

**Exercise 8: Continuous-Time Optimal Control**

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Consider the following continuous-time optimal control problem:

$$\begin{aligned} \min_{x(\cdot), u(\cdot)} \quad & \int_0^T L(x(t), u(t))dt + E(x(T)) \\ \text{s.t.} \quad & x(0) = \bar{x}_0 \\ & \dot{x}(t) = f(x(t), u(t)), \quad t \in [0, T]. \end{aligned} \tag{1}$$

1. (a) Discretize problem (1) using the explicit Euler integrator with step-size  $h$  over  $N$  intervals. Write on paper the obtained discrete-time optimal control problem.

$$\begin{aligned} \min_{x, u} \quad & h \sum_{i=0}^{N-1} L(x_i, u_i) + E(x_N) \\ \text{s.t.} \quad & x_0 = \bar{x}_0 \\ & x_{i+1} = x_i + hf(x_i, u_i), \quad i = 0, \dots, N-1 \end{aligned}$$

- (b) Write the first-order optimality conditions for the discretized problem obtained in (a). Use the Hamiltonian function defined as

$$H(x, u, \lambda) := L(x, u) + \lambda^T f(x, u) \tag{2}$$

to simplify these conditions.

$$\begin{aligned} r_{E_0} &:= \bar{x}_0 - x_0 &= 0 \\ r_{Sx_0} &:= h\nabla_{x_0} H(x_0, u_0, \lambda_1) - \lambda_0 + \lambda_1 &= 0 \\ r_{Su_0} &:= h\nabla_{u_0} H(x_0, u_0, \lambda_1) &= 0 \\ r_{E_1} &:= x_0 + hf(x_0, u_0) - x_1 &= 0 \\ r_{Sx_1} &:= h\nabla_{x_1} H(x_1, u_1, \lambda_2) - \lambda_1 + \lambda_2 &= 0 \\ r_{Su_1} &:= h\nabla_{u_1} H(x_1, u_1, \lambda_2) &= 0 \\ \vdots &:= \vdots &= \vdots \\ r_{E_N} &:= x_{N-1} + hf(x_{N-1}, u_{N-1}) - x_N &= 0 \\ r_{Sx_N} &:= \nabla_{x_N} E(x_N) - \lambda_N &= 0 \end{aligned}$$

- (c) Now let  $N \rightarrow \infty$  and  $h \rightarrow 0$ . What type of problem do the conditions derived in (b) converge to?

$$\begin{aligned} x(0) &= \bar{x}_0 \\ \dot{\lambda} &= -\nabla_x H(x, u, \lambda) \\ \dot{x} &= f(x, u) \\ 0 &= \nabla_u H(x, u, \lambda) \\ \lambda(T) &= \nabla_x E(x(T)) \end{aligned}$$



where

$$P_1 := Q_1 + A_1^T Q_2 A_1 - (S_1^T + A_1^T Q_2 B_1)(R_1 + B_1^T Q_2 B_1)^{-1}(S_1 + B_1^T Q_2 A_1).$$

Noting that the structure of (1e) is the same in (1e), a recursion can be defined that can be used to progressively reduce the system for an arbitrary number of stages  $N$ :

$$P_k := Q_k + A_k^T P_{k+1} A_k - (S_k^T + A_k^T P_{k+1} B_k)(R_k + B_k^T P_{k+1} B_k)^{-1}(S_k + B_k^T P_{k+1} A_k).$$

- (f) **[Bonus]** What kind of matrix ODE does the difference equation derived in (e) converge to for  $N \rightarrow \infty$  and  $h \rightarrow 0$ ?

*Hint: if you have not solved the bonus point (e) you can refer to equation 8.5 from the course's script.*

The difference equation has the form

$$P_k := hQ_c + (I + hA_c)^T P_{k+1} (I + hA_c) - (hS_c^T + (I + hA_c)^T P_{k+1} hB_c)(hR_c + hB_c^T P_{k+1} hB_c)^{-1}(hS_c + hB_c^T P_{k+1} (I + hA_c)).$$

Expanding and eliminating the terms of order 2 or higher, we obtain

$$P_k := hQ_c + P_{k+1} + hA_c^T P_{k+1} + hP_{k+1} A_c - (hS_c^T + P_{k+1} hB_c) \frac{1}{h} (R_c)^{-1} (hS_c + hB_c^T P_{k+1}).$$

dividing by  $h$  and for  $h \rightarrow 0$  we obtain:

$$-\dot{P} := Q_c + A_c^T P + P A_c - (S_c^T + P B_c) R_c^{-1} (S_c + B_c^T P).$$