Lecture 1: Recap on theory and algorithms for nonlinear programming

Moritz Diehl and Armin Nurkanović

Systems Control and Optimization Laboratory (syscop)
Summer School on Direct Methods for Optimal Control of Nonsmooth Systems
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universität freiburg

Outline of the lecture

What is an optimization problem?

Minimize (or maximize) an objective function $F(w)$ depending on deceision variables w subject to equality and/or inequality constrains

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An optimization problem

$$
\begin{aligned}
\min_{w} \quad & F(w) && \text{(1a)} \\
\text{s.t.} \quad & G(w) = 0 && \text{(1b)} \\
& H(w) \geq 0 && \text{(1c)}
\end{aligned}
$$

Terminology

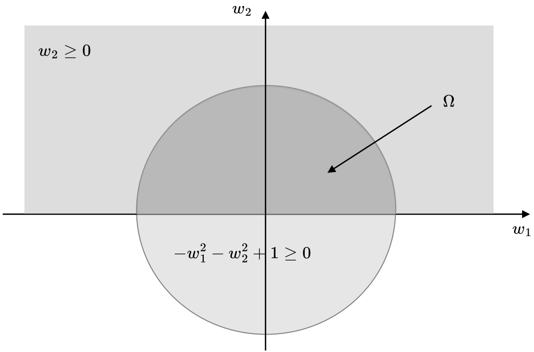
- w - decision variable
- F: objective/cost function
- G, H: equality and inequality constraint functions

- Optimization is a powerful tool used in all quantitative sciences
- Only in few special cases a closed form solution exist
- Use an iterative algorithm to find solution
- The optimization problem may be parametric, and all functions depend on a fixed parameter p

Basic definitions: the feasible set

Definition

The feasible set of the optimization problem (1) is defined as
$\Omega = \{w \in \mathbb{R}^n \mid G(w) = 0, H(w) \geq 0\}$. A point $w \in \Omega$ is is called a feasible point.



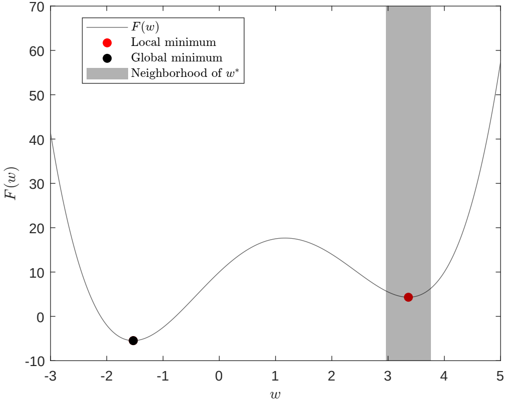
The feasible set is the intersection of the two grey areas (halfspace and circle)

Basic definitions: local and global minimizer

Definition

- A point $w^* \in \Omega$ is called a **local minimizer** of the NLP (1) if there exists an open ball $\mathcal{B}_\epsilon(w^*)$ with $\epsilon > 0$, such that for all $w \in \mathcal{B}_\epsilon(w^*) \cap \Omega$ it holds that $F(w) \geq F(w^*)$.
- A point $w^* \in \Omega$ is called a **global minimizer** of the NLP (1) if for all $w \in \Omega$ it holds that $F(w) \geq F(w^*)$.

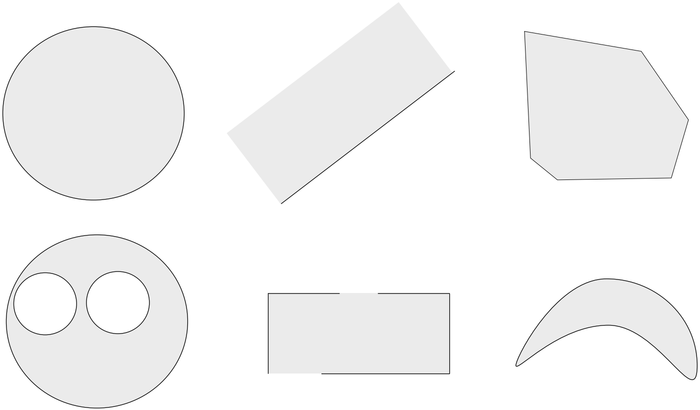
The value $F(w^*)$ at a local/global minimizer w^* is called local/global minimum.



$$F(w) = \frac{1}{2}w^4 - 2w^3 - 3w^2 + 12w + 10$$

Convex sets

A key concept in optimization is convexity



A set Ω is said to be convex if for any w_1, w_2 and any $\theta \in [0,1]$ it holds $\theta w_1 + (1-\theta) w_2 \in \Omega$

Convex functions

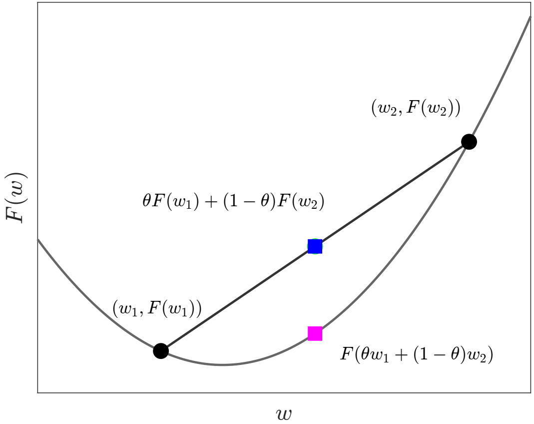
- A function F is convex if for every $w_1, w_2 \in \mathbb{R}^n$ and $\theta \in [0,1]$ it holds that

$$F(\theta w_1 + (1-\theta) w_2) \leq \theta F(w_1) + (1-\theta) F(w_2)$$

- F is concave if and only if $-F$ is convex
- F is convex if and only if the epigraph

$$\mathrm{epi} F = \{(w,t) \in \mathbb{R}^{n_w+1} \mid F(w) \leq t\}$$

is a convex set



Convex optimization problems

A convex optimization problem

$$\begin{aligned} \min_{w} \quad & F(w) \\ \text{s.t.} \quad & G(w) = 0 \\ & H(w) \geq 0 \end{aligned}$$

An optimization problem is convex if the objective function F is convex and the feasible set Ω is convex.

- Every locally optimal solution is global
- First order conditions are necessary and sufficient (we come back to this)
- *"...in fact, the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."* R. T. Rockafellar, SIAM Review, 1993

Outline of the lecture

Some classifications of optimization problems

Optimization problems can be:

- unconstrained ($\Omega = \mathbb{R}^n$) or constrained ($\Omega \subset \mathbb{R}^n$)
- convex or nonconvex
- linear or nonlinear
- differentiable or nonsmooth
- continuous or (mixed-)integer
- finite or infinite dimensional

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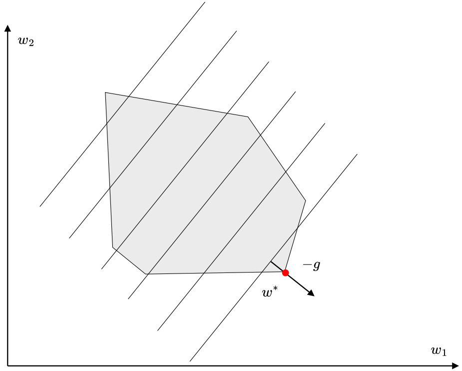
"... the main fact, which should be known to any person dealing with optimization models, is that in general, optimization problems are unsolvable."
Yurii Nesterov, Lectures on Convex Optimization, 2018.

Class 1: Linear Programming (LP)

Linear program

$$\min_{w} \; g^\top w$$
$$\text{s.t.} \quad Aw - b = 0$$
$$Cw - d \geq 0$$



- convex optimization problem
- 1947: simplex method by Dantzig, 1984: polynomial time interior-point method by Karmarkar
- if not unbounded, the solution is always at edge or vertex of the feasible set
- today very mature and reliable

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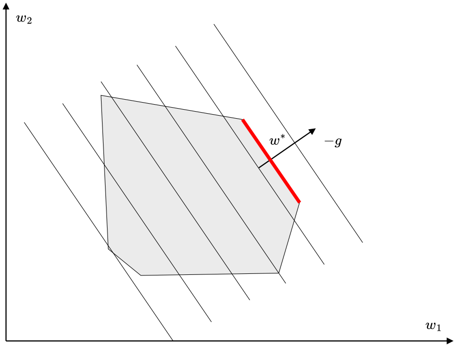

- convex optimization problem
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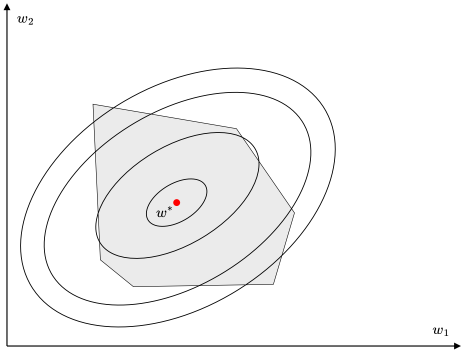
Class 2: Quadratic Programming (QP)

Quadratic program

$$\min_{w} \; \frac{1}{2} w^\top Q w + g^\top w$$
$$\text{s.t.} \quad Aw - b = 0$$
$$Cw - d \geq 0$$

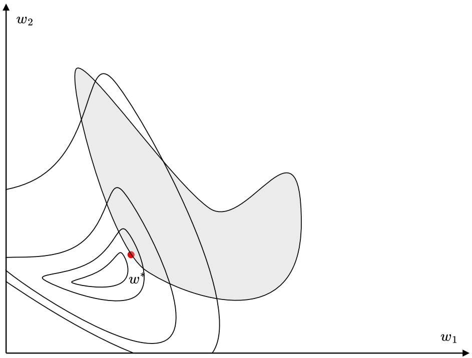


- depending on Q, can be convex and nonconvex
- solved online in linear model predictive control
- many good solvers: `Gurobi`, `OSQP`, `HPIPM`, `qpOASES`, `OOQP`, ...
- subsproblems in nonlinear optimization

Class 3: Nonlinear Program (NLP)

Nonlinear programming problem

$$\begin{aligned} \min_{w} \ & F(w) \\ \text{s.t.} \quad & G(w) = 0 \\ & H(w) \geq 0 \end{aligned}$$



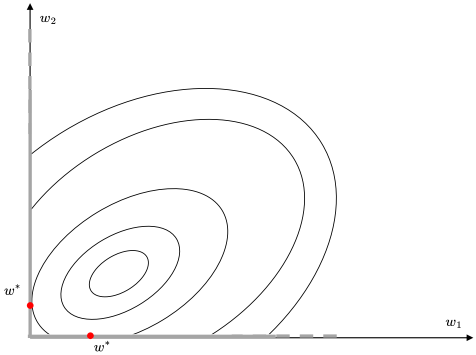
- can be convex and nonconvex
- solved with iterative Newton-type algorithms
- solved in nonlinear model predictive control

Class 4: Mathematical programs with Complementarity Constraints (MPCC)

MPCC

$$\begin{aligned} \min_{w_0, w_1, w_2} \quad & F(w) \\ \text{s.t.} \quad & G(w) = 0 \\ & H(w) \geq 0 \\ & 0 \leq w_1 \perp w_2 \geq 0 \end{aligned}$$

$w = [w_0^\top, w_1^\top, w_2^\top]^\top$



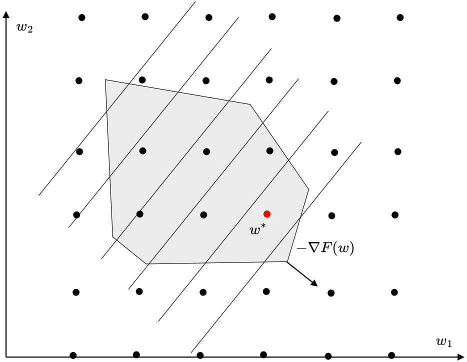
- Special case of nonlinear programs treated extensively in this course
- Standard constraint qualifications fail to holds
- Very powerful modeling concept
- Requires specialized theory and algorithms (Lectures by C. Kirches)

Class 5: Mixed-integer programming

Mixed-integer nonlinear program (MINLP)

$$\begin{aligned} \min_{w_0 \in \mathbb{R}^p, w_1 \in \mathbb{Z}^q} \quad & F(w) \\ \text{s.t.} \quad & G(w) = 0 \\ & H(w) \geq 0 \end{aligned}$$

$w = [w_0^\top, w_1^\top]^\top, n = p + q$



- Combinatorial problem, feasible set is finite
- Branch and bound, brunch and cut methods
- Requires solution of many relaxed continuous convex or nonconvex problems
- Optimization problems treated in this course can always be reformulate into MINLPs (but not very efficient)

Class 6: Continuous time optimal control problems

Continuous-time Optimal Control Problem

$$\begin{aligned}\min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_{\mathrm{c}}(x(t),u(t))\,\mathrm{d}t + E(x(T))\\ \text{s.t.} \quad & x(0) = \bar{x}_0\\ & \dot{x}(t) = f_{\mathrm{c}}(x(t),u(t))\\ & 0 \geq h(x(t),u(t)),\ t \in [0,T]\\ & 0 \geq r(x(T))\end{aligned}$$

- Infinite dimensional problem, can be convex or nonconvex
- Dynamic constraint can be replaced by $\dot{x}(t) = f_{\mathrm{c}}(x(t),u(t))$:
  - DAE
  - PDE
  - stoachstic ODE/PDE
  - **Nonsmooth ODE** - this course
- All or some components of $u_i(t)$ may take values in $\mathbb{Z}$ (mixed-integer OCP)

Direct optimal control methods solve Nonlinear Programs (NLP)

Continuous time OCP

$$\begin{aligned} \min_{x(\cdot),u(\cdot)} \quad & \int_0^T L_c(x(t),u(t))\,\mathrm{d}t + E(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = f_c(x(t),u(t)) \\ & 0 \geq h(x(t),u(t)),\ t \in [0,T] \\ & 0 \geq r(x(T)) \end{aligned}$$

Direct methods like direct collocation, multiple shooting first discretize, then optimize.

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Direct methods like direct collocation, multiple shooting first discretize, then optimize.

Discrete time OCP (an NLP)

$$
\begin{aligned}
\min_{x,u} \quad & \sum_{k=0}^{N-1} \ell(x_k,u_k) + E(x_N) \\
\text{s.t.} \quad & x_0 = \bar{x}_0 \\
& x_{k+1} = f(x_k,u_k) \\
& 0 \geq h(x_k,u_k), \; k = 0,\dots,N-1 \\
& 0 \geq r(x_N)
\end{aligned}
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Variables $x = (x_0,\dots,x_N)$ and $u = (u_0,\dots,u_{N-1})$ can be summarized in vector $w = (x,u) \in \mathbb{R}^n$.

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Nonlinear MPC solves Nonlinear Programs (NLP)

Discrete time NMPC Problem (an NLP)

$$
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Outline of the lecture

Algebraic characterization of **unconstrained** local optima

Consider the unconstrained problem: $\min_{w \in \mathbb{R}^n} \quad F(w)$

First-Order **Necessary** Condition of Optimality (FONC)

w^* local optimum $\quad \Rightarrow \quad \nabla F(w^*) = 0, \ w^*$ stationary point

Second-Order **Necessary** Condition of Optimality (SONC)

w^* local optimum $\quad \Rightarrow \quad \nabla^2 F(w^*) \succeq 0$

Algebraic characterization of **unconstrained** local optima

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Second-Order **Necessary** Condition of Optimality (SONC)

$$w^* \text{ local optimum} \quad \Rightarrow \quad \nabla^2 F(w^*) \succeq 0$$

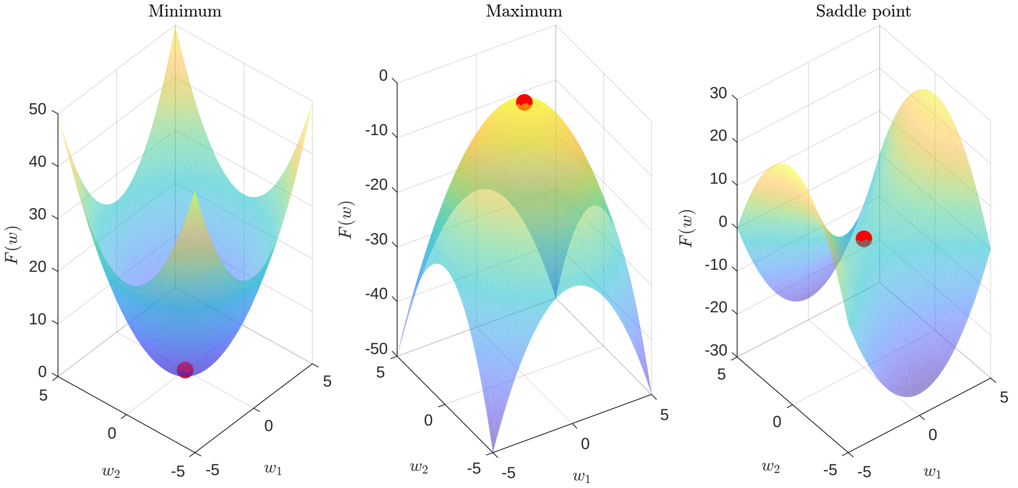
Second-Order **Sufficient** Conditions of Optimality (SOSC)

$$\nabla F(w^*) = 0 \text{ and } \nabla^2 F(w^*) \succ 0 \quad \Rightarrow \quad x^* \text{ strict local minimum}$$

$$\nabla F(w^*) = 0 \text{ and } \nabla^2 F(w^*) \prec 0 \quad \Rightarrow \quad x^* \text{ strict local maximum}$$

No conclusion can be drawn in the case $\nabla^2 F(w^*)$ is indefinite!

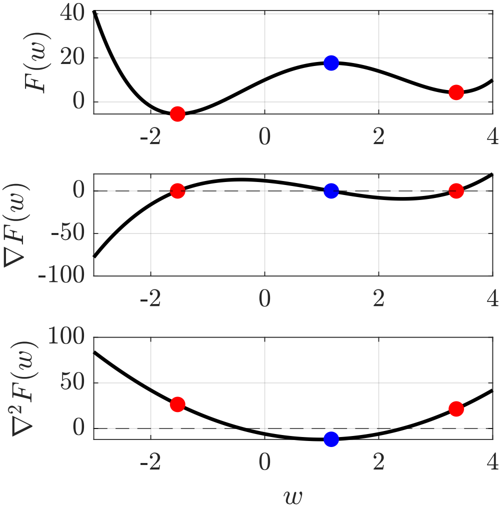
Type of stationary points



A stationary point can be a minimum, maximum or a saddle point

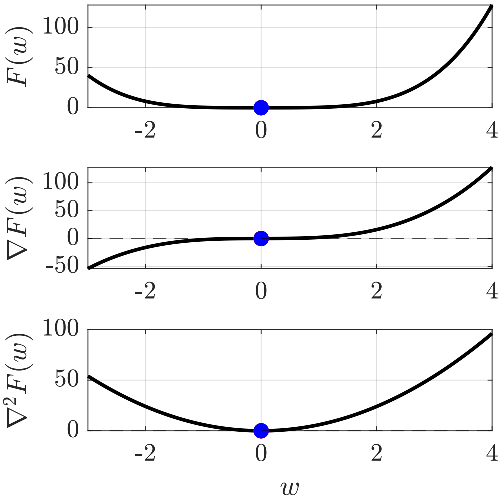
Optimality conditions - unconstrained

- Necessary conditions: find a candidate point (or to exclude points)
- Sufficient conditions: verify optimality of a candidate point



Optimality conditions - unconstrained

- Necessary conditions: find a candidate point (or to exclude points)
- Sufficient conditions: verify optimality of a candidate point
- A minimizer must satisfy SONC, but does not have to satisfy SOSC



FONC for equality constraints

Nonlinear Program (NLP)

$$\min_{w \in \mathbb{R}^n} \; F(w)$$
$$\text{s.t. } G(w) = 0$$

$\mathcal{L}(w, \lambda) = F(w) - \lambda^\top G(w)$ is the Lagrangian

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Definition (LICQ)

A point w satisfies Linear Independence Constraint Qualification LICQ if and only if $\nabla G\left(w\right)$ is full column rank

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First-order Necessary Conditions

Let F, G in $\mathcal{C}^1$. If w^* is a (local) minimizer, **and** w^* satisfies LICQ, then there is a unique vector λ such that:

$$\nabla_w \mathcal{L}(w^*, \lambda^*) = \nabla F(w^*) - \nabla G(w^*)\lambda = 0 \qquad \text{Dual feasibility}$$
$$\nabla_\lambda \mathcal{L}(w^*, \lambda^*) = G(w^*) = 0 \qquad \text{Primal feasibility}$$

The KKT conditions

Nonlinear Program (NLP)

$$\min_{w \in \mathbb{R}^n} \; F(w)$$
$$\text{s.t.} \; G(w) = 0$$
$$H(w) \geq 0$$

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$$[\nabla G(w), \quad \nabla H_{\mathcal{A}}(w)]$$

is full column rank

Active set $\mathcal{A} = \{i \mid H_i(w) = 0\}$

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Theorem (KKT conditions)

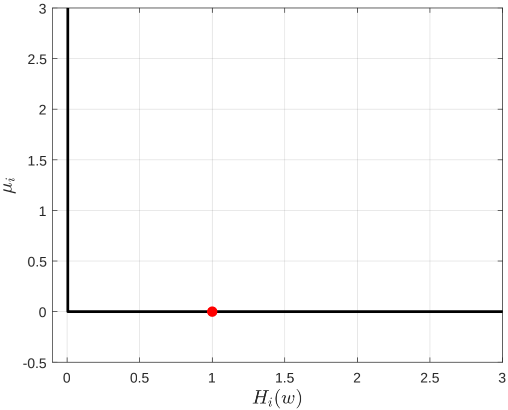
Let F, G, H be $\mathcal{C}^1$. If w^* is a (local) minimizer ***and*** *satisfies LICQ, then there are unique vectors λ^* and μ^* such that (w^*, λ^*, μ^*) satisfies:*

$\nabla_w \mathcal{L}(w^*, \mu^*, \lambda^*) = 0, \quad \mu^* \geq 0,$	*Dual feasibility*
$G(w^*) = 0, \quad H(w^*) \geq 0$	*Primal feasibility*
$\mu_i^* H_i(w^*) = 0, \quad \forall\, i$	*Complementary slackness*

The complementarity slackness condition

Active constraints:

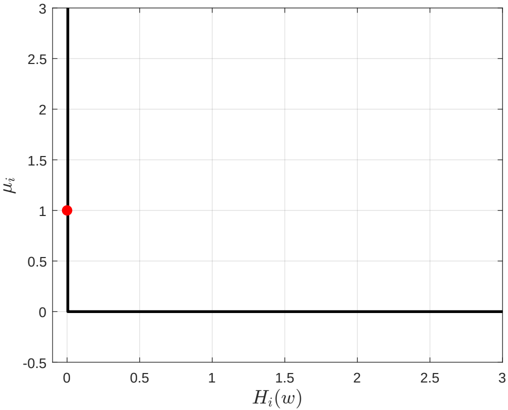
- $H_i(w^*) > 0$ then $\mu_i^* = 0$, and H_i is inactive



The complementarity slackness condition

Active constraints:

- $H_i(w^*) > 0$ then $\mu_i^* = 0$, and H_i is inactive
- $\mu_i^* > 0$ and $H_i(w) = 0$ then $H_i(w)$ is strictly active



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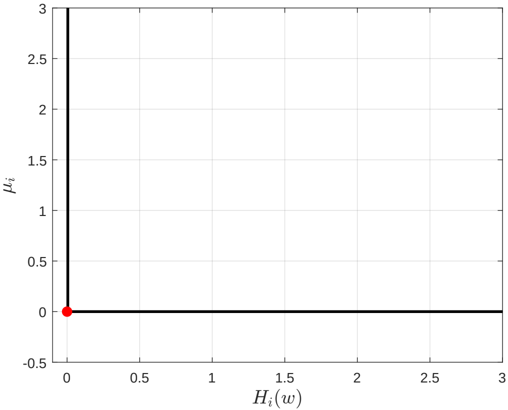
- $\mu_i^* = 0$ and $H_i(w) = 0$ then then $H_i(w)$ is weakly active



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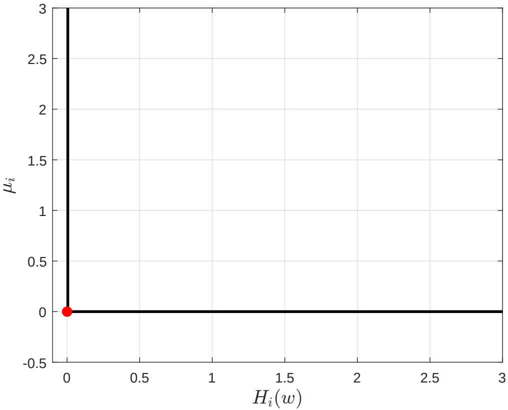
- $\mu_i^* = 0$ and $H_i(w) = 0$ then then $H_i(w)$ is weakly active
- We define the **active set** $\mathbb{A}^*$ as the set of indices i of the active constraints



$$\min_{w \in \mathbb{R}^n} \; F(w)$$
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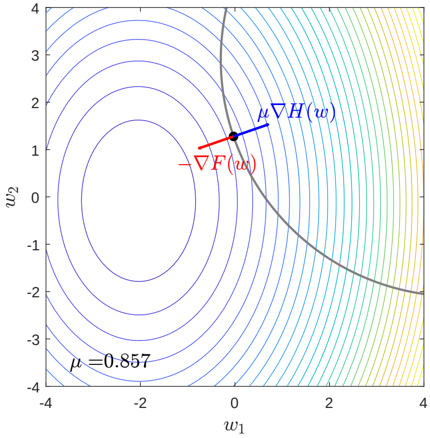
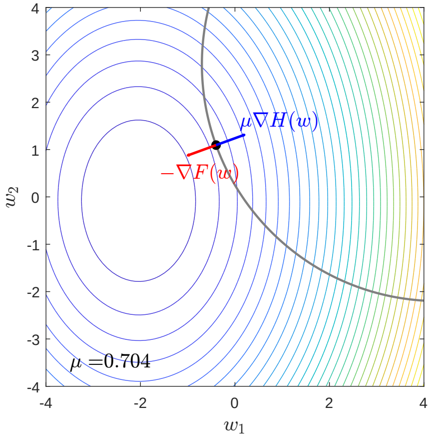


Some intuitions on the KKT conditions

Ball rolling down a valley blocked by a fence

$$\min_{w \in \mathbb{R}^n} \; F(w)$$
$$\text{s.t. } H(w) \geq 0$$

- $-\nabla F$ is the gravity



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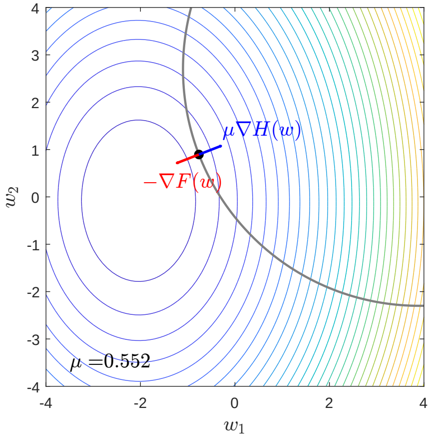

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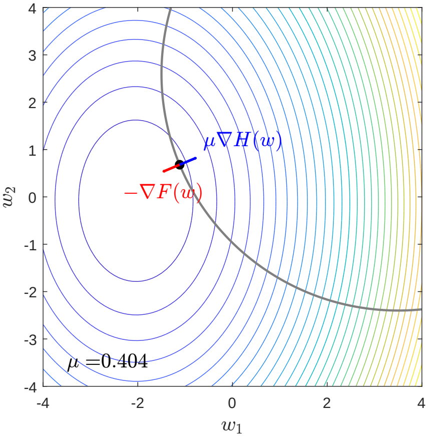
- $\mu \nabla H$ is the force of the fence. Sign $\mu \geq 0$ means the fence can only "push" the ball



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- $-\nabla F$ is the gravity
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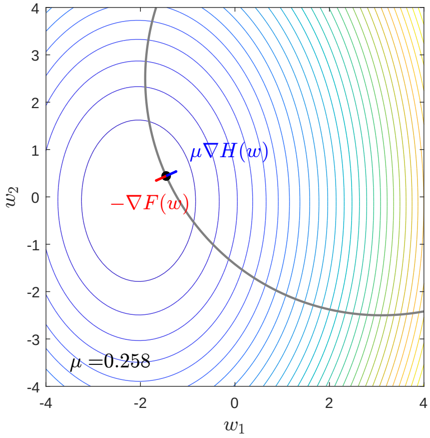
- ∇H gives the direction of the force and μ adjusts the magnitude.



Some intuitions on the KKT conditions

Ball rolling down a valley blocked by a fence

$$\min_{w \in \mathbb{R}^n} \; F(w)$$
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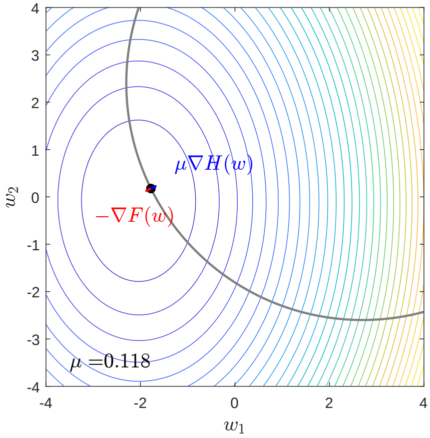

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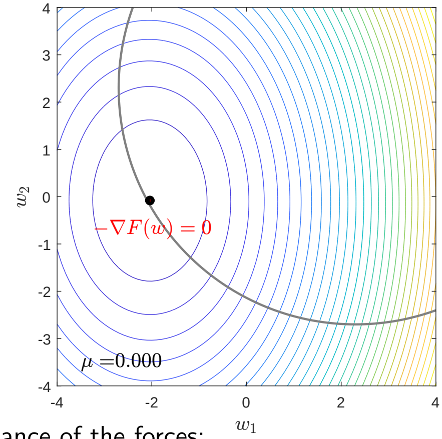

Balance of the forces:

$$\nabla \mathcal{L}(w, \mu) = \nabla F(w) - \mu \nabla H(w) = 0$$

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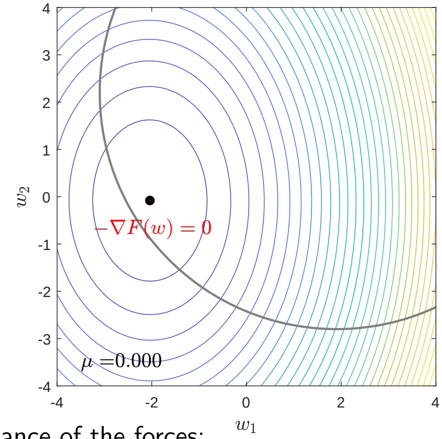
- Inactive constraint $H(w) > 0, \;\; \mu = 0$

$$H(w) > 0, \quad \mu = 0$$

- Complementary slackness $\mu H = 0$ describes a contact problem



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Summary of optimality conditions

Optimality conditions for NLP with equality and/or inequality constraints:

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Nonconvex problem ⇒ minimum is not necessarily globa.
But some nonconvex problems have a unique minimum

Some important practical consequences...

- A local (global) optimum **may not** be a KKT point
- A KKT point **may not** be a local (global) optimum
  ... the lack of equivalence results from a lack of regularity and/or SOSC
- A local (global) optimum **may not** be a KKT point
  ... due to violation of constraint qualifications, e.g. LICQ violated (Covered by C. Kirches)

Outline of the lecture

Newton's method

To solve a nonlinear system, solve a sequence of linear systems

Linearization of F at linearization point $\bar{w}$

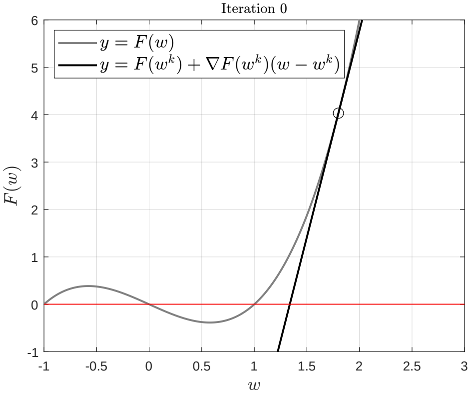
equals

First order Taylor series at $\bar{w}$

equals

$$F_{\mathrm{L}}(w;\bar{w}) := F(\bar{w}) + \frac{\partial F}{\partial w}(\bar{w}) \quad (w - \bar{w})$$

(for continuously differentiable $F : \mathbb{R}^n \to \mathbb{R}^n$)



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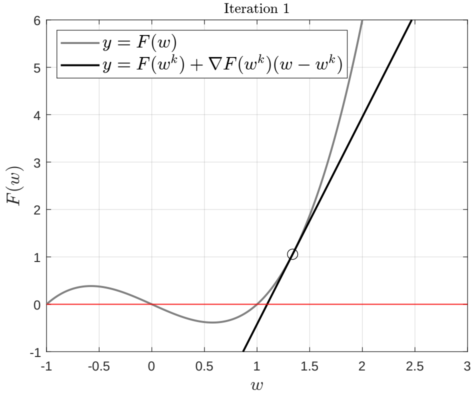

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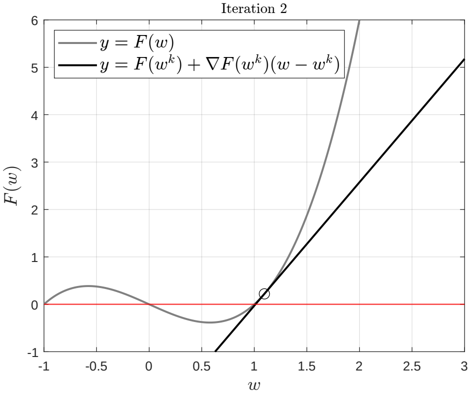

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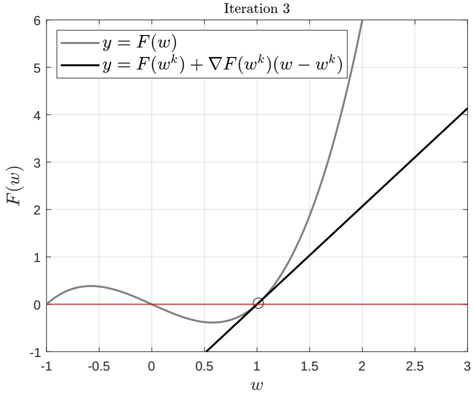


General Nonlinear Program (NLP)

In direct methods, we have to solve the discretized optimal control problem, which is a Nonlinear Program (NLP)

General Nonlinear Program (NLP)

$$\min_{w} F(w) \;\; \text{s.t.} \;\; \left\{ \begin{array}{rcl} G(w) & = & 0 \\ H(w) & \geq & 0 \end{array} \right.$$

We first treat the case without inequalities

NLP only with equality constraints

$$\min_{w} F(w) \;\; \text{s.t.} \quad G(w) \;\; = \;\; 0$$

Lagrange function and optimality conditions

Lagrange function

$$\mathcal{L}(w, \lambda) = F(w) - \lambda^T G(w)$$

Then for an optimal solution w^* exist multipliers λ^* such that

Nonlinear root-finding problem

$$\begin{aligned} \nabla_w \mathcal{L}(w^*, \lambda^*) &= 0 \\ G(w^*) &= 0 \end{aligned}$$

Newton's Method on optimality conditions

How to solve nonlinear equations

$$\begin{aligned} \nabla_w \mathcal{L}(w^*, \lambda^*) &= 0 \\ G(w^*) &= 0 \quad ? \end{aligned}$$

Linearize!

$$\begin{array}{rrrcl} \nabla_w \mathcal{L}(w^k, \lambda^k) & +\nabla_w^2 \mathcal{L}(w^k, \lambda^k)\Delta w & -\nabla_w G(w^k)\Delta\lambda & = & 0 \\ G(w^k) & +\nabla_w G(w^k)^T \Delta w & & = & 0 \end{array}$$

This is equivalent, due to $\nabla \mathcal{L}(w^k, \lambda^k) = \nabla F(w^k) - \nabla G(w^k)\lambda^k$ with the shorthand $\lambda^+ = \lambda^k + \Delta\lambda$ to

$$\begin{array}{rrrcl} \nabla_w F(w^k) & +\nabla_w^2 \mathcal{L}(w^k, \lambda^k)\Delta w & -\nabla_w G(w^k)\lambda^+ & = & 0 \\ G(w^k) & +\nabla_w G(w^k)^T \Delta w & & = & 0 \end{array}$$

Newton Step = Quadratic Program

Conditions

$$\begin{array}{rcccl} \nabla_w F(w^k) & +\nabla_w^2 \mathcal{L}(w^k, \lambda^k)\Delta w & -\nabla_w G(w^k)\lambda^+ & = & 0 \\ G(w^k) & +\nabla_w G(w^k)^T \Delta w & & = & 0 \end{array}$$

are optimality conditions of a quadratic program (QP), namely:

Quadratic program

$$\begin{array}{rl} \min\limits_{\Delta w} & \nabla F(w^k)^T \Delta w + \frac{1}{2}\Delta w^T A^k \Delta w \\ \text{s.t.} & G(w^k) + \nabla G(w^k)^T \Delta w \quad = \quad 0, \end{array}$$

with

$$A^k = \nabla_w^2 \mathcal{L}(w^k, \lambda^k)$$

Newton's method

The full step Newton's Method iterates by solving in each iteration the Quadratic Progam

$$\begin{aligned} \min_{\Delta w} \quad & \nabla F(w^k)^T \Delta w + \frac{1}{2} \Delta w^T A^k \Delta w \\ \text{s.t.} \quad & G(w^k) + \nabla G(w^k)^T \Delta w = 0, \end{aligned}$$

with $A^k = \nabla_w^2 \mathcal{L}(w^k, \lambda^k)$. This obtains as solution the step Δw^k and the new multiplier $\lambda_{\mathrm{QP}}^+ = \lambda^k + \Delta\lambda^k$

New iterate

$$\begin{aligned} w^{k+1} &= w^k + \Delta w^k \\ \lambda^{k+1} &= \lambda^k + \Delta\lambda^k = \lambda_{\mathrm{QP}}^+ \end{aligned}$$

This Newton's method is also called "Sequential Quadratic Programming (SQP) for equality constrained optimization" (with "exact Hessian" and "full steps")

NLP with Inequalities

Regard again NLP with both, equalities and inequalities:

NLP with equality and inequality constraints

$$\min_{w} F(w) \quad \text{s.t.} \quad \begin{cases} G(w) & = & 0 \\ H(w) & \geq & 0 \end{cases}$$

Lagrangian function for NLP with equality and inequality constraints

$$\mathcal{L}(w, \lambda, \mu) = F(w) - \lambda^T G(w) - \mu^T H(w)$$

Optimality conditions with inequalities

Theorem (Karush-Kuhn-Tucker (KKT) conditions)

Let F, G, H be $\mathcal{C}^2$. If w^* is a (local) minimizer and satisfies LICQ, then there are unique vectors λ^* and μ^* such that (w^*, λ^*, μ^*) satisfies:

$$
\begin{aligned}
&\nabla_w \mathcal{L}(w^*, \mu^*, \lambda^*) = 0 \\
&G(w^*) = 0 \\
&H(w^*) \geq 0 \\
&\mu^* \geq 0 \\
&H(w^*)^\top \mu^* = 0
\end{aligned}
$$

- These contain nonsmooth conditions (the last three) which are called *complementarity conditions*
- This system cannot be solved by Newton's Method. But still with SQP...

Sequential Quadratic Programming (SQP)

By Linearizing all functions within the KKT Conditions, and setting $\lambda^+ = \lambda^k + \Delta\lambda$ and $\mu^+ = \mu^k + \Delta\mu$, we obtain the KKT conditions of a Quadratic Program (QP) (we omit these conditions).

QP with inequality constraints

$$\min_{\Delta w} \quad \nabla F(w^k)^T \Delta w + \frac{1}{2} \Delta w^T A^k \Delta w$$
$$\text{s.t.} \quad \begin{cases} G(w^k) + \nabla G(w^k)^T \Delta w &= 0 \\ H(w^k) + \nabla H(w^k)^T \Delta w &\geq 0 \end{cases}$$

with

$$A^k = \nabla_w^2 \mathcal{L}(w^k, \lambda^k, \mu^k)$$

and its solution delivers

$$\Delta w^k, \quad \lambda_{\mathrm{QP}}^+, \quad \mu_{\mathrm{QP}}^+$$

Constrained Gauss-Newton Method

In special case of least squares objectives

Least squares objective function

$$F(w) = \frac{1}{2}\|R(w)\|_2^2$$

can approximate Hessian $\nabla_w^2 \mathcal{L}(w^k, \lambda^k, \mu^k)$ by much cheaper

$$A^k = \nabla R(w) \nabla R(w)^T.$$

Need no multipliers to compute A^k! QP$=$ linear least squares:

Gauss-Newton QP

$$\begin{array}{rll} \displaystyle\min_{\Delta w} & \multicolumn{2}{l}{\frac{1}{2}\|R(w^k) + \nabla R(w^k)^T \Delta w\|_2^2} \\ \text{s.t.} & G(w^k) + \nabla G(w^k)^T \Delta w & = 0 \\ & H(w^k) + \nabla H(w^k)^T \Delta w & \geq 0 \end{array}$$

Convergence: linear (better if $\|R(w^*)\|$ small)

Interior point methods

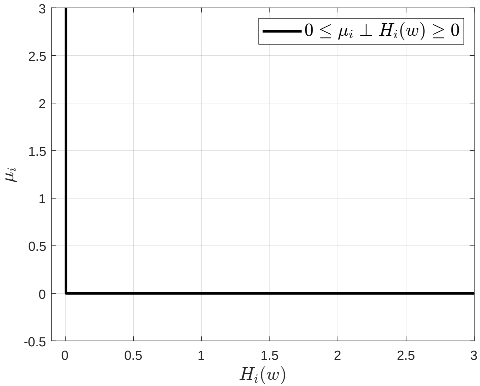
NLP with inequalites

$$\min_{w} \; F(w)$$
$$\text{s.t.} \quad H(w) \geq 0$$

KKT conditions

$$\nabla F(w) - \nabla H(w)^\top \mu = 0$$
$$0 \leq \mu \perp H(w) \geq 0$$

Main difficulty: inequality conditions introduce nonsmoothness in the KKT conditions



The barrier problem

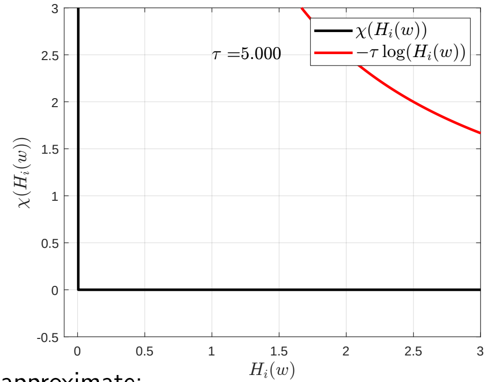
NLP with inequalites

$$\min_{w} \; F(w) \quad \text{s.t.} \quad H(w) \geq 0$$

Barrier problem

$$\min_{w} \; F(w) - \tau \sum_{i=1}^{m} \log(H_i(w)) =: F_\tau(w)$$

Main idea: put inequality constraint into objective



approximate:

$$\chi(H_i(w)) = \begin{cases} 0 & \text{if } H_i(w) \geq 0 \\ \infty & \text{if } H_i(w) < 0 \end{cases}$$

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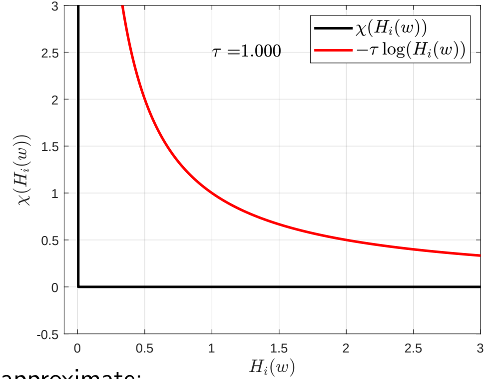

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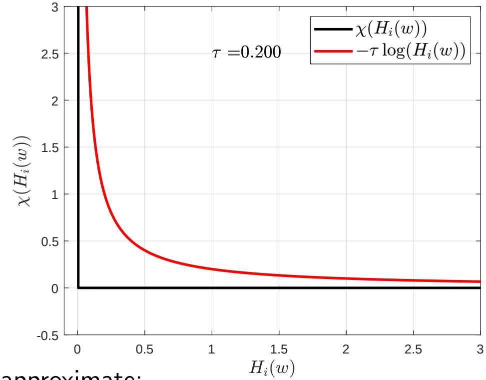

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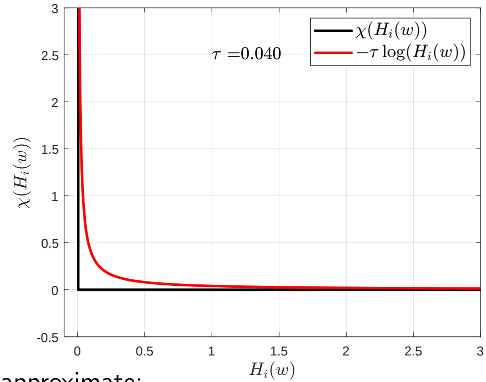

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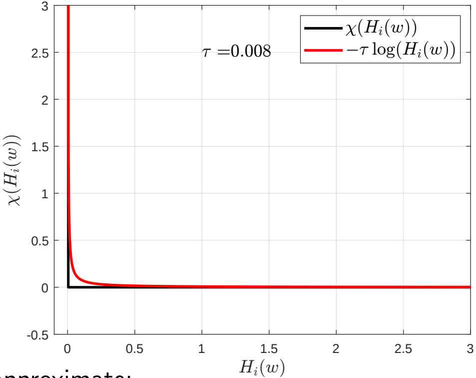

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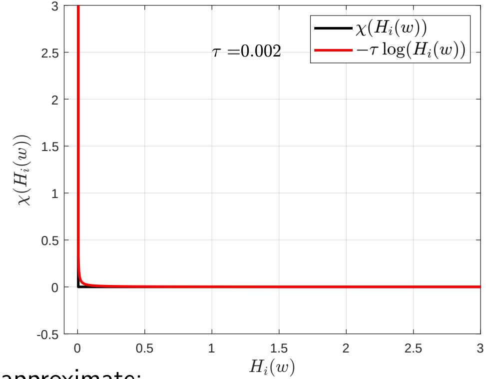

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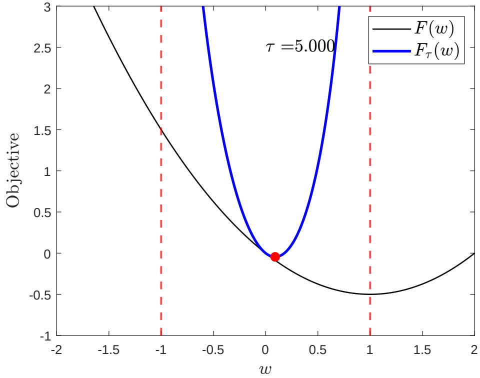
An example of the barrier problem

Example NLP

$$\min_{w} \; 0.5w^2 - 2w$$
$$\text{s.t.} \quad -1 \leq w \leq 1$$

Barrier problem

$$\min_{w} \; 0.5w^2 - 2 - \tau \log(w+1) - \tau \log(1-w)$$



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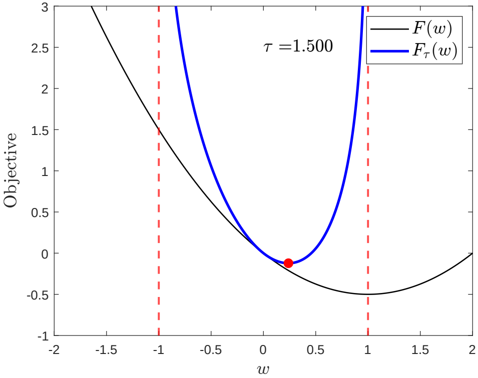

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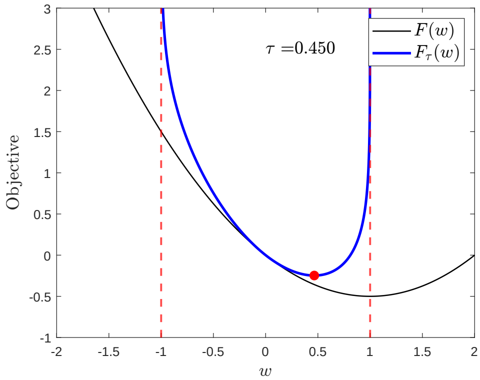

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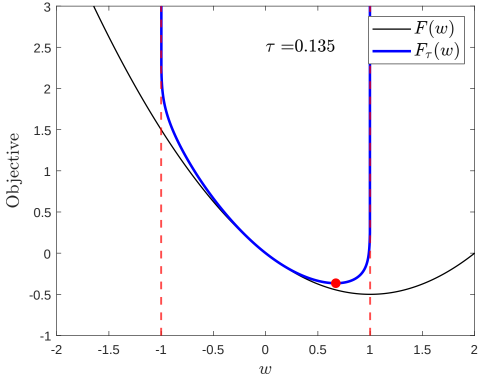

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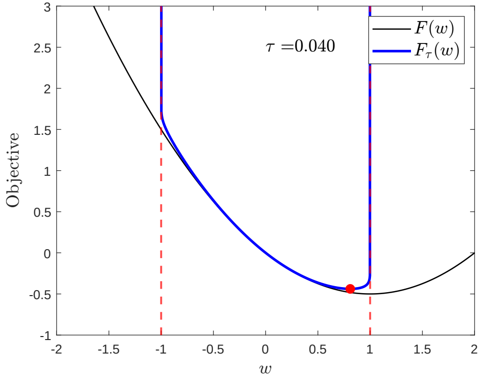

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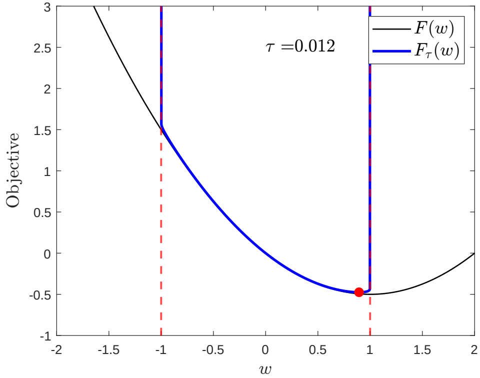

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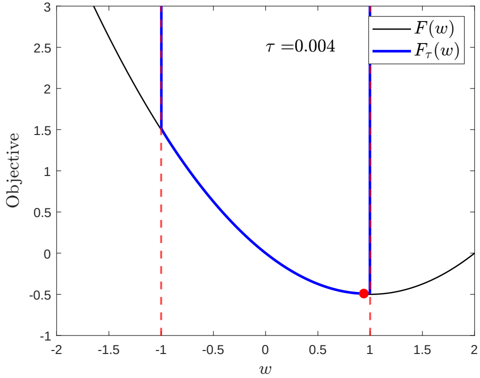

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KKT conditions

$$\nabla F(w) - \tau \sum_{i-1}^{m} \frac{1}{H_i(w)} \nabla H_i(w) = 0$$

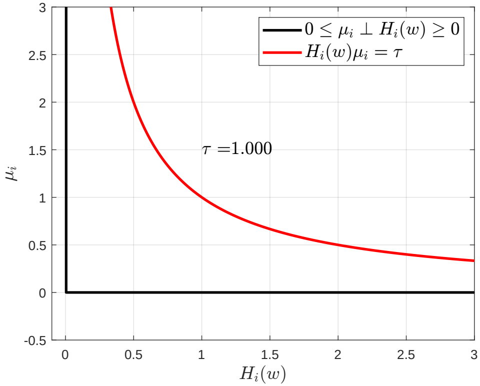
Introduce variable $\mu_i = \frac{\tau}{H_i(w)}$

Smoothed KKT conditions

$$\nabla F(w) - \nabla H(w)^\top \mu = 0$$

$$H_i(w)\mu_i = \tau$$

$$(H_i(w) > 0, \mu_i > 0)$$



Primal-dual interior point methods

Alternative interpretation

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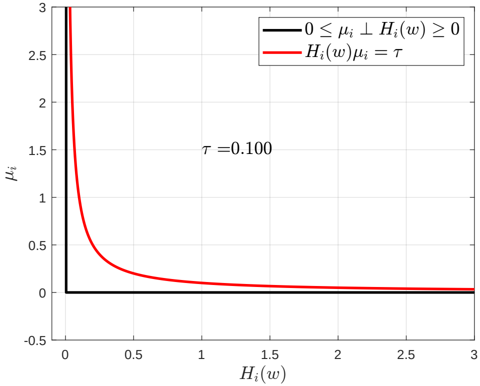

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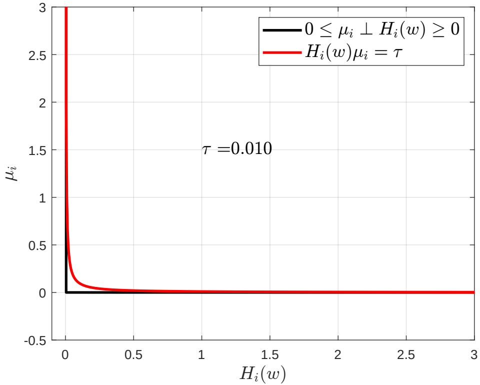
Introduce variable $\mu_i = \frac{\tau}{H_i(w)}$

Smoothed KKT conditions

$$\nabla F(w) - \nabla H(w)^\top \mu = 0$$
$$H_i(w)\mu_i = \tau$$
$$(H_i(w) > 0, \mu_i > 0)$$



Primal-dual interior point methods

Alternative interpretation

Barrier problem

$$\min_{w} \; F(w) - \tau \sum_{i=1}^{m} \log(H_i(w)) =: F_\tau(w)$$

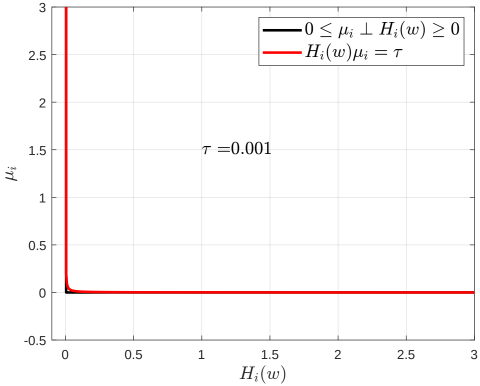
KKT conditions

$$\nabla F(w) - \tau \sum_{i-1}^{m} \frac{1}{H_i(w)} \nabla H_i(w) = 0$$

Introduce variable $\mu_i = \frac{\tau}{H_i(w)}$

Smoothed KKT conditions

$$\nabla F(w) - \nabla H(w)^\top \mu = 0$$
$$H_i(w)\mu_i = \tau$$
$$(H_i(w) > 0, \mu_i > 0)$$



Primal-dual interior point method

Nonlinear programming problem

$$\begin{aligned} \min_{w} \quad & F(w) \\ \text{s.t.} \quad & G(w) = 0 \\ & H(w) \geq 0 \end{aligned}$$

Smoothed KKT conditions

$$R_\tau(w, s, \lambda, \mu) = \begin{bmatrix} \nabla_w \mathcal{L}(w, \lambda, \mu) \\ G(w) \\ H(w) - s \\ \mathrm{diag}(s)\mu - \tau e \end{bmatrix} = 0$$

$(s, \mu > 0)$

$e = (1, \dots, 1)$

Solve approximately with Newton's method for fixed τ

$$R_\tau(w, s, \lambda, \mu) + \nabla R_\tau(w, s, \lambda, \mu)\Delta z = 0$$

with $z = (w, s, \lambda, \mu)$

Line-serach

Find $\alpha \in (0, 1)$

$$\begin{aligned} w^{k+1} &= w^k + \alpha \Delta w \\ s^{k+1} &= s^k + \alpha \Delta s \\ \lambda^{k+1} &= \lambda^k + \alpha \Delta \lambda \\ \mu^{k+1} &= \mu^k + \alpha \Delta \mu \end{aligned}$$

such that $s^{k+1} > 0, \mu^{k+1} > 0$

and reduce τ...

Summary Newton-type optimization

- Newton type optimization solves the necessary optimality conditions
- Newton's method linearizes the nonlinear system in each iteration
- for constraints, need Lagrangian function, and KKT conditions
- for equalities KKT conditions are smooth, can apply Newton's method
- for inequalities KKT conditions are non-smooth, can apply Sequential Quadratic Programming (SQP)
- QPs with inequalities can be solved with interior point methods
- Also NLPs with inequalities can be solved with interior point methods (e.g. by the IPOPT solver)

References

- Moritz Diehl, Sébastien Gros. "Numerical optimal control (Draft)," Lecture notes, 2019.
- Jorge Nocedal, Stephen J. Wright, Numerical optimization. New York, NY: Springer New York, 2006.