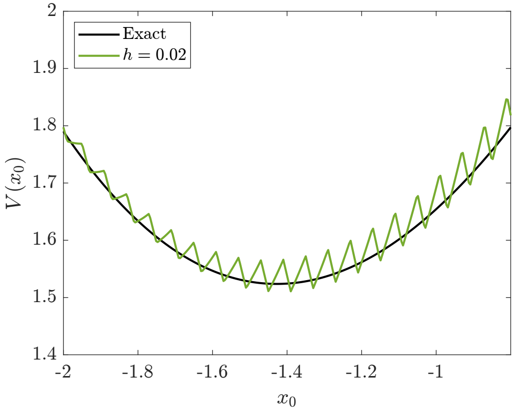
Discrete-time OCP

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2$$

$$\text{s.t.} \quad x_{n+1} = \phi_f(x_n, z_n)$$

$$0 = \phi_{\text{int}}(x_n, z_n), \; n = 0, \dots N-1$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\mathrm{s}} = 1$, accuracy $O(h^2)$
- step size $h = 0.02$, i.e., $N = 100$ integration steps



Many artificial local minima and wrong derivatives.

Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]

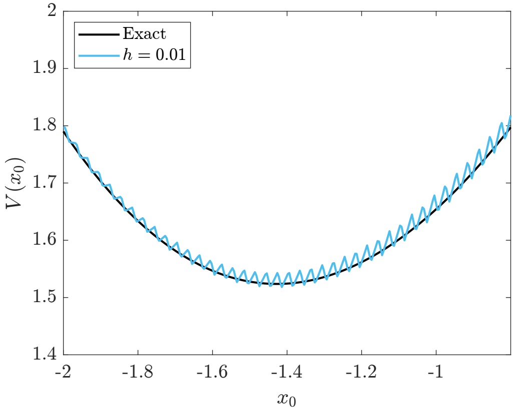
Discrete-time OCP

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2$$

$$\text{s.t.} \quad x_{n+1} = \phi_f(x_n, z_n)$$

$$0 = \phi_{\text{int}}(x_n, z_n), \;\; n = 0, \ldots N-1$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_\mathrm{s} = 1$, accuracy $O(h^2)$
- step size $h = 0.01$, i.e., $N = 200$ integration steps



Many artificial local minima and wrong derivatives.

Direct optimal control with a time stepping IRK discretization

Tutorial example inspired by [Stewart & Anitescu, 2010]

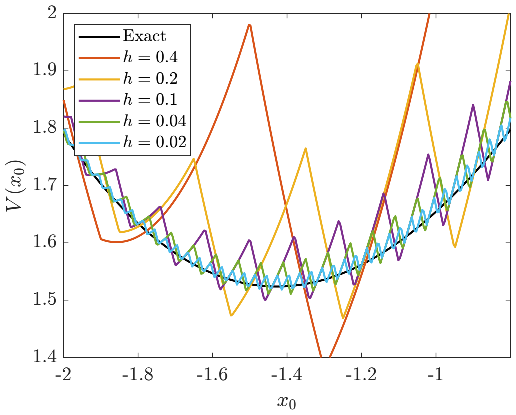
Discrete-time OCP

$$\min_{\mathbf{x},\mathbf{z}} \quad \sum_{n=0}^{N-1} \ell_n(x_n) + (x_N - 5/3)^2$$

$$\text{s.t.} \quad x_{n+1} = \phi_f(x_n, z_n)$$

$$0 = \phi_{\text{int}}(x_n, z_n), \;\; n = 0, \dots N-1$$

- discretize the DCS with fixed step size IRK methods
- e.g., midpoint rule, Gauss-Legendre IRK with $n_{\text{s}} = 1$, accuracy $O(h^2)$
- decreasing the step size might worsen the situation



Many artificial local minima and wrong derivatives.

What happens if we use smoothed models in direct optimal control?

Direct optimal control with a standard IRK discretization - smoothing

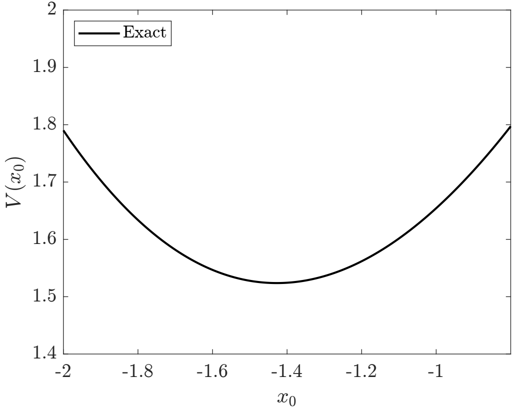
Tutorial example inspired by [Stewart & Anitescu, 2010]

Continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^0([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2)-5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \operatorname{sign}(x(t)), \quad t \in [0,2]$$

- midpoint rule, with $h = 0.05;\ N = 40$



Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired by [Stewart & Anitescu, 2010]

Smoothed continuous-time OCP

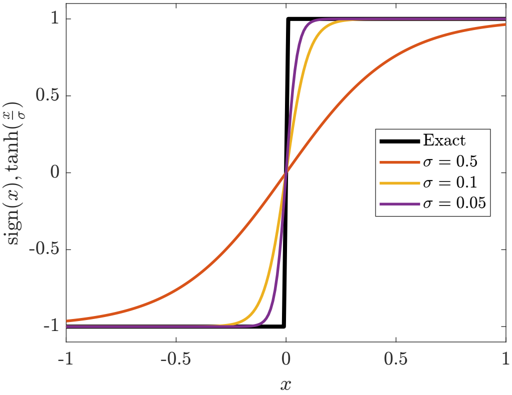
$$\min_{x(\cdot)\in\mathcal{C}^{\infty}([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05$; $N = 40$
- solve smoothed OCP for different σ



Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired by [Stewart & Anitescu, 2010]

Smoothed continuous-time OCP

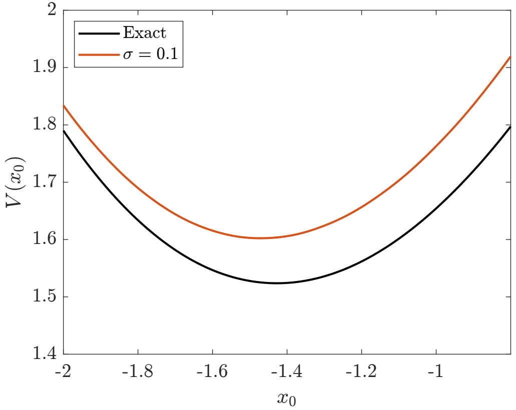
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0, 2]$$

Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05;\ N = 40$
- solve smoothed OCP with $\sigma = 0.1$



Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired by [Stewart & Anitescu, 2010]

Smoothed continuous-time OCP

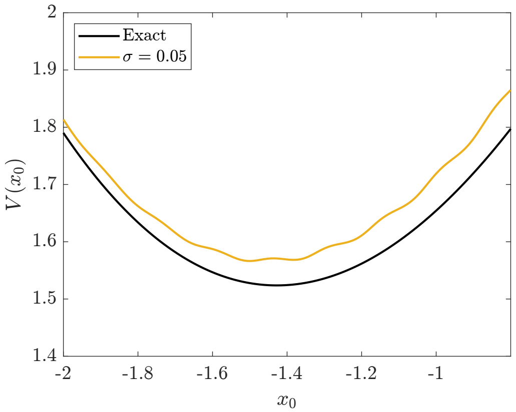
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05;\ N = 40$
- solve smoothed OCP with $\sigma = 0.05$



Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired by [Stewart & Anitescu, 2010]

Smoothed continuous-time OCP

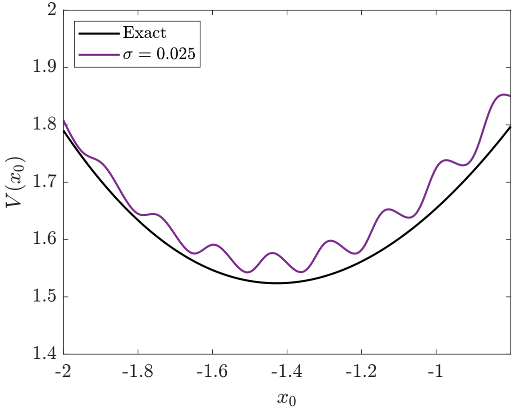
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0 \in \mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05;\ N = 40$
- solve smoothed OCP with $\sigma = 0.025$



Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired by [Stewart & Anitescu, 2010]

Smoothed continuous-time OCP

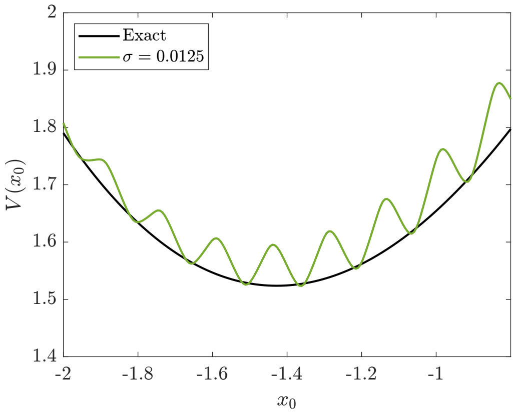
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2\mathrm{d}t + (x(2)-5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05$; $N = 40$
- solve smoothed OCP with $\sigma = 0.0125$



Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired by [Stewart & Anitescu, 2010]

Smoothed continuous-time OCP

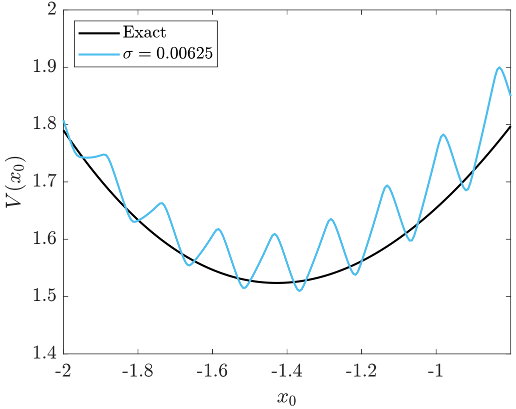
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05;\ N = 40$
- solve smoothed OCP with $\sigma = 0.00625$



Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired by [Stewart & Anitescu, 2010]

Smoothed continuous-time OCP

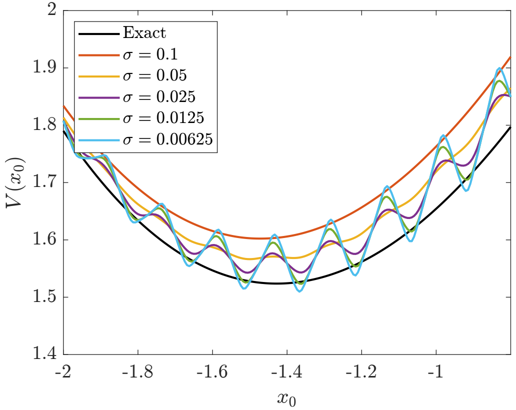
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

- midpoint rule, with $h = 0.05;\ N = 40$



If $h \gg \sigma$, then the smooth approximation behaves **the same as the nonsmooth problem!**

Direct optimal control with a standard IRK discretization - smoothing

Tutorial example inspired by [Stewart & Anitescu, 2010]

Smoothed continuous-time OCP

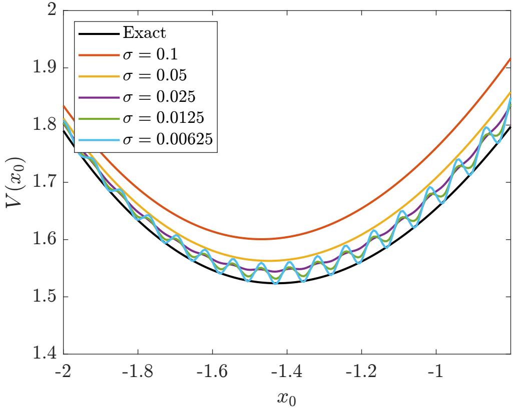
$$\min_{x(\cdot)\in\mathcal{C}^\infty([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2)-5/3)^2$$

$$\text{s.t.} \quad \dot{x}(t) = 2 - \tanh\left(\frac{x(t)}{\sigma}\right), \quad t \in [0,2]$$

Equivalent reduced problem

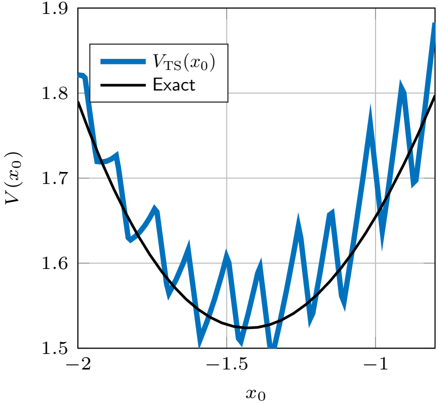
$$\min_{x_0\in\mathbb{R}} V_\sigma(x_0)$$

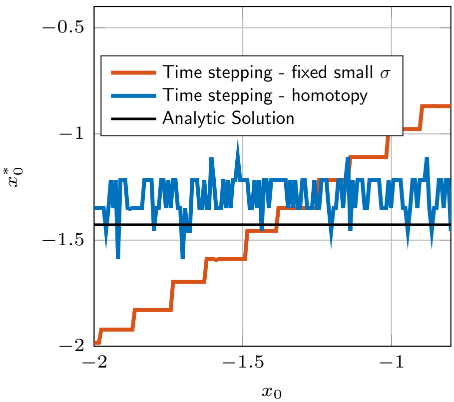
- midpoint rule, with $h = 0.025$; $N = 80$



If $h \gg \sigma$, then the smooth approximation behaves **the same as the nonsmooth problem!**

Direct optimal control with a standard time-stepping IRK discretization





- spurious local minima, optimizer gets trapped close to initialization
- sensitivity only correct if step sizes are smaller than smoothing parameter [Stewart & Anitescu, 2010]: homotopy improves convergence
- even for the best local minimizer, only $O(h)$ accuracy can be expected

Outline

Main ideas of FESD

Based on [Baumrucker & Biegler, 2009], [N. et. al, 2022, 2022a, 2023]

FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)

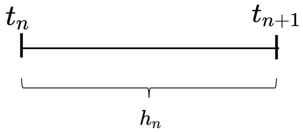
$$\dot{x} \in F_{\mathrm{F}}(x,u) \quad \iff \quad \begin{aligned} \dot{x} &= F(x,u)\theta \\ 0 &= G_{\mathrm{DCS}}(x,z,\theta) \end{aligned}$$

Main ideas of FESD

Based on [Baumrucker & Biegler, 2009], [N. et. al, 2022, 2022a, 2023]

FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements



Main ideas of FESD

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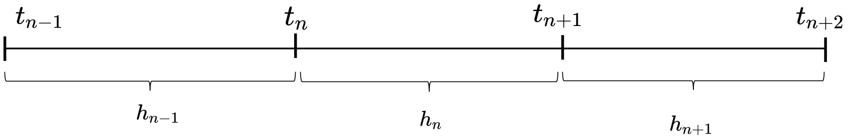
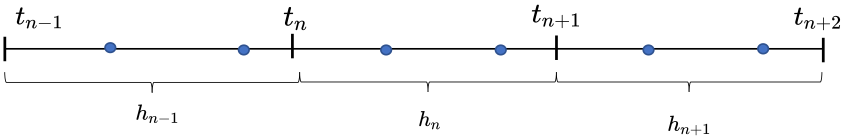


Main ideas of FESD

Based on [Baumrucker & Biegler, 2009], [N. et. al, 2022, 2022a, 2023]

FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2 and 3)

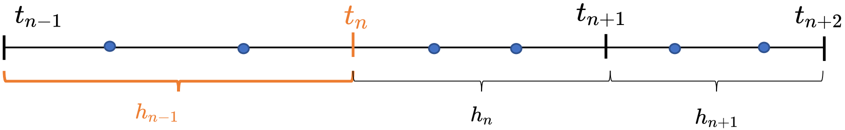


Main ideas of FESD

Based on [Baumrucker & Biegler, 2009], [N. et. al, 2022, 2022a, 2023]

FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2 and 3)
4. Let step sizes h_n be degrees of freedom (under-determined system)



Main ideas of FESD

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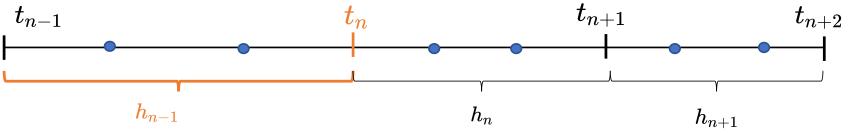
FESD overview

1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2 and 3)
4. Let step sizes h_n be degrees of freedom
5. Cross complementarity conditions - adapt h_n for switch detection

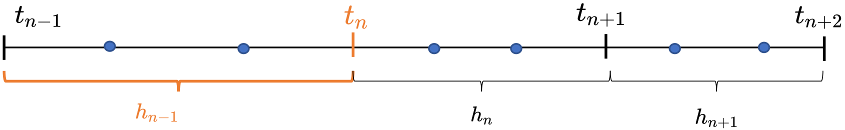


Main ideas of FESD

Based on [Baumrucker & Biegler, 2009], [N. et. al, 2022, 2022a, 2023]

FESD overview

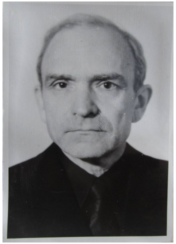
1. Transform Filippov DI into equivalent DCS - Stewart or Heaviside step (Lecture 5)
2. Consider at least two integration intervals = finite elements
3. Use general implicit Runge-Kutta methods (Lectures 2 and 3)
4. Let step sizes h_n be degrees of freedom
5. Cross complementarity conditions - adapt h_n for switch detection
6. Step equilibration - remove degrees of freedom if no switch



Switched ODE not well-defined on region boundaries ∂R_i. Idea by A.F. Filippov (1923-2006): replace ODE by differential inclusion, using convex combination of neighboring vector fields.

Filippov Differential Inclusion

$$\dot{x} \in F_{\mathrm{F}}(x,u) := \Big\{ \sum_{i=1}^{n_f} f_i(x,u)\,\theta_i \;\Big|\; \sum_{i=1}^{n_f} \theta_i = 1, \quad \theta_i \geq 0, \quad i = 1,\dots n_f, \quad \theta_i = 0, \quad \text{if } x \notin \overline{R_i} \Big\}$$



Aleksei F. Filippov (1923-2006)
image source: wikipedia

Recap on Filippov Convexification

Switched ODE not well-defined on region boundaries ∂R_i. Idea by A.F. Filippov (1923-2006): replace ODE by differential inclusion, using convex combination of neighboring vector fields.

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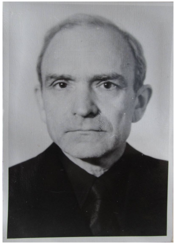

Aleksei F. Filippov
(1923-2006)
image source: wikipedia

- for interior points $x \in R_i$ nothing changes: $F_{\mathrm{F}}(x,u) = \{f_i(x,u)\}$
- Provides meaningful generalization on region boundaries.
  E.g. on $\overline{R_1} \cap \overline{R_2}$ both θ_1 and θ_2 can be nonzero

From Filippov to dynamic complementarity systems

Using the KKT conditions of the parametric LP

LP representation

$$\dot{x} = F(x,u)\,\theta$$

$$\text{with} \quad \theta \in \underset{\tilde{\theta} \in \mathbb{R}^{n_f}}{\operatorname{argmin}} \quad g(x)^\top \tilde{\theta}$$
$$\text{s.t.} \quad 0 \leq \tilde{\theta}$$
$$1 = e^\top \tilde{\theta}$$

where

$$F(x,u) := [f_1(x,u), \dots, f_{n_f}(x,u)] \in \mathbb{R}^{n_x \times n_f}$$
$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$
$$e := [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}$$

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$$e := [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}$$

Express equivalently by optimality conditions:

Dynamic Complementarity System (DCS)

$$\dot{x} = F(x,u)\,\theta \tag{1a}$$
$$0 = g(x) - \lambda - e\mu \tag{1b}$$
$$0 \leq \theta \perp \lambda \geq 0 \tag{1c}$$
$$1 = e^\top \theta \tag{1d}$$

Compact notation

$$\dot{x} = F(x,u)\,\theta$$
$$0 = G_{\mathrm{LP}}(x,\theta,\lambda,\mu),$$

- $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^{n_f}$ are Lagrange multipliers
- (1c) $\Leftrightarrow \min\{\theta,\lambda\} = 0 \in \mathbb{R}^{n_f}$
- Together, (1b), (1c), (1d) determine the $(2n_f + 1)$ variables (θ,λ,μ) uniquely

Conventional discretization by Implicit Runge Kutta (IRK) method

Continuous time DCS

$$
\begin{aligned}
x(0) &= \bar{x}_0, \\
\dot{x}(t) &= v(t) \\
v(t) &= F(x(t), u(t))\, \theta(t) \\
0 &= g(x(t)) - \lambda(t) - e\mu(t) \\
0 &\leq \theta(t) \perp \lambda(t) \geq 0 \\
1 &= e^\top \theta(t), \quad t \in [0, T]
\end{aligned}
$$

Conventional discretization by Implicit Runge Kutta (IRK) method

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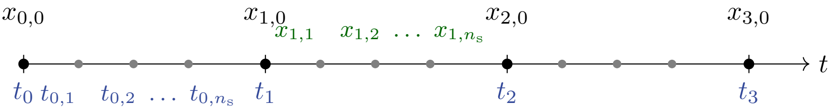
1 &= e^\top \theta(t), \quad t \in [0, T]
\end{aligned}
$$

Discrete time IRK-DCS equation

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \quad x_{n+1,0} = x_{n,0} + h \textstyle\sum_{i=1}^{n_\mathrm{s}} b_i v_{n,i} \\
x_{n,i} &= x_{n,0} + h \textstyle\sum_{j=1}^{n_\mathrm{s}} a_{i,j} v_{n,j} \\
v_{n,i} &= F(x_{n,i}, u_{n,i})\,\theta_{n,i} \\
0 &= g(x_{n,i}) - \lambda_{n,i} - e\mu_{n,i} \\
0 &\leq \theta_{n,i} \perp \lambda_{n,i} \geq 0 \\
1 &= e^\top \theta_{n,i}, \quad i = 1, \ldots, n_\mathrm{s}, \quad n = 0, \ldots, N-1
\end{aligned}
$$

Notation: $x_{n,i} \in \mathbb{R}^{n_x}, \theta_{n,i} \in \mathbb{R}^m$ etc. RK stage values with:

- $n \in \{0, 1, \ldots, N\}$ - index of integration step; step length $h := T/N$
- $i, j \in \{0, 1, \ldots, n_\mathrm{s}\}$ - index of intermediate IRK stage / collocation point
- $a_{i,j}$ and b_i - Butcher tableau entries of Implicit Runge Kutta method



Conventional time stepping - illustrative example

Regard example with $x \in \mathbb{R}^2$ and constants $a, k, c > 0$:

$$\dot{x} = \begin{cases} f_1(x), & x_1 > 0, \\ f_2(x), & x_1 < 0. \end{cases}$$

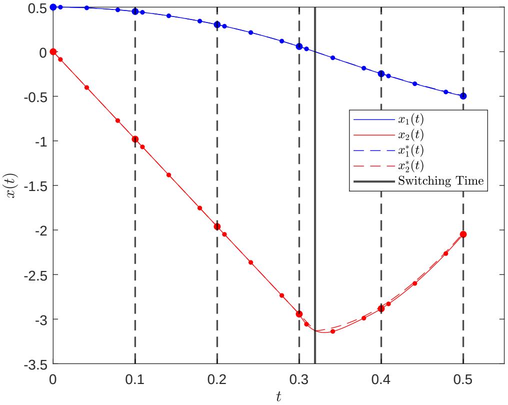
$$f_1(x) = \begin{pmatrix} x_2 \\ -a \end{pmatrix}, \; f_2(x) = \begin{pmatrix} x_2 \\ -kx_1 - cx_2 \end{pmatrix}$$

$$g_1(x) = -x_1,$$
$$g_2(x) = x_1,$$

$$\bar{x}_0 = [0.5\,,\,0]^\top.$$

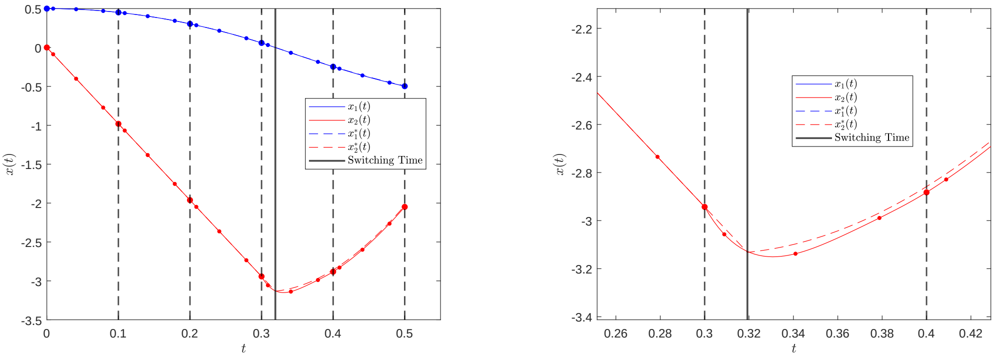
Solve with IRK Radau IIA method of **order 7**

$s = 4, \; N = 5, \; T = 0.5, \; h = 0.1$



Conventional time stepping - illustrative example

Zoom in



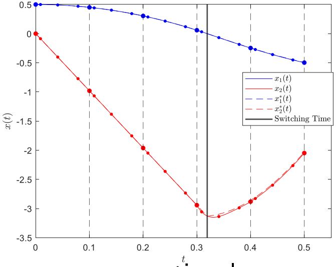
High integration accuracy of 7th order IRK method is lost in fourth time step.
Reason: we try to approximate a nonsmooth function by a (smooth) polynomial.

Question: could we ensure that switches happen only at element boundaries?
$\rightarrow$ Finite Elements with Switch Detection (FESD)

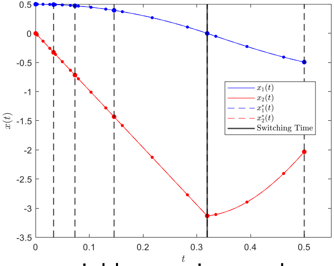
Finite Elements with Switch Detection (FESD)

FESD is a novel DCS discretization method based on three ideas:

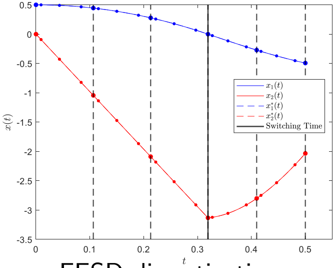
- make stepsizes h_n free, ensure $\sum_{n=0}^{N-1} h_n = T$ [cf. Baumrucker & Biegler, 2009]
- allow switches only at element boundaries, enforce via *cross-complementarities*
- remove spurious degrees of freedom via *step equilibration*



conventional discretization



variable stepsizes and cross-complementarities



FESD discretization with step equilibration

Conventional DCS and FESD discretization without step equilibration

Time-stepping discretization

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \quad h = T/N \\
x_{n+1,0} &= x_{n,0} + h \sum_{i=1}^{n_\mathrm{s}} b_i v_{n,i} \\
x_{n,i} &= x_{n,0} + h \sum_{j=1}^{n_\mathrm{s}} a_{i,j} v_{n,j} \\
v_{n,i} &= F(x_{n,i}, u_{n,i})\, \theta_{n,i} \\
0 &= g(x_{n,i}) - \lambda_{n,i} - e\mu_{n,i} \\
0 &\leq \theta_{n,i} \perp \lambda_{n,i} \geq 0 \\
1 &= e^\top \theta_{n,i}
\end{aligned}
$$

for $i = 1, \dots, n_\mathrm{s}$

and $n = 0, \dots, N-1$

FESD discretization without step equilibration

$$
\begin{aligned}
x_{0,0} &= \bar{x}_0, \ \sum_{n=0}^{N-1} h_n = T \\
x_{n+1,0} &= x_{n,0} + h_n \sum_{i=1}^{n_\mathrm{s}} b_i v_{n,i} \\
x_{n,i} &= x_{n,0} + h_n \sum_{j=1}^{n_\mathrm{s}} a_{i,j} v_{n,j} \\
v_{n,i} &= F(x_{n,i}, u_{n,i})\, \theta_{n,i} \\
0 &= g(x_{n,i'}) - \lambda_{n,i'} - e\mu_{n,i'} \\
0 &\leq \theta_{n,i} \perp \lambda_{n,i'} \geq 0 \quad \text{(cross-complementarities)} \\
1 &= e^\top \theta_{n,i}
\end{aligned}
$$

for $i = 1, \dots, n_\mathrm{s}$ and $n = 0, \dots, N-1$

and $i' = 0, 1, \dots, n_\mathrm{s}$

- N extra variables $(h_0, \dots, h_{N-1})$ restricted by one extra equality
- Additional multipliers $\lambda_{n,0}, \mu_{n,0}$ are uniquely determined

Conventional DCS and FESD discretization with step equilibration

Time-stepping discretization

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0 &\leq \theta_{n,i} \perp \lambda_{n,i'} \geq 0 \quad \text{(cross-complementarities)} \\
1 &= e^\top \theta_{n,i} \\
0 &= \nu(\theta_{n'}, \theta_{n'+1}, \lambda_{n'}, \lambda_{n'+1}) \cdot (h_{n'} - h_{n'+1})
\end{aligned}
$$

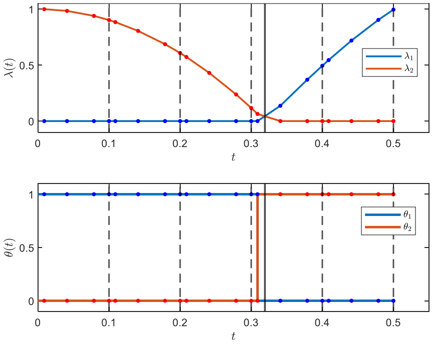
for $i = 1, \dots, n_s$ and $n = 0, \dots, N-1$

and $i' = 0, 1, \dots, n_s$ and $n' = 0, \dots, N-2$

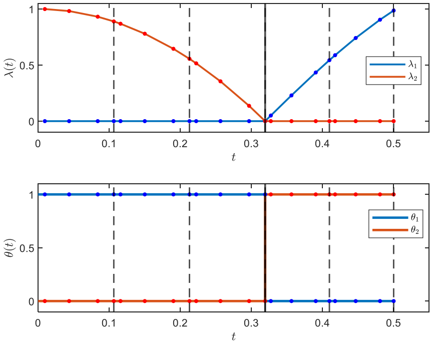
- N extra variables $(h_0, \dots, h_{N-1})$ restricted by one extra equality
- Additional multipliers $\lambda_{n,0}, \mu_{n,0}$ are uniquely determined
- Indicator function $\nu(\theta_{n'}, \theta_{n'+1}, \lambda_{k'}, \lambda_{k'+1})$ only zero if a switch occurs

Multipliers in conventional and FESD discretization

Time stepping discretization:



FESD discretization:



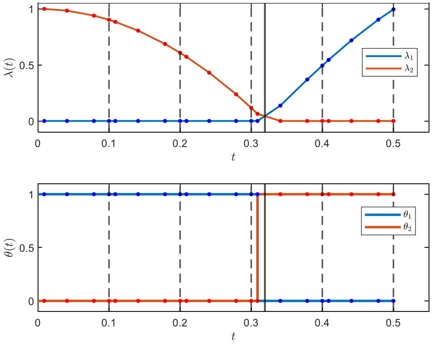
Lemma (Cross complementarity)

If any $\theta_{n,j,i}$ with $j = 1, \ldots, n_\mathrm{s}$ is positive, then all $\lambda_{n,j',i}$ with $j' = 0, \ldots, n_\mathrm{s}$ must be zero. Conversely, if any $\lambda_{n,j',i}$ is positive, then all $\theta_{n,j,i}$ are zero.

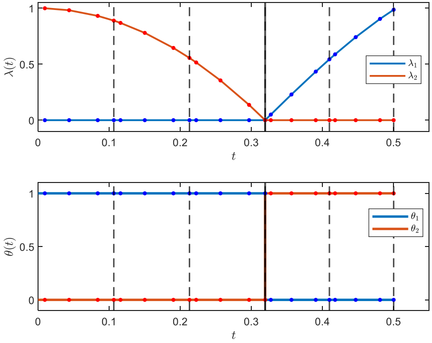
Notation $\lambda_{n,j,i}$ - n - finite element, j - RK stage, i - component of vector

Multipliers in conventional and FESD discretization

Time stepping discretization:



FESD discretization:



FESD's cross-complementarities exploit the fact that the multiplier $\lambda_i(t)$ is continuous in time.
On boundary, $\lambda_i(t_n)$ must be zero if $\theta_i(t) > 0$ for any $t \in [t_{n-1}, t_{n+1}]$ on the adjacent intervals.
This implicitly imposes the constraint $g_i(x_n) - g_j(x_n) = 0$.
$\implies$ h_n adapts for exact switch detection

Step equilibration

- if no switches happen, cross complementarity implied by standard complementarity
- spurious degrees of freedom in h_n: more degrees of freedom than equations

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- exploit complementarity of θ_n, λ_n to encode switching logic
- define (very complicated) switch indicator function ν (cf. PhD Nurkanović):

$$\nu(\theta_n, \theta_{n+1}, \lambda_n, \lambda_{n+1}) := \begin{cases} \text{positive}, & \text{if no switch at } t_{n+1} \\ 0, & \text{if switch at } t_{n+1} \end{cases}$$

Step equilibration

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- step equilibration:

$$0 = \nu(\theta_n, \theta_{n+1}, \lambda_n, \lambda_{n+1}) \cdot (h_n - h_{n+1}), \quad n = 0, \dots, N-2$$

Step equilibration

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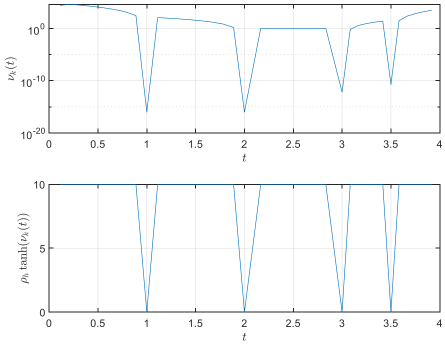
- step equilibration:

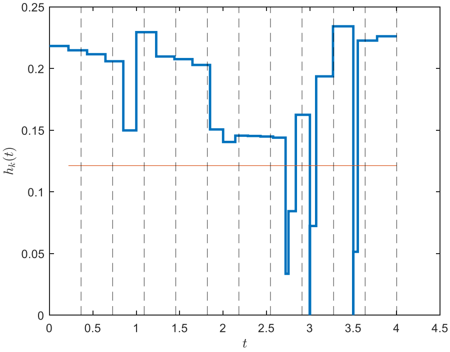
$$0 = \nu(\theta_n, \theta_{n+1}, \lambda_n, \lambda_{n+1}) \cdot (h_n - h_{n+1}), \qquad n = 0, \ldots, N-2$$

- Summary:
  - If switch happens, then h_n is determined by cross complementarity.
  - If no switch happens, then h_n is determined by step equilibration.

Indicator function over time:



Step size over time:

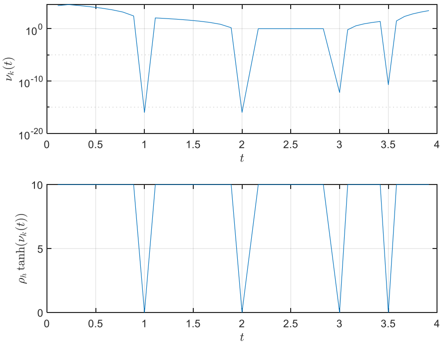


Optimizer varies step size randomly, potentially playing with integration errors.

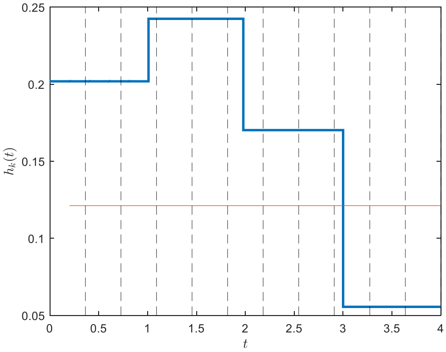
Numerical solution with equilibration

Example with four switches

Indicator function over time:



Step size over time:



Equidistant grid on each "switching stage". Jumps exactly at switching times.

Summary of theoretical results

1. An FESD problem needs to solve a nonlinear complementarity problem (NCP) to advance the integration. The solutions of these NCP are locally unique.
   - For a given point determine which constraint cross comp. and step eq. are binding, and which implicitly satisfied.
   - Obtain square system and apply implicit function theorem.

2. Convergence of the FESD method to a Filippov solution of the underlying system with accuracy $O(h^p)$ is proven. Here, p is the order of the underlying smooth IRK method.
   - Solution approximation and true solution predict same active set.
   - Switching time accuracy also $O(h^p)$.

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   - Solution approximation and true solution predict same active set.
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3. Convergence of numerical sensitivities to the true value with $O(h^p)$ is given.
   - Cross. comp. implicitly enforce switching condition and lead to correct sensitivities.
   - The Stewart & Anitescu problem is solved.

Integration order plots for FESD and IRK time stepping

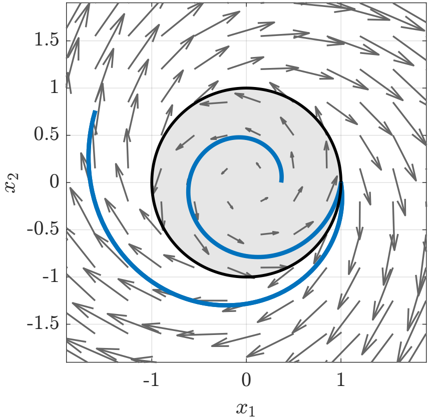
Revisit example from Lecture 4

Tutorial example

$$\dot{x} = \begin{cases} A_1 x, & \|x\|_2^2 < 1, \\ A_2 x, & \|x\|_2^2 > 1, \end{cases}$$

with $A_1 = \begin{bmatrix} 1 & 2\pi \\ -2\pi & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -2\pi \\ 2\pi & 1 \end{bmatrix}$

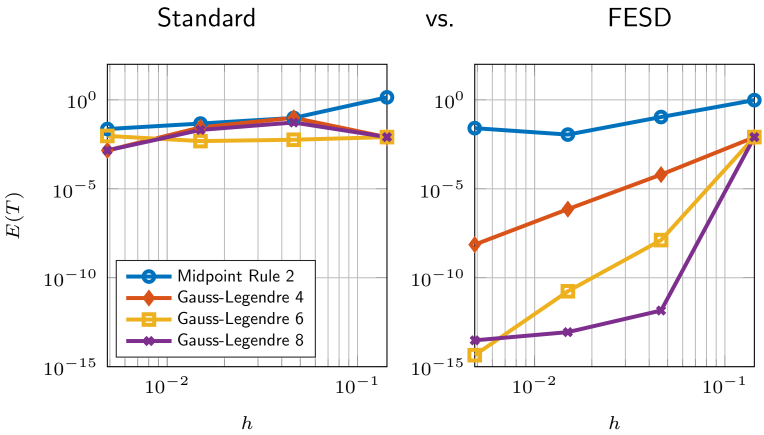
$x(0) = (e^{-1}, 0)$ for $t \in [0, \frac{\pi}{2}]$.



Compute global integration error $E(T)$ using different strategies.
Compute solution approximation:

1. With fixed step size IRK methods (time-stepping).
2. FESD with same underlying IRK methods.

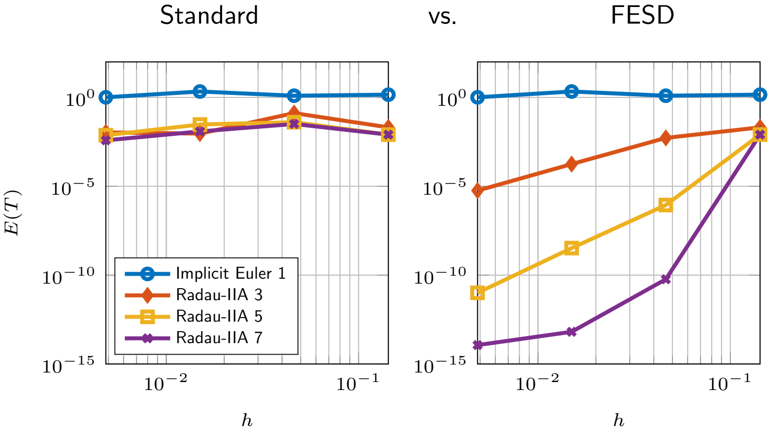
FESD recovers high integration order for switched systems



Integration error $E(T)$ at time $T = \pi/2$ vs. step-size h, for different IRK methods.

FESD discretization recovers high integration order

FESD recovers high integration order for switched systems



Integration error $E(T)$ at time $T = \pi/2$ vs. step-size h, for different IRK methods.

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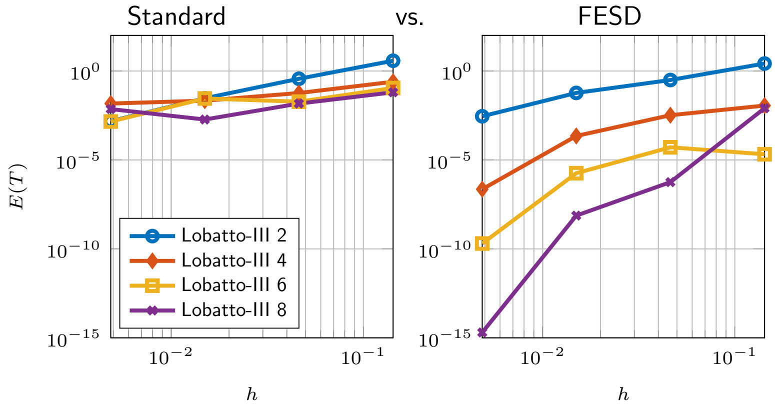


Integration error $E(T)$ at time $T = \pi/2$ vs. step-size h, for different IRK methods.
FESD discretization recovers high integration order

Outline

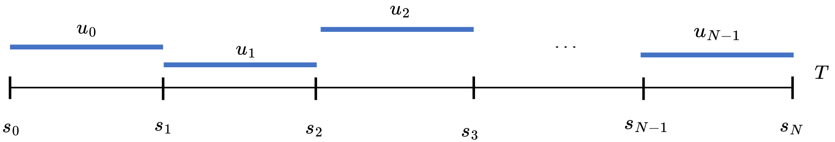
Discretizing optimal control problems with FESD

Discretized optimal control problem

$$\min_{s,z,u} \sum_{k=0}^{N-1} \Phi_L(s_k, z_k, u_k) + E(s_N)$$
$$\text{s.t.} \quad s_0 = \bar{x}_0$$
$$s_{k+1} = \Phi_f(s_k, z_k, u_k)$$
$$0 = \Phi_{\text{int}}(s_k, z_k, u_k)$$
$$0 \geq h(s_k, u_k), \ k = 0, \dots, N-1$$
$$0 \geq r(s_N)$$

- States at control grid points $s = (s_0, \dots, s_N)$
- Piecewise controls $u = (u_0, \dots, u_{N-1})$
- FESD with N_{FE} finite elements applied on every control interval

Control horizon $[0, T]$ with N control stages



Discretizing optimal control problems with FESD

Discretized optimal control problem

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- Φ_{int} summarizes all internal FESD equations: RK, cross complementarity, step equilibration,...

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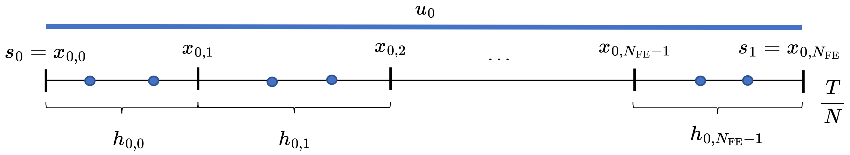

Discretizing optimal control problems with FESD

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- States at control grid points $s = (s_0, \ldots, s_N)$
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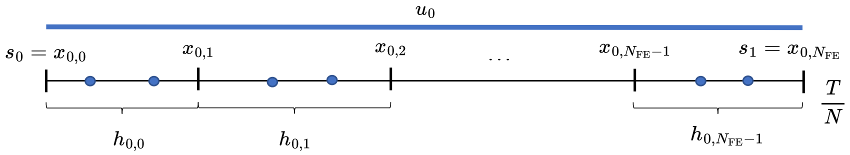
- FESD with N_{FE} finite elements applied on every control interval
- Φ_{int} summarizes all internal FESD equations: RK, cross complementarity, step equilibration,...
- $z = (z_0, \ldots, z_{N-1})$ - all interval variables: internal states, stage values of states and multipliers, step sizes, ...

Control horizon $[0, T]$ with N control stages



FESD-discretized optimal control problems are MPCC

Discretized optimal control problem

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& 0 \geq h(s_k, u_k), \; k = 0, \ldots, N-1 \\
& 0 \geq r(s_N)
\end{aligned}
$$

Collect $w = (s, z, u) \in \mathbb{R}^{n_w}$
Mathematical programs with complementarity constraints (MPCC) are more difficult than standard NLPs

NLP with Complementarity Constraints

$$
\begin{aligned}
\min_{w \in \mathbb{R}^{n_w}} \quad & F(w) \\
\text{s.t.} \quad & 0 = G(w) \\
& 0 \geq H(w) \\
& 0 \leq G_1(w) \perp G_2(w) \geq 0
\end{aligned}
$$

Standard and cross complementarity constraints summarized in

$$
0 \leq G_1(w) \perp G_2(w) \geq 0
$$

Nonlinear Programs (NLP)

Newton-type methods generate a sequence $w_0, w_1, w_2, \ldots$ by linearizing and solving convex subproblems.

Summarized NLP

$$\begin{aligned} \min_{w \in \mathbb{R}^{n_w}} \quad & F(w) \\ \text{s.t.} \quad & 0 = G(w) \\ & 0 \geq H(w) \end{aligned}$$

Still assume smooth convex F, H.
Nonlinear G makes problem nonconvex.

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There is significant nonconvex and nonsmooth structure in the NLP.

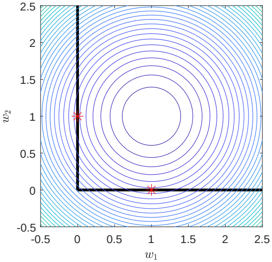
Mathematical Programs with Complementarity Constraints (MPCC)

NLP with additional constraints of complementarity type: $x \perp y \Leftrightarrow x^\top y = 0$

MPCC as an NLP

$$\min_{w \in \mathbb{R}^{n_w}} \; F(w)$$
$$\text{s.t.} \quad 0 = G(w)$$
$$0 \geq H(w)$$
$$0 \leq G_1(w)$$
$$0 \leq G_2(w)$$
$$0 \geq G_1(w)^\top G_2(w)$$

Convex J, H and smooth F. Smooth G_1, G_2.



Due to complementarity constraints, MPCC are nonsmooth and nonconvex.

Toy MPCC example:

$$\min_{w \in \mathbb{R}^2} \; (w_1 - 1)^2 + (w_2 - 1)^2$$
$$\text{s.t.} \quad 0 \leq w_1 \perp w_2 \geq 0$$

Two local minimizers. One local maximizer (without constraint qualification).

MPCCs treated in detail in three lectures by C. Kirches.

Revisiting the OCP example - now with FESD

Tutorial example inspired by [Stewart & Anitescu, 2010]

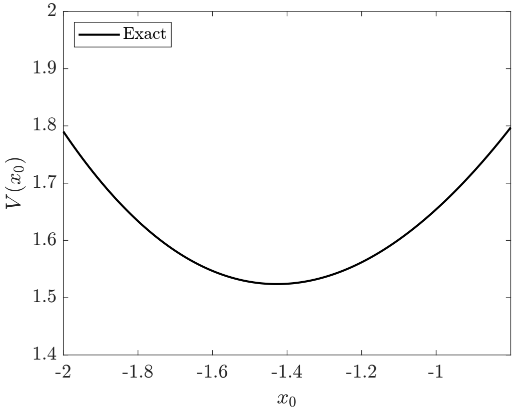
Continuous-time OCP

$$\min_{x(\cdot)\in\mathcal{C}^0([0,2])} \int_0^2 x(t)^2 \mathrm{d}t + (x(2) - 5/3)^2$$
$$\text{s.t.} \quad \dot{x}(t) = 2 - \mathrm{sign}(x(t)), \quad t \in [0,2]$$

Free initial value $x(0)$ is the effective degree of freedom.

Equivalent reduced problem

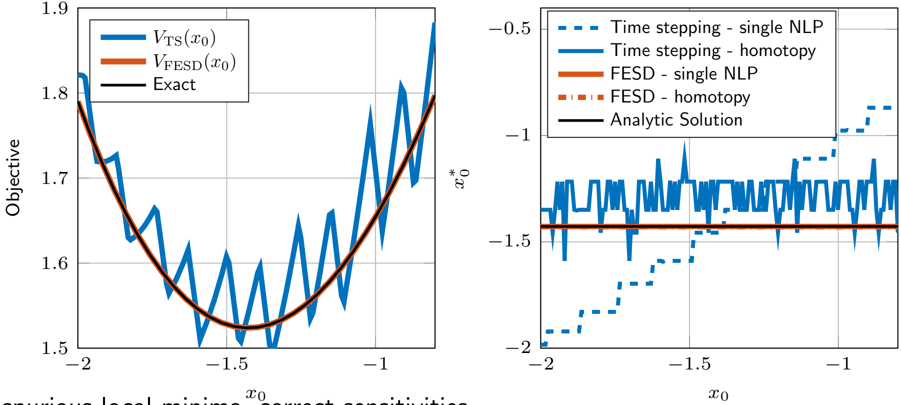
$$\min_{x_0\in\mathbb{R}} V(x_0)$$



- Denote by $V(x_0)$ the nonsmooth objective value for the unique feasible trajectory starting at $x(0) = x_0$.

Revisiting the OCP example - now with FESD

Tutorial example inspired by [Stewart & Anitescu, 2010]



- no spurious local minima, correct sensitivities
- convergence to the "true" local minimum, both with homotopy and without it
- accuracy of order $O(h^p)$, in contrast to standard approach with only $O(h)$

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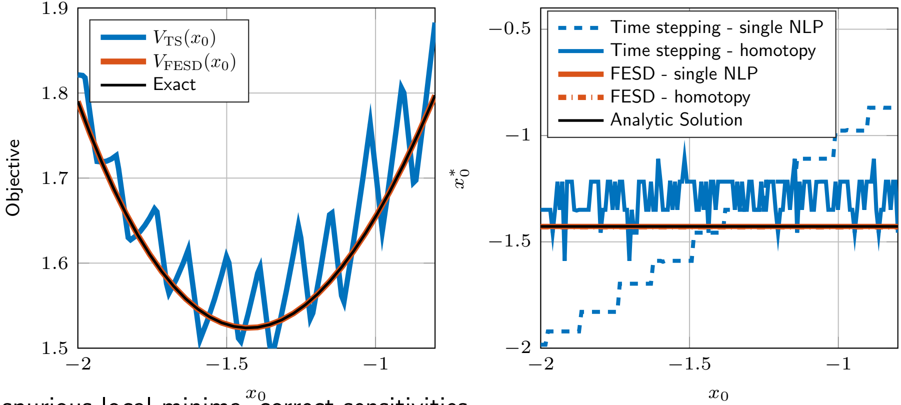


- no spurious local minima, correct sensitivities
- convergence to the "true" local minimum, both with homotopy and without it
- accuracy of order $O(h^p)$, in contrast to standard approach with only $O(h)$
- FESD solves the accuracy and convergence issues

OCP example

Benchmark example with entering/leaving sliding mode

OCP with sliding modes

$$\min_{x(\cdot),u(\cdot)} \int_0^4 u(t)^\top u(t) + v(t)^\top v(t)\,\mathrm{d}t$$

$$\text{s.t.} \quad x(0) = (\frac{2\pi}{3}, \frac{\pi}{3}, 0, 0)$$

$$\dot{x}(t) = \begin{bmatrix} -\text{sign}(c(x(t))) + v(t) \\ u(t) \end{bmatrix}$$

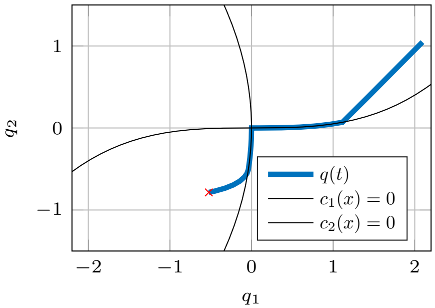
$$-2e \leq v(t) \leq 2e$$

$$-10e \leq u(t) \leq 10e \quad t \in [0, 4],$$

$$q(T) = (-\frac{\pi}{6}, -\frac{\pi}{4})$$

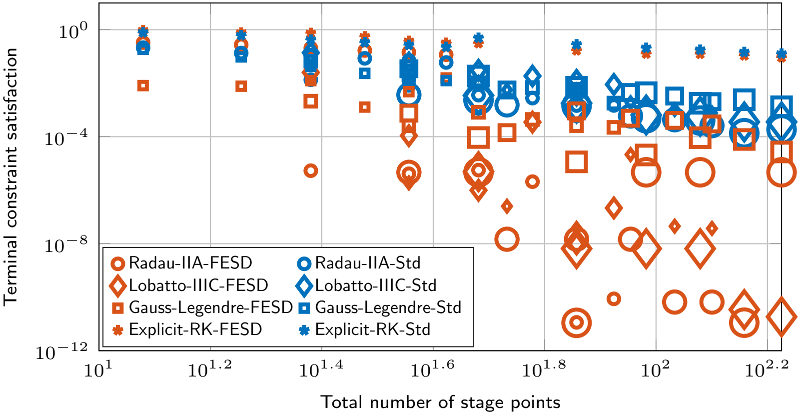
States $q, v \in \mathbb{R}^2$ and control $u \in \mathbb{R}^2$, $x = (q, v)$

Switching functions $c(x) = \begin{bmatrix} q_1 + 0.15 q_2^2 \\ 0.05 q_1^3 + q_2 \end{bmatrix}$



FESD vs standard IRK - number of function evaluations

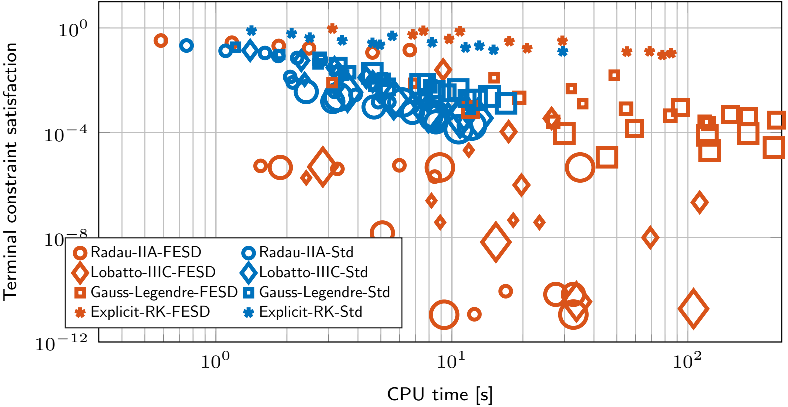
Benchmark on an optimal control problem with nonlinear sliding modes



Terminal constraint satisfaction vs. number of stage points

FESD vs standard IRK - CPU Time

Benchmark on an optimal control problem with nonlinear sliding modes



Terminal constraint satisfaction vs. CPU time

FESD one million times more accurate than Std. for CPU time of ≈ 2 s

Conclusions and summary

- Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for nonsmooth systems of level NSD2
- Following similar lines, FESD can be derived for the Heaviside step reformulation

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Conclusions and summary

- Finite Elements with Switch Detection (FESD) allow highly accurate simulation and optimal control for nonsmooth systems of level NSD2
- Following similar lines, FESD can be derived for the Heaviside step reformulation
- Key ideas: make step sizes degrees of freedom and introduce implicit relations that locate the switches
- Switch detection not only essential for high accuracy, **but also for correct sensitivities** (no spurious solutions)
- FESD solves many of the issues that standard methods have: integration accuracy, convergence of sensitivities
- Main practical difficulty: solving Mathematical Programs with Complementarity Constraints (MPCC)

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- Armin Nurkanović, Jonathan Frey, Anton Pozharskiy, and Moritz Diehl. Finite elements with switch detection for direct optimal control of nonsmooth systems with set-valued step functions. In Conference on Decision on Control, 2023.

Switch detection - example

Suppose that $x(t)$ crosses from R_1 to R_2 and recall that $\mu = \min_j g_j(x)$
Continuous time:

- Before switch: $\theta_1(t) > 0, \lambda_1(t) = 0$, and $\theta_2(t) = 0, \lambda_2 \geq 0$
- After switch: $\theta_1(t) = 0, \lambda_1(t) \geq 0$, and $\theta_2(t) > 0, \lambda_2 = 0$

Switch detection - example

Suppose that $x(t)$ crosses from R_1 to R_2 and recall that $\mu = \min_j g_j(x)$
Discrete time (switch between the n-th and $n+1$-st finite element):

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From Lemma 1 it follows that $\lambda_{n,n_\mathrm{s},1} = \lambda_{n,n_\mathrm{s},2} = 0$

Switch detection conditions

$$g_1(x_{n+1}) = \lambda_{n,n_\mathrm{s},1} - \mu_{n,n_\mathrm{s}}$$

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From Lemma 1 it follows that $\lambda_{n,n_\mathrm{s},1} = \lambda_{n,n_\mathrm{s},2} = 0$

Switch detection conditions

$$0 = g_1(x_{n+1}) - g_2(x_{n+1}) = \psi_{12}(x_{n+1})$$

Implies constraint such that h_n must adapt for exact switch detection!