Summer School on Direct Methods for Optimal Control of Nonsmooth Systems
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Exercise 1: Introduction to CasADi and direct optimal control

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The goal of this exercise is to get familiar with the open-source tool `CasADi` and to solve a first optimal control problem via direct collocation. Templates are available for Matlab and Python.

About `CasADi`

`CasADi` is an open-source tool for nonlinear optimization and algorithmic differentiation.
In preparation of the summer school, please install `CasADi`: `https://web.casadi.org/get/`
Make sure you can use it succesfully, running a small test:

```
% Matlab
import casadi.*
x = SX.sym('x')
disp(jacobian(sin(x),x))
```

```
# Python
from casadi import *
x = SX.sym("x")
print(jacobian(sin(x),x))
```

If something is unclear, consult the `CasADi` documentation at `https://web.casadi.org/docs/`.
For Python users: The solution templates borrow some functionality from `nosnoc`, please clone `https://github.com/FreyJo/nosnoc_py/` and follow the installation instructions.

Pendulum on cart with friction

Consider a model of a pendulum on a cart with friction, with state $x = (p, \theta, v, \omega)$, where p describes the position of the cart, θ the angle of the pendulum mounted on the cart, v the velocity of the cart and ω the angular velocity of the pendulum. The control action u is a force which accelerates the cart.
The differential equations corresponding to this system can be derived using Lagrange mechanics, where $q = (p, \theta)$ describe the generalized coordinates and $\dot{q} = (v, \omega)$ are its derivative. The system has the following parameters, which are assumed to be fixed, the length of the pendulum $l = 1$, the mass of the cart $m_1 = 1$, the mass at the end of the pendulum $m_2 = 0.1$ and the gravitational constant $g = 9.81$.
The differential equations can be derived as follows. First, the derivatives of the generalized coordinates are simply $\dot{p} = v$ and $\dot{\theta} = \omega$. Second, the dynamics of the generalized velocities are

$$\ddot{q} = \begin{pmatrix} \dot{v} \\ \dot{\omega} \end{pmatrix} = M(q)^{-1} f_{\text{all}}(q, \dot{q}, u) \tag{1}$$

where the inertia matrix $M(\cdot)$ of the system is given as

$$M(q) = \begin{pmatrix} m_1 + m_2 & m_2 l \cos(\theta) \\ m_2 l \cos(\theta) & m_2 l^2 \end{pmatrix} \tag{2}$$

and $f_{\text{all}}(\cdot)$ gathers the gravity, control, Coriolis and friction forces, as follows:

$$f_{\text{all}}(\cdot) = \begin{pmatrix} 0 \\ -m_2 g l \sin(\theta) \end{pmatrix} + \begin{pmatrix} u \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & -m_2 l \cos(\theta) \\ 0 & 0 \end{pmatrix} \dot{q} + \begin{pmatrix} -F_{\text{Friction}} \text{sign}(v) \\ 0 \end{pmatrix}.$$

Here, F_{Friction} is a fixed parameter which we will change in this exercise to use the model, with or without friction. We will use values $F_{\text{Friction}} \in \{0, 2\}$. For the following equations, we denote the explicit ODE as $\dot{x} = f_{\text{ODE}}(x, u)$.

1. **Modelling and simulation with CasADi** The goal is to simulate the cart pole model with different settings.

   (a) Simulate the system without friction. Fill in the missing part in the `CasADi` model description.

   (b) Extend the model to include a smoothed friction model, by replacing $\operatorname{sign}(z)$ with $\tanh(\frac{z}{\sigma})$. Model the smoothing parameter σ as a `CasADi` parameter, and run the simulation with $\sigma = 1.0$.

2. **Optimal Control with Direct collocation**
   Collocation methods belong to the class of implicit Runge-Kutta methods for solving initial value problems. To discretize an optimal control problem with *direct collocation* we replace the continuous-time dynamics

$$\dot{x}(t) = f_{\mathrm{ODE}}(x(t), u(t)),$$

by the discrete-time collocation equations. Thereby, we split the control horizon $[0, T]$ into N control intervals with a uniform time discretization grid $t_n = nh$, $n = 0, \dots, N$, where h is the step size and the corresponding state values are $x_n = x(t_n)$. For the control discretization we use $u(t) = u_n, t \in [t_n, t_{n+1}]$, $n = 1, \dots, N$. On every control interval the state trajectory is approximated by polynomials $q_n(t)$, $n = 1, \dots, N$. Note that in every control interval we may have multiple integration steps, but for simplicity we take only one integration step per control interval.

Next, on each control interval $[t_n, t_{n+1}]$, we compute the coefficients of these polynomials to ensure that the ODE is exactly satisfied at the *collocation points* $t_{n,i} = t_n + hc_i, i = 1, \dots, n_{\mathrm{s}}$, where, n_{s} is the number of stages. The choice of the points $0 = c_0 < c_1 < \ldots < c_{n_{\mathrm{s}}} \leq 1$ determines the accuracy and stability properties of the resulting method. Popular choices for c_i are the Radau IIA or Gauss-Legendre points. In the lecture, we found the interpolating polynomial $\dot{q}_n(t)$ through the state derivatives $k_{n,1}, \dots, k_{n,n_{\mathrm{s}}}$. Here, we implement a collocation method by finding the interpolating polynomial $q_n(t)$ through the initial value x_n and *state values* $x_{n,1}, \dots, x_{n,n_{\mathrm{s}}}$ at the stage points.

For the implementation, we make use of the Lagrangian polynomial basis. Using these time points, we define a basis for our polynomials:

$$\ell_i(\tau) = \prod_{j=0,\, i \neq j}^{n_{\mathrm{s}}} \frac{\tau - c_j}{c_i - c_j}, \qquad i = 0, \dots, n_{\mathrm{s}}. \tag{3}$$

Note that, in contrast to the lecture, the counter starts from $i = 0$, as we include the point $c_0 = 0$, since we interpolate through x_n.

We approximate the state trajectory on $[t_n, t_{n+1}]$ by a linear combination of the basis functions:

$$q_n(t) = \sum_{j=0}^{n_{\mathrm{s}}} \ell_j \left(\frac{t - t_n}{h} \right) x_{n,j}. \tag{4}$$

By differentiation, we obtain an approximation of the time derivative at each collocation point:

$$\dot{q}_n(t_{n,i}) = \frac{1}{h} \sum_{j=0}^{n_{\mathrm{s}}} \dot{\ell}_j(c_i)\, x_{n,j} := \frac{1}{h} \sum_{j=0}^{n_{\mathrm{s}}} C_{j,i}\, x_{n,j}, \quad i = 0, \dots, n_{\mathrm{s}}. \tag{5}$$

The expression for the state at the end of an interval reads as:

$$x_{n+1} = \sum_{i=0}^{n_{\mathrm{s}}} \ell_i(1)\, x_{n,i} := \sum_{i=0}^{n_{\mathrm{s}}} D_i\, x_{k,i} \tag{6}$$

Moreover, using the obtained approximation $q_n(t)$ we can integrate the *stage cost*

$$\int_0^T L(x(t), u(t))\mathrm{d}t,$$

over every control interval and obtain a formula for *quadratures*:

$$\int_{t_n}^{t_{n+1}} \sum_{j=0}^{n_\mathrm{s}} \ell_j\left(\frac{t-t_n}{h}\right) L(x_{n,j}, u_n)\,\mathrm{d}t = h \sum_{j=0}^{n_\mathrm{s}} \int_0^1 \ell_j(t)\,\mathrm{d}t\, L(x_{n,j}, u_n) := h \sum_{j=0}^{n_\mathrm{s}} B_j\, L(x_{n,j}, u_n). \quad (7)$$

Tasks:

(a) Using the derived formulae above, write down on paper the collocation equations for a single integration interval.
Solution:

$$\sum_{j=0}^{n_\mathrm{s}} C_{j,i} x_{n,j} = h\, f(x_{n,i}, u_n), \qquad i = 1, \dots, n_\mathrm{s}$$

$$x_{n+1} = \sum_{i=0}^{n_\mathrm{s}} D_i\, x_{k,i}.$$

(b) We want to solve the continuous time optimal control problem (OCP)

$$\begin{aligned}
\underset{x(\cdot),\, u(\cdot)}{\text{minimize}} \quad & \int_0^T f_q(x(\cdot), u(\cdot)) + f_{q,T}(x(T)) \\
\text{subject to} \quad & x(0) = \bar{x}_0, \\
& \dot{x}(t) = f_\mathrm{ODE}(x(t), u(t)),\ t \in [0, T), \\
& l_\mathrm{bu} \le u(t) \le u_\mathrm{bu}, \qquad t \in [0, T], \\
& l_\mathrm{bx} \le x(t) \le u_\mathrm{bx}, \qquad t \in [0, T]
\end{aligned}$$

where the initial state is $\bar{x}_0 = [1, 0, 0, 0]$, the control bounds are $l_\mathrm{bx} = -30,\ u_\mathrm{bu} = 30$, the state bounds are $l_\mathrm{bx} = [-5, -\infty, -\infty, -\infty]$, $u_\mathrm{bx} = [5, \infty, \infty, \infty]$ and the objective function terms are:

$$f_q(x, u) = (x - x_\mathrm{ref})^\top Q (x - x_\mathrm{ref}) + u^\top R u$$
$$f_{q,T}(x, u) = (x - x_\mathrm{ref})^\top Q_\mathrm{terminal} (x - x_\mathrm{ref})$$

with $Q = \mathrm{diag}(10, 100, 1, 1), R = 1, Q_\mathrm{terminal} = \mathrm{diag}(500, 100, 10, 10)$, and reference state $x_\mathrm{ref} = (0, \pi, 0, 0)$ describing the unstable equilibrium point.

(c) Define the ingredients listed above to specify the OCP and solve it with direct collocation in `CasADi`. Note: in Matlab, the OCP formulation is part of the model.
Complete the implementation of direct collocation to discretize the OCP problem. The matrices B, C and D are already provided, use your solution from (a).

(d) **Bonus**: Form the Jacobian of the constraints and inspect the sparsity pattern using the `spy` command. The following is a hint:

```
% MATLAB
J = jacobian(vertcat(g{:}), vertcat(w{:}));
spy(sparse(DM.ones(J.sparsity())));
```

```
# Python
J = jacobian(vertcat(g), vertcat(w))
import matplotlib.pylab as plt
plt.spy(DM.ones(J.sparsity()).sparse())
```

Repeat the same for the Hessian of the Lagrangian function $\mathcal{L}(w, \lambda) = f_\mathrm{objective}(w) + \lambda^\mathrm{T} g(w)$.