Lecture 5: Modeling with Filippov Systems - Stewart and Step Formulation

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Systems Control and Optimization Laboratory (syscop)
Summer School on Direct Methods for Optimal Control of Nonsmooth Systems
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universität freiburg

Outline of the lecture

Motivating examples - crossing a discontinuity

Consider the ODE

$$\dot{x} = 2 - \mathrm{sign}(x)$$

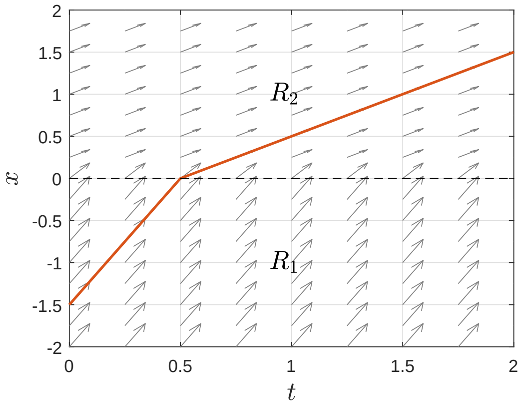
Motivating examples - crossing a discontinuity

Consider the ODE

$$\dot{x} = 2 - \text{sign}(x)$$

More explicitly...

$$\dot{x} = \begin{cases} 3, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$



Motivating examples - sliding mode (simpler)

Consider the ODE

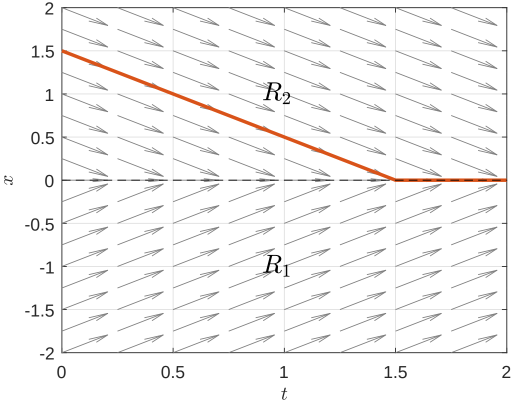
$$\dot{x} = -\mathrm{sign}(x)$$

And let

$$\mathrm{sign}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

Then...

$$\dot{x} = \begin{cases} 1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x > 0 \end{cases}$$



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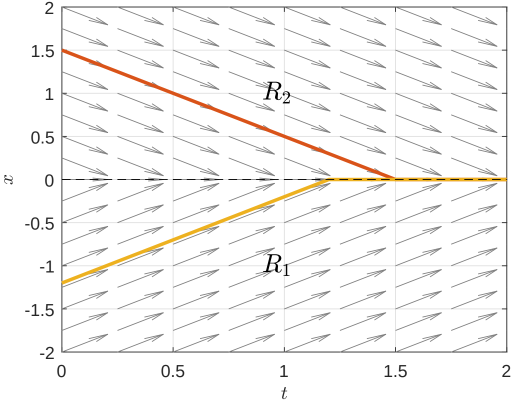


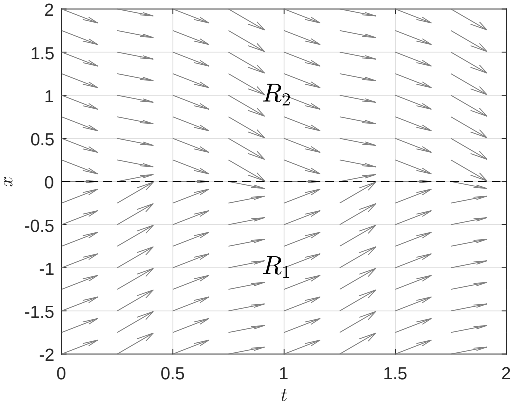
Motivating examples - sliding mode

Consider the ODE

$$\dot{x} = -\mathrm{sign}(x) + 0.5\sin(t)$$

And let

$$\mathrm{sign}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$



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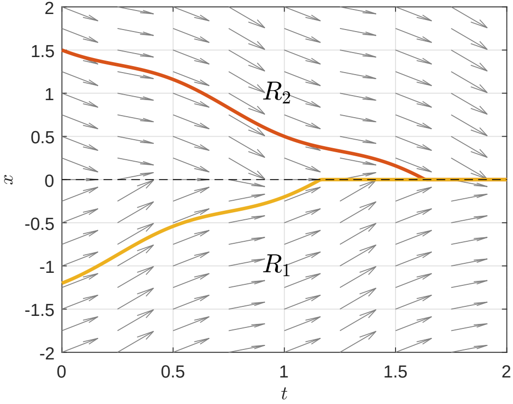

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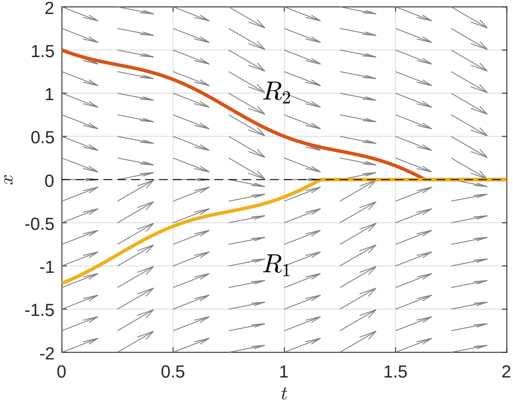
We have for some $t > t^*$ that $x(t) = 0$ and $\dot{x}(t) = 0$



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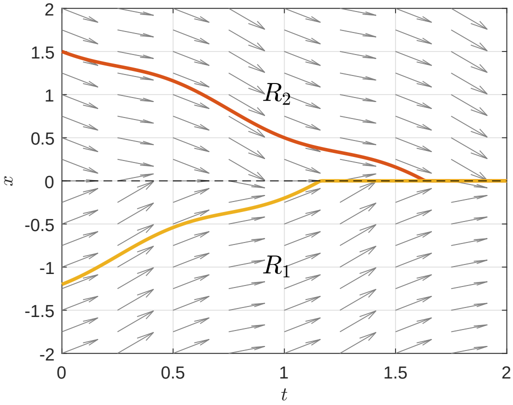
That is $\mathrm{sign}(0) = 0 = 0.5\sin(t)$



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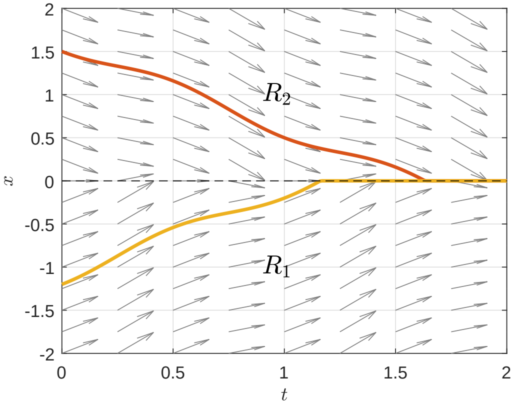
Something went wrong...



Motivating examples - sliding mode - fixed

Consider the ODE

$$\dot{x} \in -\mathrm{sign}(x) + 0.5\sin(t)$$

And let

$$\mathrm{sign}(x) \in \begin{cases} \{-1\}, & \text{if } x < 0 \\ [-1, 1], & \text{if } x = 0 \\ \{1\}, & \text{if } x > 0 \end{cases}$$

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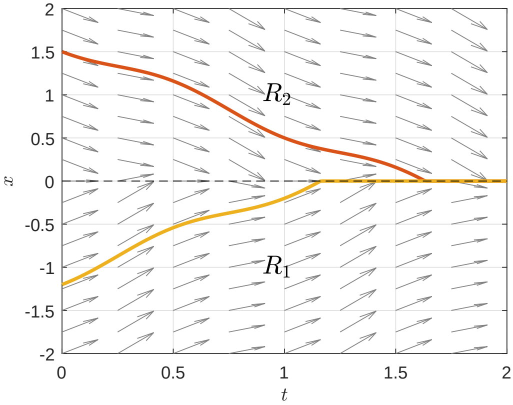

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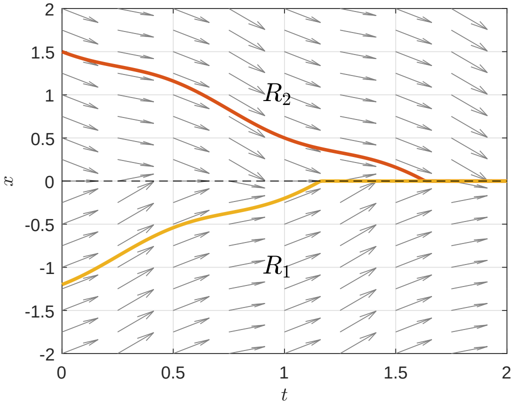

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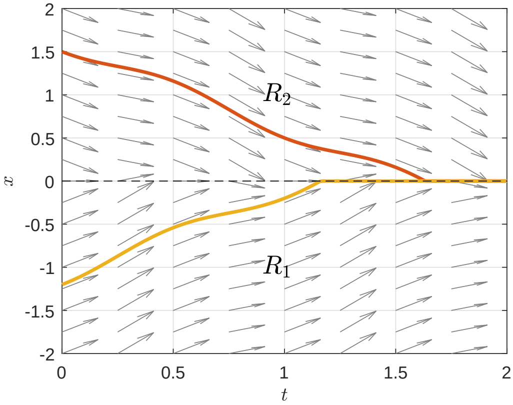
We have for some $t > t^*$ that $x(t) = 0$ and $\dot{x}(t) = 0$

That is $\text{sign}(0) = [-1,1] \ni 0.5\sin(t)$

It works! Thanks to A.F. Filippov



Outline of the lecture

Filippov's convexification for ODEs with discontinuous right-hand side

Filippov differential inclusion

Replace ODE with a discontinuous right-hand side

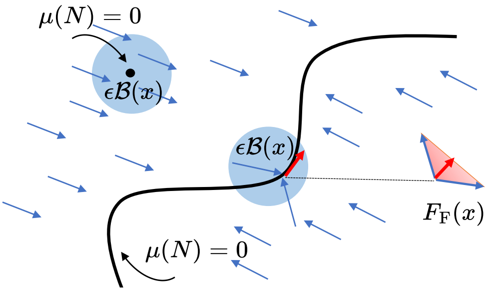
$$\dot{x}(t) = f(x(t))$$

by

$$\dot{x}(t) \in F_{\mathrm{F}}(x(t))$$

where $F_{\mathrm{F}}(x) : \mathbb{R}^{n_x} \to \mathcal{P}(\mathbb{R}^{n_x})$ is defined as:

$$F_{\mathrm{F}}(x) := \bigcap_{\epsilon>0} \bigcap_{\mu(N)=0} \overline{\mathrm{conv}}\, f(x + \epsilon\mathcal{B}(x) \setminus N)$$



- $f(x)$ continuous at x: $F_{\mathrm{F}}(x) = \{f(x)\}$

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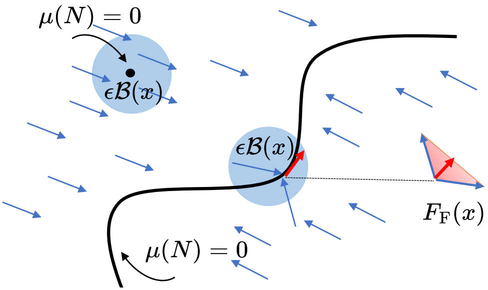


- at discontinuity: convex combination of neighboring vector fields and ignore what is at the discontinuity

Piecewise smooth systems (PSS)

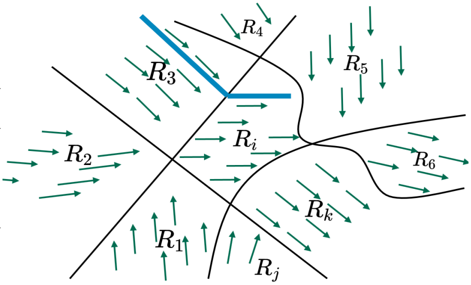
Regard **discontinuous** right-hand side, **piecewise smooth** on disjoint open regions $R_i \subset \mathbb{R}^{n_x}$

Discontinuous ODE (NSD2)

$$\dot{x} = f_i(x,u), \text{ if } x \in R_i, \;\; i = 1,\dots,n_f$$

$$
\begin{aligned}
R_1 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) > 0\} \\
R_2 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) < 0\} \\
&\vdots \\
R_{n_f} &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) < 0, \psi_2(x) < 0, \dots \psi_{n_\psi}(x) < 0\}
\end{aligned}
$$

- zero level sets of $\psi_i(x) = 0$ - region boundaries
- n_ψ smooth scalar switching functions define 2^{n_f} regions

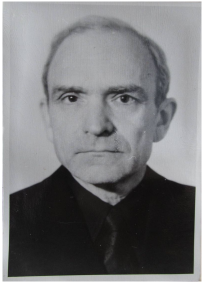


Filippov convexification for piecewise smooth systems

The "structured" discontinuous right-hand side in PSS enables to define convex multipliers θ_i to define the convex set $F_{\mathrm{F}}(x, u)$

Filippov Differential Inclusion

$$\dot{x} \in F_{\mathrm{F}}(x,u) := \Big\{ \sum_{i=1}^{n_f} f_i(x,u)\,\theta_i \;\Big|\; \sum_{i=1}^{n_f} \theta_i = 1, \\ \theta_i \geq 0, \quad i = 1, \dots n_f, \\ \theta_i = 0, \quad \text{if } x \notin \overline{R_i} \Big\}$$



Aleksei F. Filippov
(1923-2006)
image source: wikipedia

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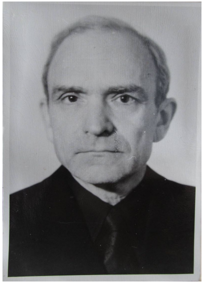

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- for interior points $x \in R_i$ nothing changes: $F_{\mathrm{F}}(x,u) = \{f_i(x,u)\}$
- Provides meaningful generalization on region boundaries
  E.g. on $\overline{R_1} \cap \overline{R_2}$ both θ_1 and θ_2 can be nonzero

Filippov's convexification for sums of discontinuous ODEs

[Stewart1996]

Sum of disc. functions

$$\dot{x} = \sum_{i=1}^{n_{\mathrm{sys}}} f_i(x)$$

Sum of piecewise smooth systems

$$\dot{x} = \sum_{i=1}^{n_{\mathrm{sys}}} f_i(x),\ \ f_i(x) = f_{i,j}(x) \text{ if } x \in R_{i,j}, j = 1, \dots, n_{f,i}$$

Sum of Filippov systems

$$\dot{x} \in \sum_{i=1}^{n_{\mathrm{sys}}} F_{\mathrm{F,i}}(x)$$

Sum of Filippov systems

$$\dot{x} \in \sum_{i=1}^{n_{\mathrm{sys}}} F_{\mathrm{F,i}}(x) = \left\{ \sum_{i=1}^{n_{\mathrm{sys}}} \sum_{j=1}^{n_{f,i}} f_{i,j}(x)\theta_{i,j} \mid \theta_i \geq 0,\ e_i^\top \theta_i = 1 \right\}$$

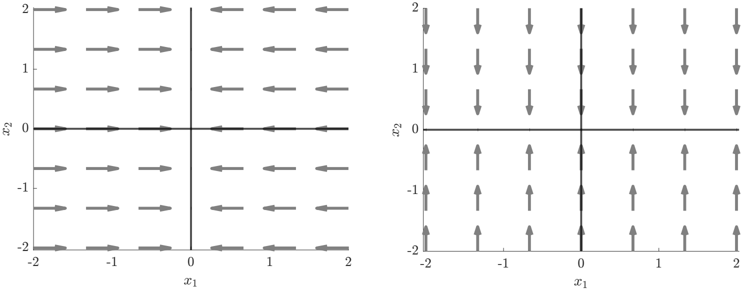
- We regard n_{sys} independent subsystems and their Filippov convexification.
- Often reduces computational complexity.
- In fact, aggregated consideration often impossible.

Illustrative example for sum of Filippov systems

Regard: $x \in \mathbb{R}^2$,

$$\dot{x}_1 = -\mathrm{sign}(x_1),\ \dot{x}_2 = -\mathrm{sign}(x_2)$$

$$\dot{x} = \begin{bmatrix} -\mathrm{sign}(x_1) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathrm{sign}(x_2) \end{bmatrix} = \begin{bmatrix} -\mathrm{sign}(x_1) \\ -\mathrm{sign}(x_2) \end{bmatrix}$$



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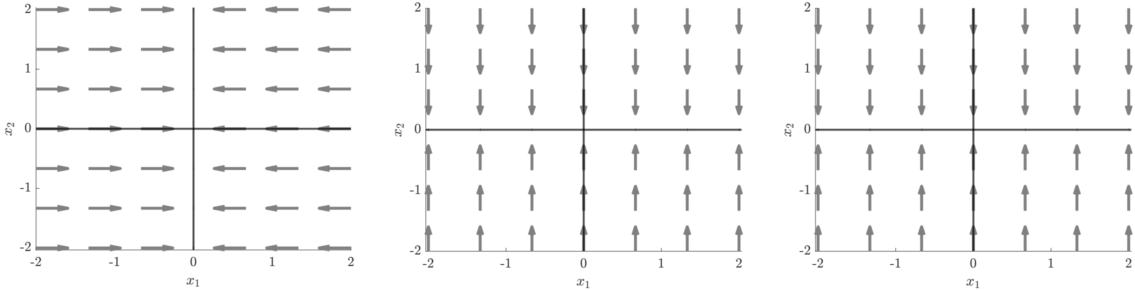

Outline of the lecture

How to compute convex multipliers θ?

One answer in a remarkable paper by David E. Stewart from 1990

Numer. Math. 58, 299–328 (1990)

Numerische Mathematik

A high accuracy method for solving ODEs with discontinuous right-hand side

David Stewart

Department of Mathematics, University of Queensland, St. Lucia, Australia 4067

Received August 1, 1987/January 16, 1990

Summary. Ordinary Differential Equations with discontinuities in the state variables require a differential inclusion formulation to guarantee existence [8]. From this formulation a high accuracy method for solving such initial value problems is developed which can give any order of accuracy and "tracks" the discontinuities. The method uses an "active set" approach, and determines appropriate active sets from solutions to Linear Complementarity Problems. Convergence results are established under some non-degeneracy assumptions. The method has been implemented, and results compare favourably with previously published methods [7, 21].

Stewart's representation

Assume sets R_i given by $\boxed{R_i = \left\{x \in \mathbb{R}^{n_x} \middle| g_i(x) < \min_{j \neq i} g_j(x) \right\}}$

- How to obtain it from $R_i = \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \ldots \psi_{n_\psi}(x) > 0\}$?
- How to find the functions $g_i(x)$?

Stewart's representation

Assume sets R_i given by $\boxed{R_i = \{x \in \mathbb{R}^{n_x} | g_i(x) < \min_{j \neq i} g_j(x)\ \}}$

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- How to find the functions $g_i(x)$?

Definition of regions via switching functions

$$
\begin{aligned}
R_1 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) > 0\} \\
R_2 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) < 0\} \\
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R_{n_f} &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) < 0, \psi_2(x) < 0, \dots \psi_{n_\psi}(x) < 0\}
\end{aligned}
$$

$$\psi(x) := \begin{bmatrix} \psi_1(x) & \psi_2(x) & \dots & \psi_{n_\psi}(x) \end{bmatrix}^\top \in \mathbb{R}^{n_\psi}$$

Sign matrix

$$S = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & -1 \end{bmatrix}$$

Definition via i-th row $S_{i,\bullet}$:

$$R_i = \{x \in \mathbb{R}^{n_x} \mid S_{i,\bullet}\psi(x) > 0\}$$

$$\boxed{g(x) = -S\psi(x)}$$

Examples for finding switching function

- In Stewart's representation sets R_i given by $R_i = \left\{x \in \mathbb{R}^{n_x} \middle| g_i(x) < \min_{j \neq i} g_j(x) \right\}$
- From switching functions $\psi(x) \in \mathbb{R}^{n_\psi}$ obtain *Stewart's indicator functions* $g(x) \in \mathbb{R}^{n_f}$ via $g(x) = -S\psi(x)$

Example 1 - single switching function

$$
\begin{aligned}
R_1 &= \{x \in \mathbb{R}^{n_x} \mid \psi(x) > 0\} \\
R_2 &= \{x \in \mathbb{R}^{n_x} \mid \psi(x) < 0\} \\
S &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
g(x) &= \begin{bmatrix} -\psi(x) \\ \psi(x) \end{bmatrix}
\end{aligned}
$$

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Example 1 - single switching function

$$R_1 = \{x \in \mathbb{R}^{n_x} \mid \psi(x) > 0\}$$
$$R_2 = \{x \in \mathbb{R}^{n_x} \mid \psi(x) < 0\}$$
$$S = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$g(x) = \begin{bmatrix} -\psi(x) \\ \psi(x) \end{bmatrix}$$

Example 2 - two switching function

$$\psi(x) = (\psi_1(x), \psi_2(x))$$
$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$$
$$g(x) = \begin{bmatrix} -\psi_1(x) - \psi_2(x) \\ -\psi_1(x) + \psi_2(x) \\ \psi_1(x) - \psi_2(x) \\ \psi_1(x) + \psi_2(x) \end{bmatrix}$$

How to compute convex multipliers θ?

Assume sets R_i given by [cf. Stewart, 1990]

$$\boxed{R_i = \left\{x \in \mathbb{R}^n \middle| g_i(x) < \min_{j \neq i} g_j(x) \right\}}$$

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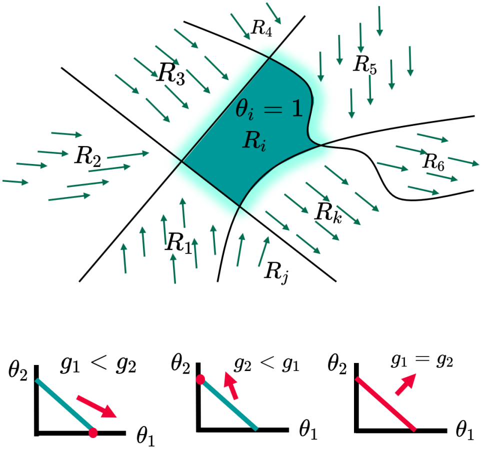
Assume sets R_i given by [cf. Stewart, 1990]

$$\boxed{R_i = \left\{x \in \mathbb{R}^n \middle| g_i(x) < \min_{j\neq i} g_j(x)\ \right\}}$$

Linear program (LP) Representation

$$\dot{x} = \sum_{i=1}^{n_f} f_i(x,u)\,\theta_i \quad \text{with}$$

$$\begin{aligned} \theta \in \arg\min_{\tilde{\theta}\in\mathbb{R}^{n_f}} \quad & \sum_{i=1}^{n_f} g_i(x)\,\tilde{\theta}_i \\ \text{s.t.} \quad & \sum_{i=1}^{n_f} \tilde{\theta}_i = 1 \\ & \tilde{\theta} \geq 0 \end{aligned}$$



Note that the boundary between R_i and R_j is defined by $\{x \in \mathbb{R}^n \mid 0 = g_i(x) - g_j(x)\}$.

From Filippov to dynamic complementarity systems

Using the KKT conditions of the parametric LP

LP representation

$$
\begin{aligned}
\dot{x} &= F(x,u)\,\theta \\
\text{with} \quad \theta \in \underset{\tilde{\theta}\in\mathbb{R}^{n_f}}{\arg\min} \quad & g(x)^\top \tilde{\theta} \\
\text{s.t.} \quad & 0 \leq \tilde{\theta} \\
& 1 = e^\top \tilde{\theta}
\end{aligned}
$$

where

$$
\begin{aligned}
F(x,u) &:= [f_1(x,u), \dots, f_{n_f}(x,u)] \in \mathbb{R}^{n_x \times n_f} \\
g(x) &:= [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f} \\
e &:= [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}
\end{aligned}
$$

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Using the KKT conditions of the parametric LP

LP representation

$$\dot{x} = F(x,u)\,\theta$$

$$\text{with} \quad \theta \in \underset{\tilde{\theta} \in \mathbb{R}^{n_f}}{\operatorname{argmin}} \quad g(x)^\top \tilde{\theta}$$

$$\text{s.t.} \quad 0 \leq \tilde{\theta}$$

$$1 = e^\top \tilde{\theta}$$

where

$$F(x,u) := [f_1(x,u), \dots, f_{n_f}(x,u)] \in \mathbb{R}^{n_x \times n_f}$$

$$g(x) := [g_1(x), \dots, g_{n_f}(x)]^\top \in \mathbb{R}^{n_f}$$

$$e := [1, 1, \dots, 1]^\top \in \mathbb{R}^{n_f}$$

Express equivalently by optimality conditions:

Dynamic Complementarity System (DCS)

$$\dot{x} = F(x,u)\,\theta \tag{1a}$$

$$0 = g(x) - \lambda - e\mu \tag{1b}$$

$$0 \leq \theta \perp \lambda \geq 0 \tag{1c}$$

$$1 = e^\top \theta \tag{1d}$$

Compact notation

$$\dot{x} = F(x,u)\,\theta$$

$$0 = G_{\mathrm{LP}}(x, \theta, \lambda, \mu),$$

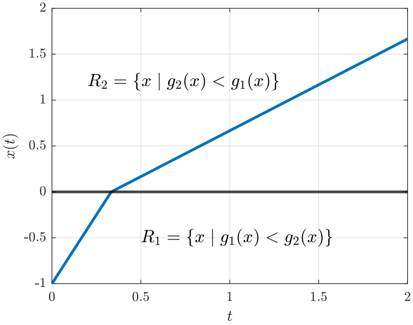
- $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^{n_f}$ are Lagrange multipliers
- (1c) $\Leftrightarrow \min\{\theta, \lambda\} = 0 \in \mathbb{R}^{n_f}$
- Together, (1b), (1c), (1d) determine the $(2n_f + 1)$ variables θ, λ, μ uniquely

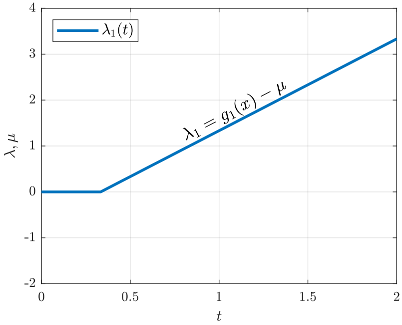
Interpretation of the DCS multipliers

Dynamic complementarity system

$$\dot{x} = F(x,u)\,\theta$$
$$0 = g_i(x) - \lambda_i - \mu, \; i = 1,\dots,n_f$$
$$0 \leq \theta \perp \lambda \geq 0$$
$$1 = e^\top \theta$$

- If $x \in R_i$, then $\theta_i > 0$, $\lambda_i = 0$ (from complementarity)
- $\lambda_i = g_i(x) - \mu$ (from $\nabla_x \mathcal{L}(x,\lambda,\mu) = 0$)





Interpretation of the DCS multipliers

Dynamic complementarity system

$$\dot{x} = F(x,u)\,\theta$$
$$0 = g_i(x) - \lambda_i - \mu,\ i = 1,\dots,n_f$$
$$0 \leq \theta \perp \lambda \geq 0$$
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- If $x \in R_i$, then $\theta_i > 0$, $\lambda_i = 0$ (from complementarity)
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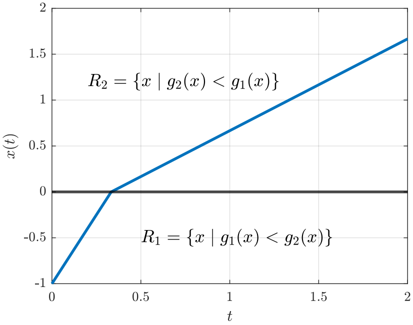

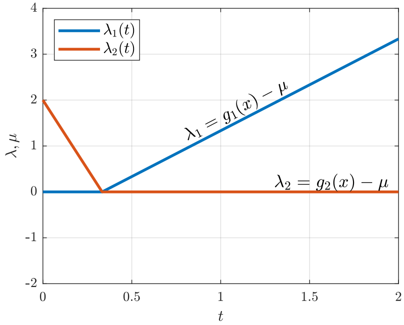

Interpretation of the DCS multipliers

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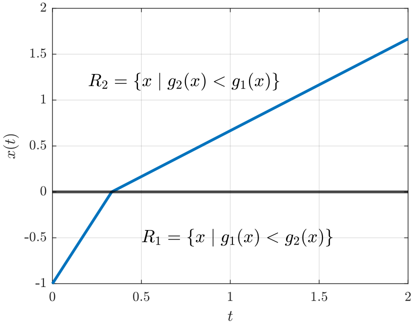
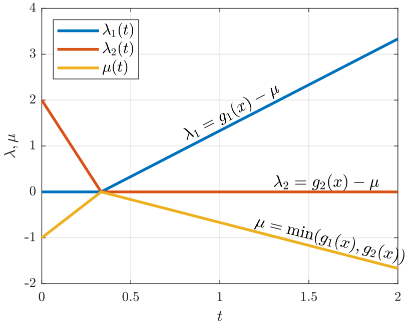
- $\lambda_i = g_i(x) - \mu$ (from $\nabla_x \mathcal{L}(x,\lambda,\mu) = 0$)
- $\mu = \min_j g_j(x)$ (from definition of R_i)
- $\lambda_i = g_i(x) - \min_j g_j(x)$ continuous functions!





Interpretation of the DCS multipliers

Dynamic complementarity system

$$\dot{x} = F(x,u)\,\theta$$
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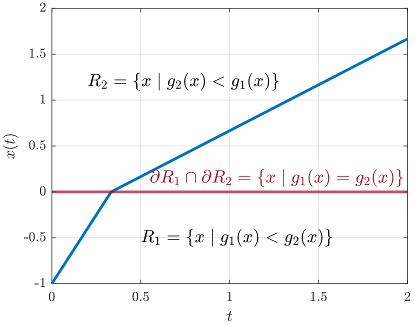
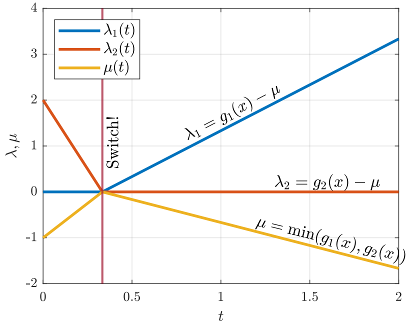
- $\lambda_i = g_i(x) - \mu$ (from $\nabla_x \mathcal{L}(x,\lambda,\mu) = 0$)
- $\mu = \min_j g_j(x)$ (from definition of R_i)
- $\lambda_i = g_i(x) - \min_j g_j(x)$ continuous functions!
- At switch $\lambda_i = \lambda_j = 0 \implies g_i(x) - g_j(x) = 0$ (region boundary)

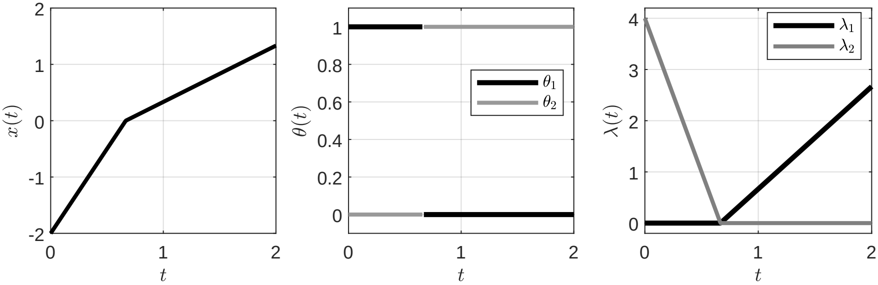




Example: continuity of multipliers in different switching cases

Different switching cases

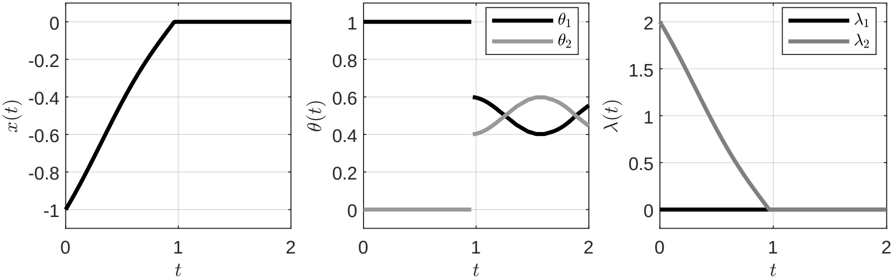
1. Crossing a surface of discontinuity, $\dot{x}(t) \in 2 - \mathrm{sign}(x(t))$,



Example: continuity of multipliers in different switching cases

Different switching cases

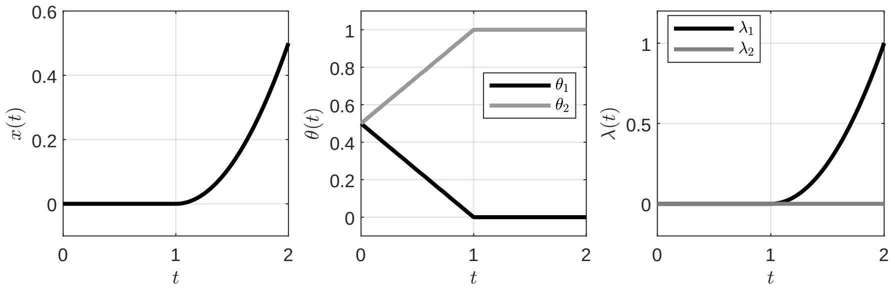
2. Sliding mode, $\dot{x}(t) \in -\mathrm{sign}(x(t)) + 0.2\sin(5t)$,



Example: continuity of multipliers in different switching cases

Different switching cases

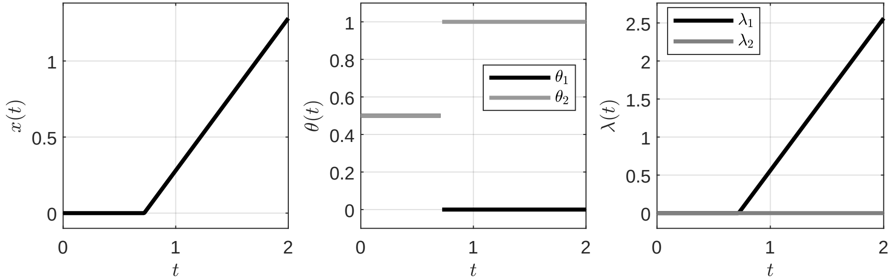
3. Leaving sliding mode $\dot{x}(t) \in -\mathrm{sign}(x(t)) + t$.



Example: continuity of multipliers in different switching cases

Different switching cases

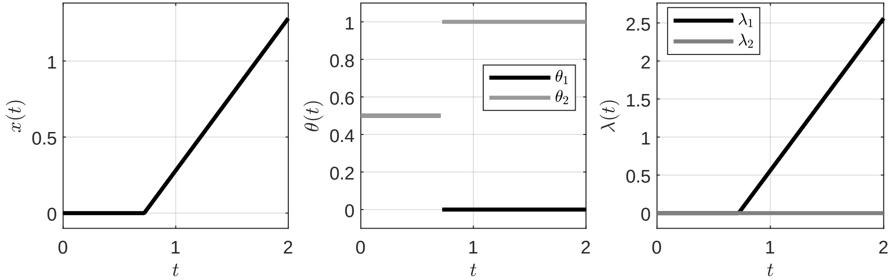
4. Spontaneous switch, $\dot{x}(t) \in \mathrm{sign}(x(t))$,



Example: continuity of multipliers in different switching cases

Different switching cases

1. Crossing a surface of discontinuity, $\dot{x}(t) \in 2 - \mathrm{sign}(x(t))$,
2. Sliding mode, $\dot{x}(t) \in -\mathrm{sign}(x(t)) + 0.2\sin(5t)$,
3. Leaving sliding mode $\dot{x}(t) \in -\mathrm{sign}(x(t)) + t$.
4. Spontaneous switch, $\dot{x}(t) \in \mathrm{sign}(x(t))$,



The active set of the DCS

Dynamic complementarity system

$$\dot{x} = F(x,u)\,\theta$$
$$0 = g_i(x) - \lambda_i - \mu,\ i = 1,\dots,n_f$$
$$0 \leq \theta \perp \lambda \geq 0$$
$$1 = e^\top \theta$$

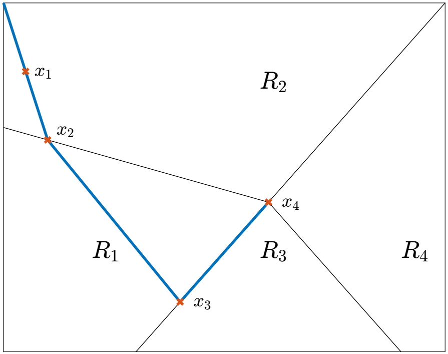
DAE with fixed $\mathcal{I}$

$$\dot{x} = F_{\mathcal{I}}(x,u)\,\theta_{\mathcal{I}}$$
$$0 = g_{\mathcal{I}}(x) - \mu e,$$
$$1 = e^\top \theta_{\mathcal{I}}$$

- Locally well-behaved smooth ODE or DAE

Active set

$$\mathcal{I}(x) := \left\{ i \mid g_i(x) = \min_{j \in \mathcal{J}} g_j(x) \right\} = \left\{ i \mid \theta_i > 0 \right\}$$



$$\mathcal{I}(x_1) = \{2\},\ \mathcal{I}(x_2) = \{1,2\},\ \mathcal{I}(x_3) = \{1,3\}$$
$$\mathcal{I}(x_4) = \{1,2,3,4\}$$

Properties of the DCS

Sufficient conditions for the uniqueness of the solution

DAE with fixed $\mathcal{I}$

$$\dot{x} = F_{\mathcal{I}}(x,u)\,\theta_{\mathcal{I}} \tag{2a}$$

$$0 = g_{\mathcal{I}}(x) - \mu e, \tag{2b}$$

$$1 = e^\top \theta_{\mathcal{I}} \tag{2c}$$

Given $|\mathcal{I}| \geq 1$, define the matrix

$$M_{\mathcal{I}}(x) = \nabla g_{\mathcal{I}}(x)^\top F_{\mathcal{I}}(x,u) \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{I}|}.$$

Proposition

Suppose that for a fixed active set $\mathcal{I}(x(t)) = \mathcal{I}$ *for* $t \in [0,T]$, *it holds that the matrix* $M_{\mathcal{I}}(x(t))$ *is invertible and* $e^\top M_{\mathcal{I}}(x(t))^{-1} e \neq 0$ *for all* $t \in [0,T]$. *Given the initial value* $x(0)$, *then the DAE* (2) *has a unique solution for all* $t \in [0,T]$.

Proof. Index reduction and implicit function theorem. □

Outline of the lecture

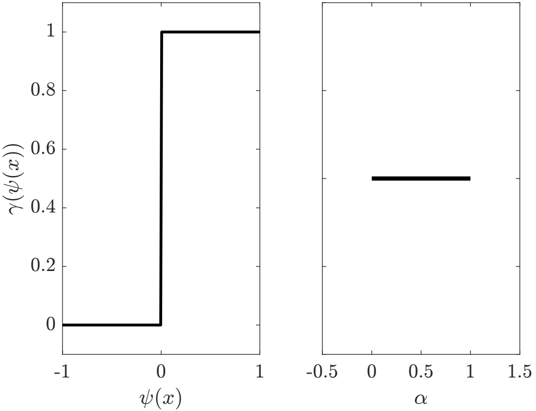
Heaviside step function

Set-valued step function

$$\gamma(\psi(x)) = \begin{cases} \{1\}, & \psi(x) > 0, \\ [0,1], & \psi(x) = 0, \\ \{0\}, & \psi(x) < 0. \end{cases}$$

LP representation

$$\begin{aligned} \gamma(\psi(x)) = \arg\min_{\alpha \in \mathbb{R}} \quad & -\psi(x)\alpha \\ \text{s.t.} \quad & 0 \leq \alpha \leq 1. \end{aligned}$$



Heaviside step function

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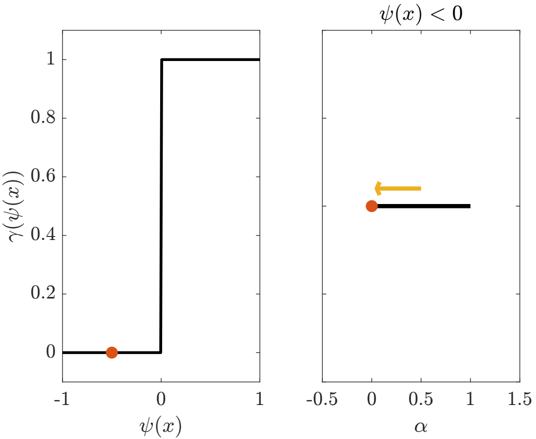

$$\psi(x) < 0 \implies \alpha = \{0\}$$

Heaviside step function

Set-valued step function

$$\gamma(\psi(x)) = \begin{cases} \{1\}, & \psi(x) > 0, \\ [0,1], & \psi(x) = 0, \\ \{0\}, & \psi(x) < 0. \end{cases}$$

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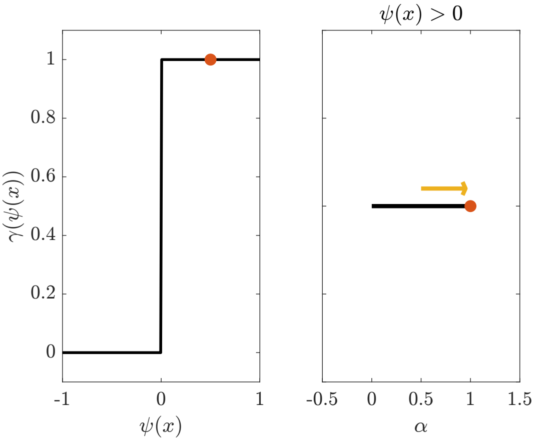


$$\psi(x) > 0 \implies \alpha = \{1\}$$

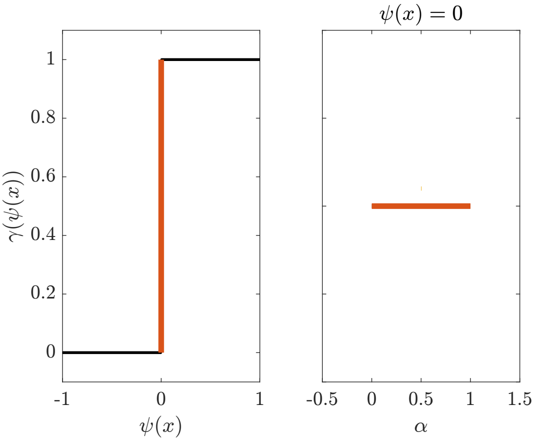
Heaviside step function

Set-valued step function

$$\gamma(\psi(x)) = \begin{cases} \{1\}, & \psi(x) > 0, \\ [0,1], & \psi(x) = 0, \\ \{0\}, & \psi(x) < 0. \end{cases}$$

LP representation

$$\gamma(\psi(x)) = \arg\min_{\alpha \in \mathbb{R}} \; -\psi(x)\alpha$$
$$\text{s.t.} \quad 0 \leq \alpha \leq 1.$$



$$\psi(x) = 0 \implies \alpha = [0,1]$$

Motivating example

Consider two switching functions $\psi_1(x)$ and $\psi_2(x)$ and four regions

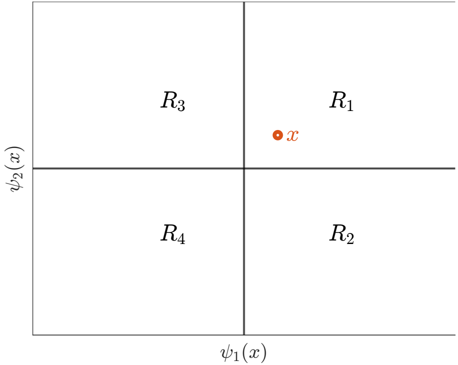
Nonsmooth system

$$\dot{x} = \alpha_1 \alpha_2 f_1(x)$$

Step representation

$\theta_i = 1$ if $x \in R_i$:

$$\psi_1(x) > 0,\ \psi_2(x) > 0 \implies$$
$$\alpha_1 = 1, \alpha_2 = 1,\ \theta_1 = \alpha_1\alpha_2 = 1$$



$$R_1 = \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0,\ \psi_2(x) > 0\}$$

Motivating example

Consider two switching functions $\psi_1(x)$ and $\psi_2(x)$ and four regions

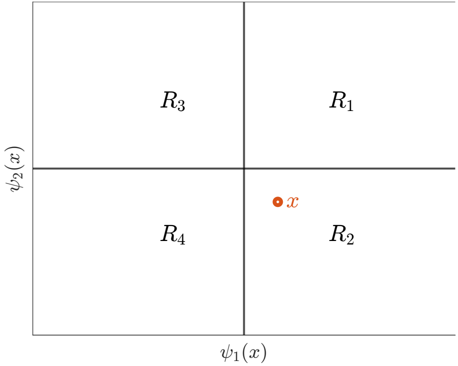
Nonsmooth system

$$\dot{x} = \alpha_1 \alpha_2 f_1(x) + \alpha_1 (1 - \alpha_2) f_2(x)$$

Step representation

$\theta_i = 1$ if $x \in R_i$:

$$\psi_1(x) > 0,\ \psi_2(x) < 0 \implies$$
$$\alpha_1 = 1, \alpha_2 = 0,\ \theta_2 = \alpha_1(1 - \alpha_2) = 1$$



$$R_2 = \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0,\ \psi_2(x) < 0\}$$

Motivating example

Consider two switching functions $\psi_1(x)$ and $\psi_2(x)$ and four regions

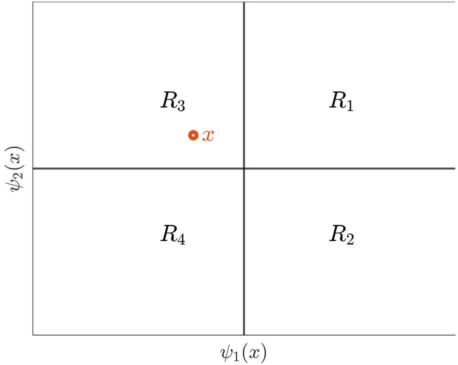
Nonsmooth system

$$\dot{x} = \alpha_1 \alpha_2 f_1(x) + \alpha_1 (1-\alpha_2) f_2(x) + (1-\alpha_1)\alpha_2 f_3(x)$$

Step representation

$\theta_i = 1$ if $x \in R_i$:

$$\psi_1(x) < 0,\ \psi_2(x) > 0 \implies \alpha_1 = 0, \alpha_2 = 1,\ \theta_3 = (1-\alpha_1)\alpha_2 = 1$$



$$R_3 = \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) < 0,\ \psi_2(x) > 0\}$$

Motivating example

Consider two switching functions $\psi_1(x)$ and $\psi_2(x)$ and four regions

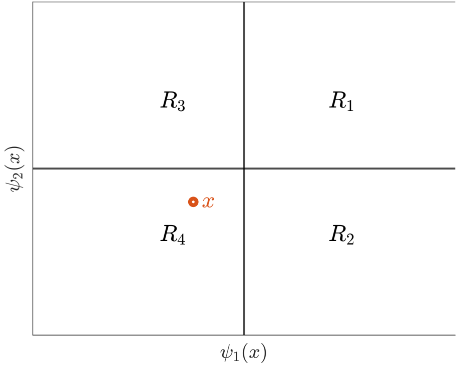
Nonsmooth system

$$\begin{aligned}\dot{x} = &\alpha_1\alpha_2 f_1(x) + \alpha_1(1-\alpha_2) f_2(x)\\ &+ (1-\alpha_1)\alpha_2 f_3(x) + (1-\alpha_1)(1-\alpha_2) f_4(x)\end{aligned}$$

Step representation

$\theta_i = 1$ if $x \in R_i$:

$$\begin{aligned}&\psi_1(x) < 0,\ \psi_2(x) < 0 \implies\\ &\alpha_1 = 0, \alpha_2 = 0,\ \theta_4 = (1-\alpha_1)(1-\alpha_2) = 1\end{aligned}$$



$$R_4 = \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) < 0,\ \psi_2(x) < 0\}$$

Motivating example

Consider two switching functions $\psi_1(x)$ and $\psi_2(x)$ and four regions

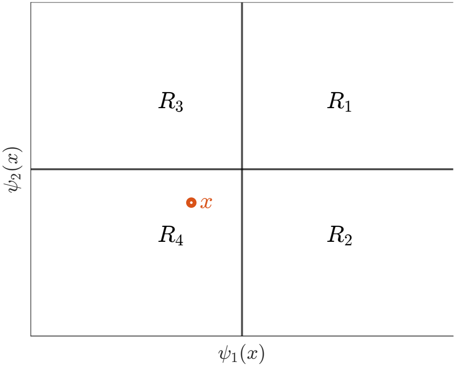
Nonsmooth system

$$\dot{x} = \alpha_1\alpha_2 f_1(x) + \alpha_1(1-\alpha_2)f_2(x) + (1-\alpha_1)\alpha_2 f_3(x) + (1-\alpha_1)(1-\alpha_2)f_4(x)$$

Step representation

$\theta_i = 1$ if $x \in R_i$:

$$\begin{aligned}\theta_1 &= \alpha_1\alpha_2\\ \theta_2 &= \alpha_1(1-\alpha_2)\\ \theta_3 &= (1-\alpha_1)\alpha_2\\ \theta_4 &= (1-\alpha_1)(1-\alpha_2)\end{aligned}$$



$$R_4 = \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) < 0,\ \psi_2(x) < 0\}$$

Generalizing the observed pattern

Definition of regions via switching functions

$$
\begin{aligned}
R_1 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) > 0\} \\
R_2 &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) > 0, \psi_2(x) > 0, \dots \psi_{n_\psi}(x) < 0\} \\
&\vdots \\
R_{n_f} &= \{x \in \mathbb{R}^{n_x} \mid \psi_1(x) < 0, \psi_2(x) < 0, \dots \psi_{n_\psi}(x) < 0\}
\end{aligned}
$$

$$
\psi(x) := \begin{bmatrix} \psi_1(x) & \psi_2(x) & \dots & \psi_{n_\psi}(x) \end{bmatrix}^\top \in \mathbb{R}^{n_\psi}
$$

Sign matrix

$$
S = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & -1 \end{bmatrix}
$$

Definition via i-th row $S_{i,\bullet}$:

$$
R_i = \{x \in \mathbb{R}^{n_x} \mid S_{i,\bullet}\psi(x) > 0\}
$$

We observe that

$$
\frac{1 - S_{i,j}}{2} + S_{i,j}\alpha_i = \begin{cases} \alpha_i, & \text{if } S_{i,j} = 1, \\ 1 - \alpha_i, & \text{if } S_{i,j} = -1. \end{cases}
$$

Filippov system via the step reformulation

If $x \in R_i$ then $\theta_i = 1$, hence all corresponding α_j and $1 - \alpha_k$ must be equal to one.

$$\frac{1 - S_{i,j}}{2} + S_{i,j}\alpha_i = \begin{cases} \alpha_i, & \text{if } S_{i,j} = 1, \\ 1 - \alpha_i, & \text{if } S_{i,j} = -1. \end{cases}$$

Filippov system

$$\dot{x} \in F_{\mathrm{F}}(x) := \Big\{ \sum_{i=1}^{2^{n_\psi}} \theta_i f_i(x) \,\Big|\, \theta_i = \prod_{j=1}^{n_\psi} \Big(\frac{1 - S_{i,j}}{2} + S_{i,j}\alpha_j \Big),\ i = 1, \dots, 2^{n_\psi}, \alpha_j \in \gamma(\psi_j(x)) \Big\}.$$

From differential inclusion to dynamic complementarity system

Regard the aggregated LP

$$\begin{aligned}\min_{\alpha \in \mathbb{R}^{n_\psi}} \quad & -\psi(x)^\top \alpha \\ \text{s.t.} \quad & 0 \leq \alpha_i \leq 1, \; i = 1, \dots, n_\psi\end{aligned}$$

Using its KKT conditions we pass from the DI to the DCS:

$$\begin{aligned}&\psi(x) = \lambda^{\mathrm{p}} - \lambda^{\mathrm{n}}, \\ &0 \leq \lambda^{\mathrm{n}} \perp \alpha \geq 0, \\ &0 \leq \lambda^{\mathrm{p}} \perp e - \alpha \geq 0,\end{aligned}$$

Heaviside step DCS

$$\begin{aligned}&\dot{x} = F(x,u)\,\theta, \\ &\theta_i = \prod_{j=1}^{n_\psi} \left(\frac{1 - S_{i,j}}{2} + S_{i,j}\alpha_j \right), i = 1, \dots, 2^{n_\psi} \\ &\psi(x) = \lambda^{\mathrm{p}} - \lambda^{\mathrm{n}} \\ &0 \leq \lambda^{\mathrm{n}} \perp \alpha \geq 0 \\ &0 \leq \lambda^{\mathrm{p}} \perp e - \alpha \geq 0\end{aligned}$$

Continuity of multipliers in the Heaviside step formulation

Regard the aggregated LP

$$\min_{\alpha \in \mathbb{R}^{n_\psi}} \quad -\psi(x)^\top \alpha$$
$$\text{s.t.} \quad 0 \leq \alpha_i \leq 1, \; i = 1, \ldots, n_\psi$$

Using its KKT conditions we pass from the DI to the DCS:

$$\begin{aligned} &\psi(x) = \lambda^{\mathrm{p}} - \lambda^{\mathrm{n}}, \\ &0 \leq \lambda^{\mathrm{n}} \perp \alpha \geq 0, \\ &0 \leq \lambda^{\mathrm{p}} \perp e - \alpha \geq 0, \end{aligned}$$

- From the LP and its KKT conditions: $\psi_j(x) > 0$, we have $\alpha_j = 1$
- Upper bound is active: $\lambda^{\mathrm{n}}_j = 0$ and $\lambda_{\mathrm{p},j} = \psi_j(x) > 0$

Continuity of multipliers in the Heaviside step formulation

Regard the aggregated LP

$$\min_{\alpha \in \mathbb{R}^{n_\psi}} \quad -\psi(x)^\top \alpha$$
$$\text{s.t.} \quad 0 \leq \alpha_i \leq 1, \; i = 1, \ldots, n_\psi$$

Using its KKT conditions we pass from the DI to the DCS:

$$\psi(x) = \lambda^{\mathrm{p}} - \lambda^{\mathrm{n}},$$
$$0 \leq \lambda^{\mathrm{n}} \perp \alpha \geq 0,$$
$$0 \leq \lambda^{\mathrm{p}} \perp e - \alpha \geq 0,$$

- From the LP and its KKT conditions: $\psi_j(x) > 0$, we have $\alpha_j = 1$
- Upper bound is active: $\lambda^{\mathrm{n}}_j = 0$ and $\lambda_{\mathrm{p},j} = \psi_j(x) > 0$
- Likewise, for $\psi_j(x) < 0$, we obtain $\alpha_j = 0$, $\lambda^{\mathrm{p}}_j = 0$ and $\lambda^{\mathrm{n}}_j = -\psi_j(x) > 0$

Continuity of multipliers in the Heaviside step formulation

Regard the aggregated LP

$$
\begin{aligned}
\min_{\alpha \in \mathbb{R}^{n_\psi}} \quad & -\psi(x)^\top \alpha \\
\text{s.t.} \quad & 0 \leq \alpha_i \leq 1, \; i = 1, \ldots, n_\psi
\end{aligned}
$$

Using its KKT conditions we pass from the DI to the DCS:

$$
\begin{aligned}
&\psi(x) = \lambda^{\mathrm{p}} - \lambda^{\mathrm{n}}, \\
&0 \leq \lambda^{\mathrm{n}} \perp \alpha \geq 0, \\
&0 \leq \lambda^{\mathrm{p}} \perp e - \alpha \geq 0,
\end{aligned}
$$

- From the LP and its KKT conditions: $\psi_j(x) > 0$, we have $\alpha_j = 1$
- Upper bound is active: $\lambda_j^{\mathrm{n}} = 0$ and $\lambda_{\mathrm{p},j} = \psi_j(x) > 0$
- Likewise, for $\psi_j(x) < 0$, we obtain $\alpha_j = 0$, $\lambda_j^{\mathrm{p}} = 0$ and $\lambda_j^{\mathrm{n}} = -\psi_j(x) > 0$
- $\psi_j(x) = 0$ implies that $\alpha_j \in [0, 1]$ and $\lambda_j^{\mathrm{p}} = \lambda_j^{\mathrm{n}} = 0$

Continuity of multipliers in the Heaviside step formulation

Regard the aggregated LP

$$\min_{\alpha \in \mathbb{R}^{n_\psi}} \quad -\psi(x)^\top \alpha$$
$$\text{s.t.} \quad 0 \le \alpha_i \le 1, \; i = 1, \dots, n_\psi$$

Using its KKT conditions we pass from the DI to the DCS:

$$\begin{aligned}
&\psi(x) = \lambda^{\mathrm{p}} - \lambda^{\mathrm{n}}, \\
&0 \le \lambda^{\mathrm{n}} \perp \alpha \ge 0, \\
&0 \le \lambda^{\mathrm{p}} \perp e - \alpha \ge 0,
\end{aligned}$$

- From the LP and its KKT conditions: $\psi_j(x) > 0$, we have $\alpha_j = 1$
- Upper bound is active: $\lambda_j^{\mathrm{n}} = 0$ and $\lambda_{\mathrm{p},j} = \psi_j(x) > 0$
- Likewise, for $\psi_j(x) < 0$, we obtain $\alpha_j = 0$, $\lambda_j^{\mathrm{p}} = 0$ and $\lambda_j^{\mathrm{n}} = -\psi_j(x) > 0$
- $\psi_j(x) = 0$ implies that $\alpha_j \in [0, 1]$ and $\lambda_j^{\mathrm{p}} = \lambda_j^{\mathrm{n}} = 0$

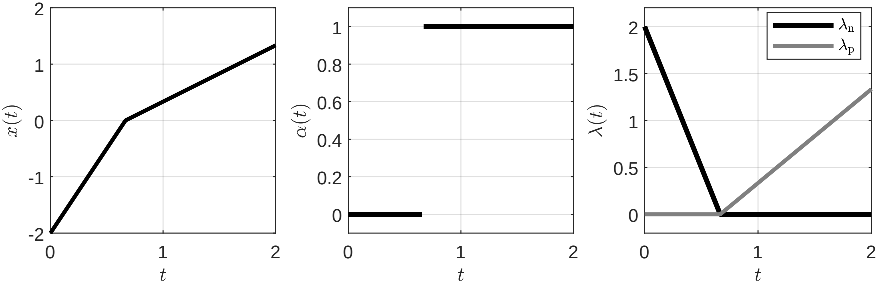
Continuity of multipliers

$$\begin{aligned}
\lambda^{\mathrm{p}} &= \max(\psi(x), 0), && \text{(positive part of } \psi(x)\text{)} \\
\lambda^{\mathrm{n}} &= -\min(\psi(x), 0), && \text{(negative part of } \psi(x)\text{)}
\end{aligned}$$

Example: continuity of multipliers in different switching cases

Different switching cases

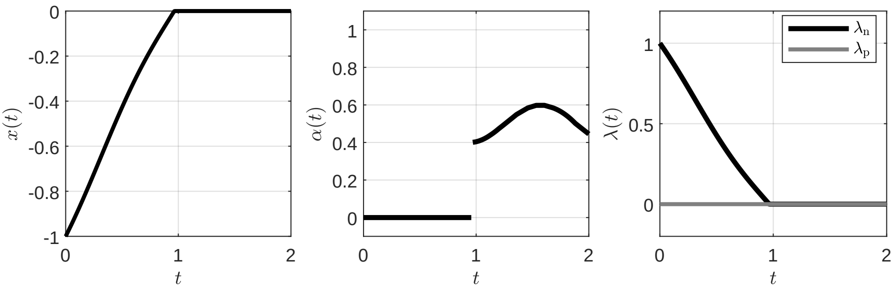
1. Crossing a surface of discontinuity, $\dot{x}(t) \in 2 - \mathrm{sign}(x(t))$,



Example: continuity of multipliers in different switching cases

Different switching cases

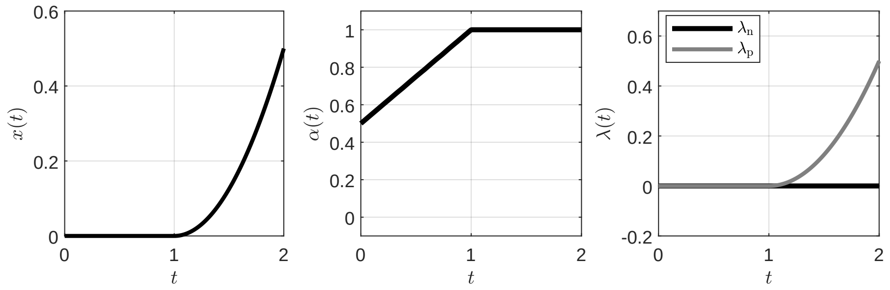
2. Sliding mode, $\dot{x}(t) \in -\mathrm{sign}(x(t)) + 0.2\sin(5t)$,



Example: continuity of multipliers in different switching cases

Different switching cases

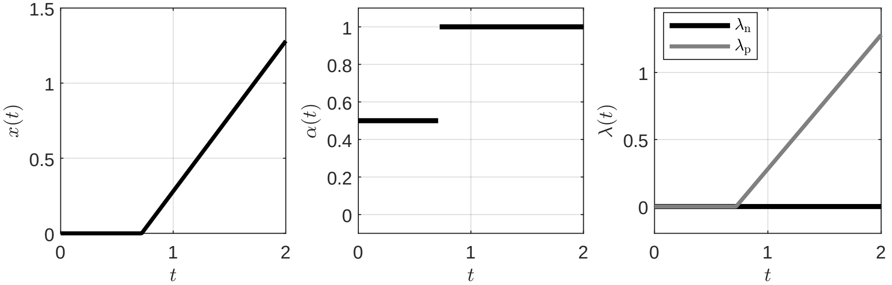
3. Leaving sliding mode $\dot{x}(t) \in -\mathrm{sign}(x(t)) + t$.



Example: continuity of multipliers in different switching cases

Different switching cases

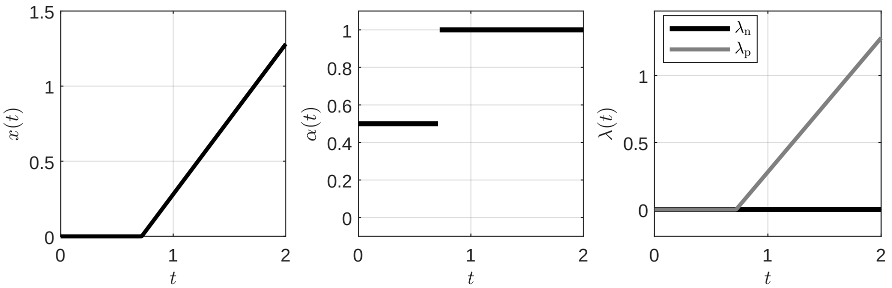
4. Spontaneous switch, $\dot{x}(t) \in \operatorname{sign}(x(t))$,

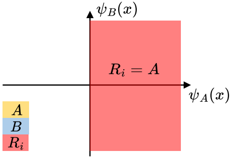
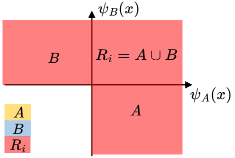
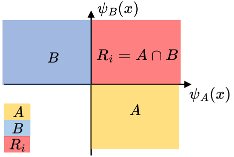


Example: continuity of multipliers in different switching cases

Different switching cases

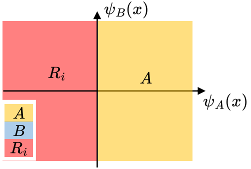
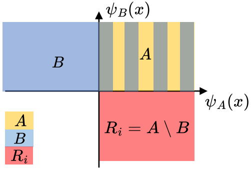
1. Crossing a surface of discontinuity, $\dot{x}(t) \in 2 - \mathrm{sign}(x(t))$,
2. Sliding mode, $\dot{x}(t) \in -\mathrm{sign}(x(t)) + 0.2\sin(5t)$,
3. Leaving sliding mode $\dot{x}(t) \in -\mathrm{sign}(x(t)) + t$.
4. Spontaneous switch, $\dot{x}(t) \in \mathrm{sign}(x(t))$,



Definition of R_i	Expression for θ_i	Sketch
$R_i = A$	$\theta_i = \alpha_A$	
$R_i = A \cup B$	$\theta_i = \alpha_A + \alpha_B$	
$R_i = A \cap B$	$\theta_i = \alpha_A \alpha_B$	

Modeling with step functions - continued

Expressions of θ_i for different definitions of R_i

Definition of R_i	Expression for θ_i	Sketch
$R_i = \mathrm{int}(\mathbb{R}^{n_x} \setminus A) = \{x \mid \psi_A(x) < 0\}$	$\theta_i = 1 - \alpha_A$	
$R_i = A \setminus B$	$\theta_i = \alpha_A - \alpha_B$	

Stewart vs. Heaviside step

Dynamic complementarity system

$$
\begin{aligned}
\dot{x} &= F(x,u)\,\theta \\
0 &= g_i(x) - \lambda_i - \mu,\ i = 1,\dots,2^{n_\psi}, \\
0 &\leq \theta \perp \lambda \geq 0 \\
1 &= e^\top \theta
\end{aligned}
$$

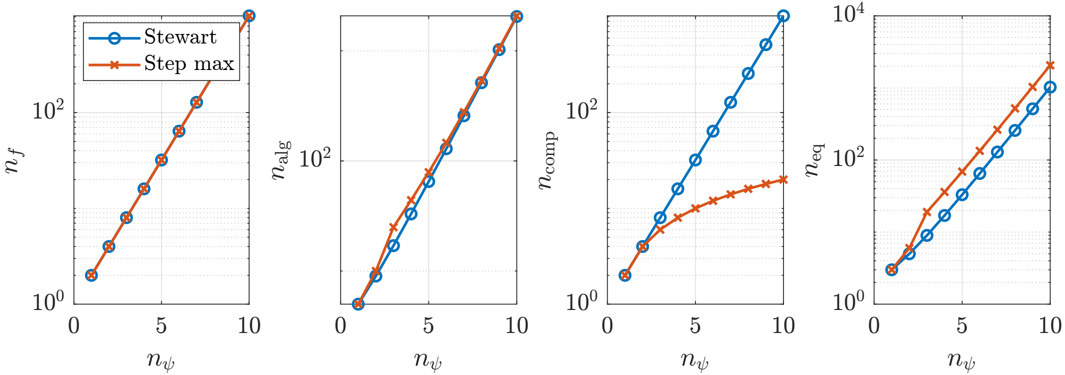
Heaviside step DCS

$$
\begin{aligned}
\dot{x} &= F(x,u)\,\theta \\
\theta_i &= \prod_{j=1}^{n_\psi} \left(\frac{1 - S_{i,j}}{2} + S_{i,j}\alpha_j \right), i = 1,\dots,2^{n_\psi} \\
\psi(x) &= \lambda^{\mathrm{p}} - \lambda^{\mathrm{n}} \\
0 &\leq \lambda^{\mathrm{n}} \perp \alpha \geq 0 \\
0 &\leq \lambda^{\mathrm{p}} \perp e - \alpha \geq 0
\end{aligned}
$$

Table: Comparisons of the problem sizes in Stewart's and the step reformulation for a fixed n_ψ.

Method	Number of systems	n_{alg}	n_{comp}	n_{eq}
Stewart	2^{n_ψ}	$2 \cdot 2^{n_\psi} + 1$	2^{n_ψ}	$2^{n_\psi} + 1$
Heaviside step	2^{n_ψ}	$2^{n_\psi} + 3n_\psi$	$2n_\psi$	$n_\psi + n_f$

Stewart vs. Heaviside step - complexity



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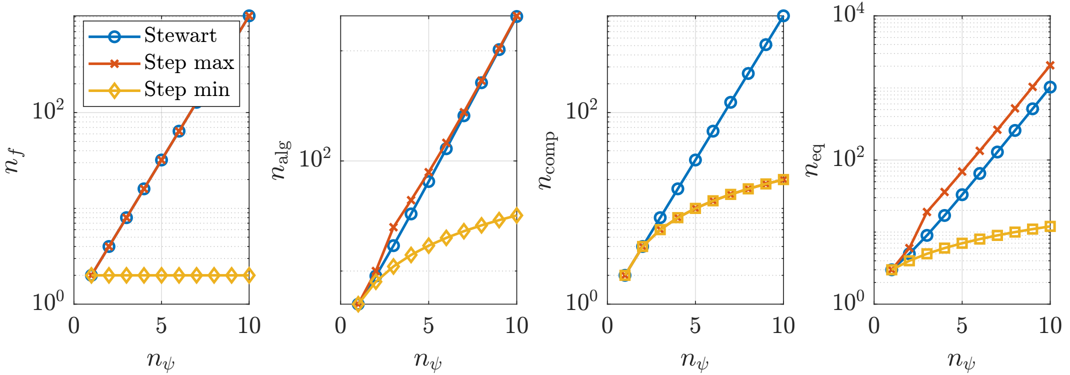
Stewart vs. Heaviside step - complexity



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Heaviside step	2^{n_ψ}	$2^{n_\psi} + 3n_\psi$	$2n_\psi$	$n_\psi + n_f$

- The set-valued step functions may be related in a more complicated and different may than in Filippov systems
- Such system are an instance of Aizerman–Pyatnitskii differential inclusions

$$\dot{x}(t) \in F_{\mathrm{AP}}(x(t), \Gamma(\psi(x(t))))$$

$$\begin{tikzcd}
 & \text{Aizerman–Pyatnitskii} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \text{DCS} \arrow[d, Rightarrow] \arrow[dd, Rightarrow, bend left] \\
\text{Filippov} \arrow[r, Rightarrow] & \text{Filippov PSS} \arrow[r, Rightarrow] \arrow[dr, Leftrightarrow] & \text{Step DCS} \\
 & \text{PSS} \arrow[u, Rightarrow] & \text{Stewart DCS}
\end{tikzcd}$$

Sensitivities w.r.t. parameters

Regard a bimodal system:

$$\dot{x}(t) = \begin{cases} f_1(x(t)), & \psi(x(t)) < 0, \\ f_2(x(t)), & \psi(x(t)) \geq 0. \end{cases} \tag{3}$$

Regard the case of crossing a switchig surface, with e.g., $\dot{x} = f_1(x)$ for $t \in [0, t_\mathrm{s})$ and after crossing at t_s we have $\dot{x} = f_2(x)$ for $t \in (t_\mathrm{s}, T]$. At t_s it holds that

$$\psi(x(t_\mathrm{s})) = 0.$$

We are interested in the exact sensitivity matrix $S^x(t, 0; x_0)$ of a solution of the system (3):

$$S^x(t, 0; x_0) = \frac{\partial x(t; x_0)}{\partial x_0} \in \mathbb{R}^{n_x \times n_x}$$

Sensitivity jump formula

Before and after the switch the $S^x(t,0;x_0)$ obey linear variational differential equation (VDE)

$$\dot{S}^x(t,0;x_0) = \frac{\partial f(x)}{\partial x} S^x(t,0;x_0),\ S^x(0,0;x_0) = I_{n_x}$$

The function $S^x(t,0;x_0)$ obeys smooth VDEs, on both sides of t_s, but exhibits a jump at t_s.

Proposition

Regard the system (3) *with* $x(0) = x_0 \in R_i$ *on an interval* $[0,T]$ *with a switch at* $t_\mathrm{s} \in (0,T)$. *Assume that the functions* $f_1(x),\ f_2(x), \psi_{i,j}(x)$ *are continuously differentiable along* $x(t), t \in [0,T]$. *Assume the solution* $x(t)$ *reaches the surface of discontinuity transversally, i.e.,* $\nabla\psi(x(t_\mathrm{s}))^\top f_1(x(t_\mathrm{s})) > 0$. *Then the sensitivity* $S^x(T,0;x_0)$ *of a solution* $x(t;x_0)$ *of the system described by the ODE* (3) *is given by*

$$S^x(T,0;x_0) = S^x(T,t_\mathrm{s}^+;x(t_\mathrm{s}))J(x(t_\mathrm{s};x_0))S^x(t_\mathrm{s}^-,0;x_0)\ \textit{with}$$

$$J(x(t_\mathrm{s};x_0)) := I + \frac{(f_2(x(t_\mathrm{s};x_0)) - f_1(x(t_\mathrm{s};x_0)))\nabla\psi(x(t_\mathrm{s};x_0))^\top}{\nabla\psi(x(t_\mathrm{s};x_0))^\top f_1(x(t_\mathrm{s};x_0))}.$$

Conclusions and summary

- Filippov system provide a handy solution concept for ODEs with a discontinuous r.h.s. (e.g., handling of sliding modes)
- For piece smooth systems, one can define multipliers θ for defining the convex Filippov set
- The multiplier θ can implicitly be computed by considering an equivalent dynamic complementarity systems
- Two approaches Stewart's and the Heaviside step formulation
- In both formulations, some algebraic variables are discontinuous, others are continuous - key for switch detection in next lecture
- Step offers more flexibility in modeling, but might be more nonlinear than Stewart's reformulation

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