Summer School on Direct Methods for Optimal Control of Nonsmooth Systems
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Exercise 2: Smoothing, Nonsmooth Systems and Finite Elements with Switch Detection

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The goal of this exercise is to get familiar with the limitations of time-stepping and smoothing used to solve optimal control problems subject to nonsmooth dynamical systems. Furthermore, the exercise provide a practical introduction to Filippov systems. Finally, we will get to use the Finite Elements with Switch Detection (FESD) for solving simulation and optimal control problems.

1. **Smoothing nonsmooth systems. Continuation of Exercise 1.2**

   (a) Solve the cart pole OCP from the previous exercise with a small smoothing parameter σ. How does the trajectory look like? Can you explain this behavior?

   (b) Solve the OCP with different σ values in the range $[10^{-8}, 10]$. Visualize how the obtained objective function value relates to σ

   (c) Run the same experiment, "warm starting" the solver, i.e. providing an initial guess from a solution with more smoothing. What do you observe?
   Solution for Matlab can be found in `nosnoc/examples/cart_pole_with_friction`
   Solution for Python can be found in `nosnoc_py/examples/cart_pole_with_friction`

2. **Simulation of a scalar nonsmooth systems – Crossing** Consider the initial value problem of a dynamic system with state $x(t) \in \mathbb{R}$ which evolves over time as follows:

$$\dot{x}(t) = \begin{cases} 1, & \text{if } x < 0 \\ 3, & \text{if } x > 0 \end{cases} \tag{1}$$

   for the initial state $x(0) = -\sqrt{2}$ and a simulation time of $T = 2$.
   Solution for Matlab can be found in
   `nosnoc/examples/different_switching_cases/main_switching_cases.m`
   Solution for Python can be found in `nosnoc_py/examples/simplest/different_switch_cases.py`

   (a) qualitatively visualize the exact solution.

   (b) derive exact switch time t_s. Solution: $\sqrt{2}$

   (c) derive the solution $x(2)$ Solution: $3 \cdot (2 - \sqrt{2})$

(d) Recall the definition of a Filippov system:

$$\dot{x} \in F_{\mathrm{F}}(x) = \{\, F(x)\theta \mid \Sigma_i^{n_f}\theta_i = 1 \ \text{ with } \ \theta_j \geq 0,\ j = 1, \ldots, n_f, \ \text{and } \theta_i = 0 \text{ if } x \notin \bar{R}_i \,\}$$
$$\text{where} \quad R_i := \{\, x \in \mathbb{R}^{n_x} \mid \mathrm{diag}(S_{i,\bullet} c(x) > 0) \,\}$$

Derive the functions $F : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_f}$ and $c : \mathbb{R}^{n_x} \to \mathbb{R}^{n_c}$, and the matrix $S \in \mathbb{R}^{n_f \times n_c}$ for the example in Eq. (1).

About `NOSNOC` and `nosnoc_py`.

For the following exercises you will need to install `NOSNOC`, respectively `nosnoc_py`.
These packages implement the Finite Elements with Switch Detection discretization (FESD) for optimal control and simulation problems with nonsmooth systems.

- **Matlab:** clone `https://github.com/nurkanovic/nosnoc` and follow the instructions
- **Python:** clone `https://github.com/FreyJo/nosnoc_py/` and follow the instructions

3. **Simulation of scalar nonsmooth systems – with `nosnoc`**

   (a) Formulate the problem in `nosnoc` by using your definitions of F, c, S

   (b) Solve the problem numerically without FESD, then with FESD. How do the numerical solutions compare to the exact one?

4. **Sliding mode variation**

   (a) Modify the Filippov system corresponding to equation (1), such that $\dot{x} = -1$ if $x > 0$.

   (b) Simulate it as in the exercise before.

   (c) Bonus: create a similar example with a spontaneous exit of a sliding mode. Initialize the `nosnoc` solver to get a result exiting on either side.

5. **Optimal control with nonsmooth systems**

   (a) Solve the cart pole OCP without smoothing with FESD in `nosnoc`.
   You will need to provide the cart pole model in the Fillipov form.
   Use the following options to discretize the OCP with `nosnoc`: 20 control intervals, with 2 finite elements and Radau-IIA Butcher tableau of size $n_s = 2$.

   (b) What do you observe compared to the solution in Task 1? Can you explain why a different behavior can be observed?