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Mathematical Programs with Complementarity Constraints Part 3: Pivoting-based algorithms

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Overview

- Linear Complementarity Problems (LCPs)
  - Lemke's Method
- Linear Programs with Complementarities (LPCCs)
  - Bound constrained LPCCs
  - A primal active set algorithm for general LPCCs
  - The dual of an LPCC
- Several active-set methods for general MPCCs
  - Scholtes' method
  - The method of Fukushima and Tseng

The Linear Complementarity Problem

The feasibility problem for an MPEC (with constraints linearized if need be) is a **Linear Complementarity Problem** (LCP):

(LCP($\boldsymbol{q}$, M))

$$\boxed{\begin{aligned} \boldsymbol{w} &= \boldsymbol{q} + M\boldsymbol{z} \\ \boldsymbol{w} &\geqslant \boldsymbol{0},\ \boldsymbol{z} \geqslant \boldsymbol{0} \\ \boldsymbol{z}^T\boldsymbol{w} &= 0 \end{aligned}}$$

$q = 0$: homogeneous LCP, z^* a solution $\Rightarrow \lambda z^*$ a solution for $\lambda \geqslant 0$.

In particular $z^* = 0$ is a solution if $q = 0$. Existence of nonzero solutions is the question.

LCPs Historically

Historically, LCPs were coined to unify the optimality systems of LPs, QPs, and bi-matrix games. Their KKT conditions form an LCP.

Example (QP):

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \boldsymbol{c}^T\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} \text{ s.t. } A\boldsymbol{x} \geqslant \boldsymbol{b}$$
$$\boldsymbol{x} \geqslant 0$$

with Q symmetric has the KKT conditions

$$\boldsymbol{\nu} := \boldsymbol{c}^T + Q\boldsymbol{x} - A^T\boldsymbol{\mu} \geqslant \boldsymbol{0},\ \boldsymbol{x} \geqslant \boldsymbol{0},\ \boldsymbol{x}^T\boldsymbol{\nu} = 0$$
$$\boldsymbol{r} := -\boldsymbol{b} + A\boldsymbol{x} \geqslant \boldsymbol{0},\ \boldsymbol{\mu} \geqslant \boldsymbol{0},\ \boldsymbol{r}^T\boldsymbol{\mu} = 0$$

Now to obtain the LCP notation we simply let

$$q = \begin{pmatrix} \boldsymbol{c} \\ -\boldsymbol{b} \end{pmatrix},\ M = \begin{pmatrix} Q & -A^T \\ A & 0 \end{pmatrix}, \boldsymbol{z} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\mu} \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} \boldsymbol{\nu} \\ \boldsymbol{r} \end{pmatrix}$$

with M positive definite if Q is.

QP view on LCPs

Vice versa, every LCP admits a QP formulation.

The LCP

$$(\mathrm{LCP}(\boldsymbol{q}, M)) \qquad \boxed{\begin{aligned} \boldsymbol{w} &= \boldsymbol{q} + M\boldsymbol{z} \\ \boldsymbol{w} &\geqslant \mathbf{0},\ \boldsymbol{z} \geqslant \mathbf{0} \\ \boldsymbol{z}^T\boldsymbol{w} &= 0 \end{aligned}}$$

with general M is the set of optimality conditions for the QP

$$\begin{aligned} &\min_{\boldsymbol{z}}\ \boldsymbol{z}^T(\boldsymbol{q} + M\boldsymbol{z}) \\ &\text{s.t. } \boldsymbol{q} + M\boldsymbol{z} \geqslant \mathbf{0}, \boldsymbol{z} \geqslant 0 \end{aligned}$$

Note that M unsymmetric is ok because it can be moved to a constraint.

$$\begin{aligned} \min_{\boldsymbol{z},\boldsymbol{z}}\ & \frac{1}{2}\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{z}\end{pmatrix}^T \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{z}\end{pmatrix} \\ \text{s.t. } & \boldsymbol{s} = \boldsymbol{q} + M\boldsymbol{z} \\ & \boldsymbol{s}, \boldsymbol{z} \geqslant 0 \end{aligned}$$

The QP is non-convex; its objective is bounded from below by 0. Solutions of (LCP) are global minimizers of the QP with objective 0.

Lemke's Method

Lemke's "Scheme I" for solving

$$\boldsymbol{w} = \boldsymbol{q} + \boldsymbol{d} z_0 + M\boldsymbol{z} \geqslant \boldsymbol{0},\ z_0 \geqslant 0, \boldsymbol{z} \geqslant \boldsymbol{0}, \boldsymbol{z}^T \boldsymbol{w} = 0.$$

Input $(\boldsymbol{q}, \boldsymbol{d}, M)$ where $\boldsymbol{d}$ ("covering vector") can be chosen, e.g. $\boldsymbol{d} = \boldsymbol{1}$.

1. Stop if $\boldsymbol{q} \geqslant \boldsymbol{0}$: $\boldsymbol{z} = \boldsymbol{0}$ is a solution for LCP(q, M).
2. Let $\bar{z}_0 \geqslant 0$ be the smallest value for which $\boldsymbol{q} + \boldsymbol{d} z_0 \geqslant \boldsymbol{0}$. Let $w_r = \operatorname{argmin}\{\boldsymbol{w} \mid z_0 = \bar{z}_0\}$. Pivot on (w_r, z_0). Let z_r be the driving variable.
3. If the driving variable's column has a negative entry, find the blocking basic variable with minimum ratio. Stop if unblocked: LCP(q, M) is infeasible if M falls into certain classes.
4. If z_0 blocked, pivot on $(z_0, \text{driving variable})$ and stop. LCP(q, M) is solved. If some other variable blocked, then pivot on $(\text{blocking variable}, \text{driving variable})$ and go to 3., using the complementary paired variable of the blocking one as new driving variable.

Matrix Classes

LCP theory introduces classifications for matrix M that characterize the difficulty of the LCP, e.g. that of finding a feasible point of a linearized MPCC:

- S-matrices (square and $\exists \boldsymbol{z} \geqslant \mathbf{0} : M\boldsymbol{z} \geqslant \mathbf{0}$): LCP$(\boldsymbol{q}, M)$ feasible for all $\boldsymbol{q}$.

  positive definite $\Rightarrow$ S-matrix

- P-matrices (all principal minors positive) $\Leftrightarrow$ LCP$(\boldsymbol{q}, M)$ has a unique solution for all $\boldsymbol{q}$.

  S-matrix $\Rightarrow$ P-matrix

- E-matrices (strictly semi-monotone):
  $\mathbf{0} \neq \boldsymbol{x} \geqslant \mathbf{0} \;\Rightarrow\; [x_k > 0$ and $(M\boldsymbol{x})_k > 0$ for some $k]$.

  P-matrix $\Rightarrow$ E-matrix. Lemke's Scheme I always succeeds for E-matrices.

There are many mmore positive results on Lemke's Scheme I in the literature.

Bound constrained LPCCs

Consider an LPCC with just two-sided bounds on the variables, i.e. without affine line constraints:

$$\begin{aligned} \min_{\boldsymbol{u}\in\mathbb{R}^c,\boldsymbol{v}\in\mathbb{R}^c} \quad & \boldsymbol{c}^T\boldsymbol{u} + \boldsymbol{d}^T\boldsymbol{v} \\ \text{s.t. } & \underline{\boldsymbol{u}} \leqslant \boldsymbol{u} \perp \boldsymbol{v} \geqslant \underline{\boldsymbol{v}} \\ & \boldsymbol{u} \leqslant \bar{\boldsymbol{u}},\ \boldsymbol{v} \leqslant \bar{\boldsymbol{v}} \end{aligned}$$

This problem is <u>not</u> combinatorial in nature, but can be trivially decomposed:

$$\left.\begin{aligned} \min_{u_i\in\mathbb{R},v_i\in\mathbb{R}} \quad & c_i u_i + d_i v_i \\ \text{s.t.} \quad & \underline{u}_i \leqslant u_i \perp v_i \geqslant \underline{v}_i \\ & u_i \leqslant \bar{u}_i,\ v_i \leqslant \bar{v}_i \end{aligned}\right\} \ 1 \leqslant i \leqslant c$$

with explicit solutions as follows:

$$\begin{aligned} \hat{u}_i &= \underline{u}_i \text{ if } c_i \geqslant 0 \text{ and } \bar{u}_i \text{ otherwise} & \hat{v}_i &= \underline{v}_i \text{ if } d_i \geqslant 0 \text{ and } \bar{v}_i \text{ otherwise} \\ u_i^* &= \hat{u}_i \text{ if } \hat{u}_i c_i \leqslant \hat{v}_i d_i \text{ and } \underline{u}_i \text{ otherwise} & v_i^* &= \hat{v}_i \text{ if } \hat{v}_i d_i < \hat{u}_i c_i \text{ and } \underline{v}_i \text{ otherwise.} \end{aligned}$$

This means the Augmented Lagrangian + SLPCC-EQP approach for solving MPCCs has cheap and regular subproblems.

Scholtes' method

A general descent-aided enumerative procedure for NLPs with structured nonconvexity:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} F(\boldsymbol{x}) \text{ s.t.} \boldsymbol{C}(\boldsymbol{x}) \in Z = \bigcup_{i=1}^{c} Z_i$$

with $Z_i \subset \mathbb{R}^n$ sufficiently regular and Z locally star-shaped, such that:

- Every stationary point $\boldsymbol{x}^*$ then $G(\boldsymbol{x}^*) \in Z_i$ for at least one index i;
- If a feasible point $\boldsymbol{x}$ is stationary for all adjacent Z_i, it is stationary for Z.

For MPCC, the Z_i can be chosen to be the feasible sets of the piece NLPs.

1. Pick Z_0, $\boldsymbol{x}_0$ with $C(\boldsymbol{x}_0) \in Z_0$, $k \leftarrow 0$.
2. Solve $\min_{\boldsymbol{x}\in\mathbb{R}^n} F(\boldsymbol{x})$ s.t.$C(\boldsymbol{x}) \in Z_k$ starting from $\boldsymbol{x}^{k-1}$.
3. Let $\boldsymbol{x}_k^*$ be the stationary point found in Z_k: Verify stationarity for all adjacent pieces and stop if successful.
4. Otherwise, choose a Z_{k_i} adjacent to $\boldsymbol{x}_k^*$ for which $\boldsymbol{x}_k^*$ is not stationary and go to 2.

General LPCCs

Consider now an LPCC with m polyhedral constraints $\boldsymbol{a}_i^T \boldsymbol{x} \geqslant b_i$ and a unified notation

$$0 \leqslant (\boldsymbol{a}_i^T \boldsymbol{x} - b_i) \perp (\boldsymbol{a}_{p+i}^T \boldsymbol{x} - b_{p+i}) \geqslant 0$$

for p complementarity pairs:

$$\boxed{\begin{aligned} &\min_{\boldsymbol{x} \in \mathbb{R}^n} \boldsymbol{g}^T \boldsymbol{x} \\ &\text{ s.t. } \boldsymbol{a}_i^T \boldsymbol{x} \geqslant b_i, && 1 \leqslant i \leqslant m \\ &\quad 0 \leqslant (\boldsymbol{a}_i^T \boldsymbol{x} - b_i) \perp (\boldsymbol{a}_{p+i}^T \boldsymbol{x} - b_{p+i}) \geqslant 0, && m+1 \leqslant i \leqslant m+p \end{aligned}}$$

We also introduce the <u>complementarity index map</u>

$$c(i) = \begin{cases} 0 & \text{if} \quad i \leqslant m \\ i+p & \text{if} \quad m+1 \leqslant i \leqslant m+p \\ i-p & \text{if} \quad m+p+1 \leqslant i \leqslant m+2p \end{cases}$$

to be able to write ($c(i) \neq 0$)

$$0 \leqslant (\boldsymbol{a}_i^T \boldsymbol{x} - b_i) \perp (\boldsymbol{a}_{c(i)}^T \boldsymbol{x} - b_{c(i)}) \geqslant 0$$

Note: Can also do this in vertical form to obtain a column basis simplex method.

General LPCCs

We introduce the set of non-strict complementarities at $\boldsymbol{x} \in \mathbb{R}^n$

$$\mathcal{D}(\boldsymbol{x}) = \left\{ m+1 \leqslant i \leqslant m+p \mid \boldsymbol{a}_i^T \boldsymbol{x} = b_i \ \wedge \ \boldsymbol{a}_{c(i)}^T \boldsymbol{x} = b_{c(i)}, \right\}$$

which only includes the smaller index of every non-stricty pair.

If we restrict ourselves to the MPEC-LICQ setting, in which S-stationarity is necessary, a feasible point $\boldsymbol{x} \in \mathbb{R}^n$ is S-stationary if $\boldsymbol{\lambda} \in \mathbb{R}^{m+2p}$ exists such that

$$\begin{aligned} \boldsymbol{g} - \sum_{i=1}^{m+2p} \lambda_i (\boldsymbol{a}_i^T \boldsymbol{x} - b_i) = \boldsymbol{0} & & \\ 0 \leqslant (\boldsymbol{a}_i^T \boldsymbol{x} - b_i) \perp \lambda_i \geqslant 0 & \qquad & 1 \leqslant i \leqslant m \\ \boldsymbol{a}_i^T \boldsymbol{x} > b_i \implies \lambda_i = 0 & \qquad & m+1 \leqslant i \leqslant m+2p \\ \lambda_i \geqslant 0 \ \wedge \ \lambda_{c(i)} \geqslant 0 & \qquad & i \in \mathcal{D}(\boldsymbol{x}) \end{aligned}$$

An LPCC active set method under MPCC-LICQ

Under MPCC-LICQ, we may assume that

- there are exactly n linearly independent active constraints at every vertex
- an initial feasible vertex is given and associated with n linearly independent active constraints.

Given an active set $\mathcal{A} = \{1 \leqslant i \leqslant m + 2p \mid \boldsymbol{a}_i^T \boldsymbol{x} = b_i\}$, complementarity feasibility requires

$$\{i, c(i)\} \cap \mathcal{A} \neq \emptyset, \;\; m + 1 \leqslant i \leqslant m + p.$$

There is no assumption on strictness, hence $\{i, c(i)\} \subseteq \mathcal{A}$ is admitted.

We introduce subsets of the active set

$$\mathcal{A}_0 = \mathcal{A} \cap \{1, \ldots, m\}, \quad \mathcal{A}_1 = \mathcal{A} \cap \{m + 1, \ldots, m + 2p\}.$$

An LPCC active set method under MPCC-LICQ

Given an active set $\mathcal{A}$, we introduce the basis matrix $A = (\boldsymbol{a}_j)_{j \in \mathcal{A}}$ and solve

$$\boldsymbol{x} = A^{-T}\boldsymbol{b}, \quad \boldsymbol{\lambda} = A^{-1}\boldsymbol{g}$$

to find $(\boldsymbol{x}, \boldsymbol{\lambda})$. Doing this efficiently is part of the mojo of all active set and simplex codes.

If we start feasible and maintain feasibility, stationarity and complementary slackness, pivoting takes place towards S-stationarity of $\boldsymbol{\lambda}$.

Let $A^{-T} = (\boldsymbol{s}_j)_{j \in \mathcal{A}}$ and move along $\boldsymbol{s}_j$ to increase $\boldsymbol{a}_j^T \boldsymbol{x}$, which becomes inactive. If $\lambda_j = \boldsymbol{s}_j^T \boldsymbol{g} < 0$, this is an improvement of the objective.

For ordinary LP, any active index j may be chosen (pricing). For LPCC, to maintain complementarity feasibility,

$$j \in \{i \mid \lambda_i < 0 \;\wedge\; (i \in \mathcal{A}_0 \;\vee\; (i \in \mathcal{A}_1 \;\wedge\; c(i) \in \mathcal{A}_1))\}$$

must hold.

Breakdown in absence of MPCC-LICQ

A degenerate example:

$$\begin{aligned} \min_{\boldsymbol{x}\in\mathbb{R}^3} \ -x_1 \text{ s.t. } x_1 - x_2 + x_3 &\geqslant 0 \\ x_1 + x_2 + x_3 &\geqslant 0 \\ -x_1 \qquad\qquad &\geqslant -1 \\ 0 \leqslant x_1 \perp x_2 &\geqslant 0 \end{aligned}$$

Assume $\boldsymbol{x} = (0,0,0)^T$, $\mathcal{A} = \{1,2,4\}$. Then

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \ \boldsymbol{g} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\lambda} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

There is only one eligible pivot, $\lambda_3 = -1 < 0$. The paired complementarity constraint $x_2 \geqslant 0$ is active but $5 \notin \mathcal{A}$ due to degeneracy. We now have two options:

- Following our algorithm we cannot make $x_1 \geqslant 0$ inactive. The method stalls.
- Deviating from our algorithm, we make $x_1 \geqslant 0$ inactive anyway. Then $x_2 \geqslant 0$ must be added to $\mathcal{A}$ to maintain complementarity feasibility. Now the method breaks down as

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

An LPCC active set method without MPCC-LICQ

We need to distinguish between active constraints in the basis matrix, and active constraints to maintain complementarity feasibility:

$$\bar{\mathcal{A}} = \mathcal{A} \cup \mathcal{E}, \quad \mathcal{A} \cap \mathcal{E} = \emptyset,$$

where $\mathcal{E}$ is an <u>extension</u> collecting complementarity constraints that <u>should be</u> active but cannot become so for rank reasons. We now ask for

$$\{i, c(i)\} \cap \bar{\mathcal{A}} \neq \emptyset$$

but allow $\{i, c(i)\} \cap \mathcal{A} = \emptyset$. This now allows to take option 2 as follows:

1. If an index $q \in \mathcal{A}$ leaves and index $c(q)$ is not in $\bar{\mathcal{A}}$, then $c(q)$ is added to $\mathcal{E}$ to maintain complementarity feasibility.
2. If a step ends up making a constraint with index $p \in \mathcal{E}$ active, we move p from $\mathcal{E}$ to $\mathcal{A}$.
3. If a constraint index $p \notin \bar{\mathcal{A}}$ enters $\mathcal{A}$, we remove $c(p)$ from $\mathcal{E}$ if found there (it could also be in $\mathcal{A}$).

This maintains regularity of the basis matrix ($\boldsymbol{a}_p^T s_q \neq 0$).

An LPCC active set method without MPCC-LICQ

Revisiting the degenerate example:

$$\begin{aligned} \min_{\boldsymbol{x} \in \mathbb{R}^3} \ -x_1 \ \text{s.t.}\ x_1 - x_2 + x_3 &\geqslant 0 \\ x_1 + x_2 + x_3 &\geqslant 0 \\ -x_1 \qquad\quad &\geqslant -1 \\ 0 \leqslant x_1 \perp x_2 &\geqslant 0 \end{aligned}$$

Assume $\boldsymbol{x} = (0,0,0)^T$, $\mathcal{A} = \{1,2,4\}$. Then

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},\ \boldsymbol{g} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix},\ \boldsymbol{\lambda} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

There is only one eligible pivot, $\lambda_4 = -1 < 0$. The paired complementarity constraint $x_2 \geqslant 0$ is active but $5 \notin \mathcal{A}$ due to degeneracy. We make $x_1 \geqslant 0$ inactive, $\boldsymbol{s}_4 = (1,0,-1)^T$ and let $\mathcal{E} = \{5\}$ immediately.

The entering constraint now is 3, resulting in $\mathcal{A} = \{1,2,3\}$ and $\boldsymbol{x} = (1,0,-1)^T$. This vertex is S-stationary with $\boldsymbol{\lambda} = (0,0,1)^T$.

An LPCC active set method without MPCC-LICQ

If we lift the CQ restriction, B-stationarity characterizes the situation that no feasible descent (first order) direction exists:

A feasible point $\boldsymbol{x}$ is B-stationary if is a minimizer of all $2^{|\mathcal{D}(\boldsymbol{x})|}$ piece LPs.

Equivalently, for all $\mathcal{P} \subseteq \mathcal{D}(\boldsymbol{x})$, a vector $\boldsymbol{\lambda}^{\mathcal{P}} \in \mathbb{R}^{m+2p}$ exists such that

$$\begin{aligned}
\boldsymbol{g} - \sum_{i=1}^{m+2p} \lambda_i^{\mathcal{P}} (\boldsymbol{a}_i^T \boldsymbol{x} - b_i) &= \boldsymbol{0} && \\
0 \leqslant (\boldsymbol{a}_i^T \boldsymbol{x} - b_i) \perp \lambda_i^{\mathcal{P}} &\geqslant 0 && 1 \leqslant i \leqslant m \\
\boldsymbol{a}_i^T \boldsymbol{x} > b_i \implies \lambda_i^{\mathcal{P}} &= 0 && m+1 \leqslant i \leqslant m+2p \\
\lambda_{c(i)}^{\mathcal{P}} &\geqslant 0 && i \in \mathcal{P} \\
\lambda_i^{\mathcal{P}} &\geqslant 0 && i \in \mathcal{D}(\boldsymbol{x}) \setminus \mathcal{P}
\end{aligned}$$

Note: MPEC-ACQ generically holds for polyhedral MPECs, and M-stationarity is necessary.

An LPCC active set method without MPCC-LICQ

A B-stationarity example:

$$\min_{\boldsymbol{x} \in \mathbb{R}^3} x_1 + x_2 + x_3 \text{ s.t. } \begin{aligned} 4x_1 - x_3 &\geqslant 0 \\ 4x_2 - x_3 &\geqslant 0 \\ 0 \leqslant x_1 \perp x_2 &\geqslant 0 \end{aligned}$$

The non-strict vertex $\boldsymbol{x} = (0, 0, 0)^T$ is the only feasible point. There,

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{pmatrix} \boldsymbol{\lambda}.$$

This has no nonnegative solution in $\boldsymbol{\lambda}$, hence $\boldsymbol{x}$ is not S-stationary.

We have $\mathcal{D} = \{3\}$. The LP pieces are $\mathcal{P} = \{3\}$ ($x_1 = 0 \leqslant x_2$) and $\mathcal{P} = \emptyset$ ($x_1 \geqslant 0 = x_2$), with multipliers $\boldsymbol{\lambda}^{\{3\}} = (\frac{3}{4}, \frac{1}{4}, -2, 0)^T$ and $\boldsymbol{\lambda}^{\emptyset} = (\frac{1}{4}, \frac{3}{4}, 0, -2)^T$. Hence $\boldsymbol{x}$ is B-stationary.

Starting our algorithm in $\boldsymbol{x}$, pivoting is possible on either index 3 or 4 and leads to an infinite loop of zero length steps without a certificate of stationarity.

An LPCC active set method without MPCC-LICQ

In absence of MPCC-LICQ, zero length steps must be handled to recognize B-stationary points that are not also S-stationary. LP anticycling methods (e.g. Bland's rule) can be shown to fail, too.

1. Monitor the sequence of the $k \geqslant 2$ most recent active set changes and detect cycles (Chvátal cycle detection). Reset the monitor as soon as a positive step is taken.
2. If a cycle is detected, solve all piece LPs hot-starting in the current vertex. This requires effort exponential in the number of non-strict complementarity pairs.
   - 2.1 Either the current vertex is confirmed optimal for all piece LPs, in which case B-stationarity is confirmed
   - 2.2 or one of the piece LPs takes a positive descent step, revealing a descent direction towards an improved vertex, and this breaks the cycle.

No useful dual problem for an LPCC

LP duality:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \boldsymbol{c}^T\boldsymbol{x} \text{ s.t. } A\boldsymbol{x} \geqslant \boldsymbol{b},\ \boldsymbol{x} \geqslant \boldsymbol{0} \quad \Longleftrightarrow \quad \max_{\boldsymbol{\lambda}\in\mathbb{R}^m} \quad \boldsymbol{b}^T\boldsymbol{\lambda} \text{ s.t. } A^T\boldsymbol{\lambda} \leqslant \boldsymbol{c},\ \boldsymbol{\lambda} \geqslant \boldsymbol{0}$$

LPCCs don't have a useful dual problem associated with them:

$$\min_{\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^n} \boldsymbol{c}^T\boldsymbol{u} + \boldsymbol{d}^T\boldsymbol{v} \text{ s.t. } A\boldsymbol{u} + B\boldsymbol{v} \geqslant \boldsymbol{b},\ \boldsymbol{0} \leqslant \boldsymbol{u} \perp \boldsymbol{v} \geqslant 0$$

Consider the case $n = 1$ and the two LP pieces ($u = 0$ or $v = 0$):

$$\min_{v}\ dv \text{ s.t. } Bv \geqslant b,\ v \geqslant 0, \qquad \min_{u}\ cu \text{ s.t. } Au \geqslant b,\ u \geqslant 0$$

with duals

$$\min_{\lambda}\ b\lambda \text{ s.t. } B^T\lambda \leqslant d,\ \lambda \geqslant 0, \qquad \min_{\lambda}\ b\lambda \text{ s.t. } A^T\lambda \leqslant c,\ \lambda \geqslant 0$$

The dual pieces <u>select</u> inequality constraints, while the paired constraint <u>drops out</u>.

No useful dual problem for an LPCC

One way of writing this as a single complementarity problem is the following:

$$
\begin{aligned}
\min_{\lambda, \boldsymbol{w}} \ & b\lambda \\
\text{s.t. } & A^T\lambda - c \leqslant w_1, \\
& B^T\lambda - d \leqslant w_2, \\
& \lambda \geqslant 0, \\
& 0 \leqslant w_1 \perp w_2 \geqslant 0.
\end{aligned}
$$

This could be generalized to more than one complementarity constraint, but requires expoentially many constraints since $\boldsymbol{w}$ are a <u>unary</u> encoding of the choice of piece.

There is no strong duality theorem, and the solution of the dual will not, in general, help to make decisions on the primal.

Hence there is no obvious path to an efficient dual active set / simplex method for LPCCs (?)

Duals of an LPCC problem

The Lagrangian dual problem for the (nonlinear) multiplicative form of the LPCC reads

$$\max_{\substack{\boldsymbol{\lambda},\boldsymbol{\mu},\\ \boldsymbol{\nu},\boldsymbol{\xi}}} \left[\inf_{\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^n} \boldsymbol{c}^T\boldsymbol{u} + \boldsymbol{d}^T\boldsymbol{v} - (A\boldsymbol{u} + B\boldsymbol{v} - \boldsymbol{b})^T\boldsymbol{\lambda} - (\boldsymbol{u}\circ\boldsymbol{v})^T\boldsymbol{\mu} - \boldsymbol{u}^T\boldsymbol{\nu} - \boldsymbol{v}^T\boldsymbol{\xi}\right] \text{ s.t. } \boldsymbol{\lambda},\boldsymbol{\nu},\boldsymbol{\xi} \geqslant \mathbf{0}.$$

The problem is non-convex. Still, the Wolfe dual problem can be written as follows:

$$\begin{aligned}\max_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\xi},\boldsymbol{u},\boldsymbol{v}}\ & \boldsymbol{c}^T\boldsymbol{u} + \boldsymbol{d}^T\boldsymbol{v} - (A\boldsymbol{u} + B\boldsymbol{v} - \boldsymbol{b})^T\boldsymbol{\lambda} - (\boldsymbol{u}\circ\boldsymbol{v})^T\boldsymbol{\mu} - \boldsymbol{u}^T\boldsymbol{\nu} - \boldsymbol{v}^T\boldsymbol{\xi}\\ \text{s.t. }& \boldsymbol{c} - A^T\boldsymbol{\lambda} - \boldsymbol{v}\circ\boldsymbol{\mu} - \boldsymbol{\nu} = \mathbf{0}\\ & \boldsymbol{d} - B^T\boldsymbol{\lambda} - \boldsymbol{u}\circ\boldsymbol{\mu} - \boldsymbol{\xi} = \mathbf{0},\\ & \boldsymbol{\lambda},\boldsymbol{\nu},\boldsymbol{\xi} \geqslant \mathbf{0}\end{aligned}$$

This can be simplified, but it remains a nonlinear problem:

$$\begin{aligned}\max_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{u},\boldsymbol{v}}\ & b^T\boldsymbol{\lambda} + (\boldsymbol{u}\circ\boldsymbol{v})^T\boldsymbol{\mu}\\ \text{s.t. }& \boldsymbol{c} - A^T\boldsymbol{\lambda} \geqslant \boldsymbol{v}\circ\boldsymbol{\mu}\\ & \boldsymbol{d} - B^T\boldsymbol{\lambda} \geqslant \boldsymbol{u}\circ\boldsymbol{\mu}\\ & \boldsymbol{\lambda} \geqslant \mathbf{0}\end{aligned}$$

The Fukushima-Tseng ϵ-active set algorithm

Pick $\mathcal{A}^0$, $\mathcal{B}^0$, ϵ^0, ν^0, $\boldsymbol{x}^0$.

1. Solve

$$\begin{aligned} \min F(\boldsymbol{x}) \text{ s.t. } G_i(\boldsymbol{x}) &= G_i(\boldsymbol{x}^k) && i \in \mathcal{A}^k \\ H_i(\boldsymbol{x}) &= H_i(\boldsymbol{x}^k) && i \in \mathcal{B}^k \\ G_i(\boldsymbol{x}) &\geqslant 0 && i \in \mathcal{B}^k \setminus \mathcal{A}^k \\ H_i(\boldsymbol{x}) &\geqslant 0 && i \in \mathcal{A}^k \setminus \mathcal{B}^k \end{aligned}$$

   for $\hat{\boldsymbol{z}}^k$ using an NLP solver up to feasibility tolerance ϵ^k.

2. If $\mathcal{A}_\epsilon(\hat{\boldsymbol{x}}^k) \cap \mathcal{B}_\epsilon(\hat{\boldsymbol{x}}^k) \subsetneq \mathcal{A}^k \cap \mathcal{B}^k$: Set $\boldsymbol{z}^{k+1} \leftarrow \hat{\boldsymbol{z}}^k$ and loop for $k \leftarrow k+1$.

3. If $\mathcal{A}_\epsilon(\hat{\boldsymbol{x}}^k) \cap \mathcal{B}_\epsilon(\hat{\boldsymbol{x}}^k) = \mathcal{A}^k \cap \mathcal{B}^k$: If $\mu_{G,i} \geqslant -\nu^k$ and $\mu_{H,i} \geqslant -\nu^k$ for all $i \in \mathcal{I}_{00}^k$, set

$$\nu^{k+1} \leftarrow \omega\nu^k \text{ for some } \omega \in (0,1), \quad \boldsymbol{z}^{k+1} \leftarrow \hat{\boldsymbol{z}}^k$$

   and go to 5.

The Fukushima-Tseng "ϵ-active set" algorithm

4. There is $i_k \in \mathcal{A}^k \cap \mathcal{B}^k$ such that $v_{i_k}^k < -\nu^k$ or $w_{i_k}^k < -\nu^k$. Then a feasible descent direction $\boldsymbol{d}$ with $\|\boldsymbol{d}\|_\infty \leqslant 1$ and $\nabla F(\boldsymbol{x}^k)^T \boldsymbol{d} \leqslant -\eta\nu^k/2$ can be found by solving a certain linear system.

   Let $\bar{t}^k = \min\{t_{\max}, \sup\{t \mid \hat{\boldsymbol{z}}^k + t\boldsymbol{d} \in \mathcal{F}_\varepsilon\}\}$ and perform an Armjo line search on F along $\boldsymbol{d}$ on the step size interval $(0, \bar{t}^k)$ to find $\boldsymbol{z}^{k+1}$.
5. Choose ϵ^{k+1} smaller than ϵ^k. Let $\mathcal{A}^{k+1} \leftarrow A_{\epsilon^{k+1}}(\boldsymbol{z}^{k+1})$, $\mathcal{B}^{k+1} \leftarrow B_{\epsilon^{k+1}}(\boldsymbol{z}^{k+1})$ and loop for $k \leftarrow k+1$.

This can be shown to convergence to S-stationary points unter MPCC-LICQ.