Exercise 1: Dynamic systems

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1. We model the amount of water $m(t)$ (in kg) in a sink, into which water flows through a faucet at the mass flow rate $u(t)$ (in kg/s), which we can control. In addition to the inflow $u(t)$ through the faucet, there is also an outflow because the plug is open. The outflow has mass flow rate $k\sqrt{m(t)}$, where k is a positive constant (with unit $\sqrt{\text{kg}}$/s) assumed to be known. We assume that the capacity of the sink is infinite, and thus overflow can never occur.

   (a) Sketch the sink with its inlet and outlet flows.

   (b) Decide which state - or states - $x(t)$ you need to describe the system completely. For this purpose, consider which quantities you need besides the input signal and the dynamic equations to predict the system behavior.
   The state of the system is completely described by the water quantity $m(t)$. Additionally, the initial state is needed.

   $$\begin{aligned} x(t) &= m(t) \\ x(0) &= x_0 \end{aligned}$$

   (c) Derive an ordinary differential equation (ODE) of the form $\dot{x}(t) = f(x(t), u(t))$ that describes the dynamic behavior of the states $x(t)$. Consider the inflow and outflow of water. Use the initial condition $m(0) = m_0$, where m_0 is a positive constant assumed to be known.
   The change in volume is determined by inflow and outflow: $\dot{m}(t) = u(t) - k\sqrt{m(t)}$

   $$\dot{x}(t) = \dot{m}(t) = u(t) - k\sqrt{m(t)}$$

   The initial condition $m(0)$, how much water is in the basin at the beginning, provides:

   $$m(0) = m_0 = x(0)$$

   The solution is therefore:

   $$\begin{gathered} \Rightarrow f\left(x\left(t\right), u\left(t\right)\right) = u(t) - k\sqrt{x(t)} \\ x(0) = m_0 \end{gathered}$$

2. Extend the setup from the previous task to include a catch basin that holds the entire volume of water that flows out of the sink. In addition, evaporation is now to be taken into account as well. Let the evaporation rate of a volume of water $m(t)$ be $v \cdot m(t)$, where v is a known constant with unit $1/\text{s}$. Formulate the differential equations that describe the amount of water in the two basins. Use $m_1(t)$ for the amount of water in the sink and $m_2(t)$ for the amount of water in the catch basin. The known initial values are $m_1(0) = m_{01}$ and $m_2(0) = m_{02}$.
   The state of the system is completely described by the amount of water $m_1(t)$ in the sink and the amount of water $m_2(t)$ in the catch basin.

   $$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}$$

   The change in the amount of water in the sink is equal to the rate of incoming water minus the rate of outgoing and evaporating water:

   $$\dot{m}_1(t) = u(t) - k\sqrt{m_1(t)} - v \cdot m_1(t)$$

   The initial condition provides:

   $$m_1(0) = m_{01}$$

   The change of the water volume in the catch basin results from the difference of the incoming water volume rate from the wash basin and the evaporation rate of the already contained water:

   $$\begin{aligned} \dot{m}_2(t) &= k\sqrt{m_1(t)} - v \cdot m_2(t) \\ m_2(0) &= m_{02} \end{aligned}$$

The solution is therefore

$$
\begin{aligned}
f\left(x\left(t\right),u\left(t\right)\right) &= \begin{bmatrix} u(t) - k\sqrt{x_1(t)} - v\cdot x_1(t) \\ k\sqrt{x_1(t)} - v\cdot x_2(t) \end{bmatrix} \\
x(0) &= \begin{bmatrix} m_{01} \\ m_{02} \end{bmatrix}
\end{aligned}
$$

3. In this task, the nonlinear sink model (without catch basin) is to be linearized around a rest position. The sink with inflow $u(t)$ and water quantity $x(t)$ is described by the ODE

$$\dot{x}(t) = u(t) - k\sqrt{x(t)}\,.$$

   (a) Calculate the equilibrium state x_{ss} as a function of the constant flow rate u_{ss}.
   In the equilibrium state: $\dot{x}_{\mathrm{ss}} = f(x_{\mathrm{ss}}, u_{\mathrm{ss}}) = 0$

$$
\begin{aligned}
0 &= u_{\mathrm{ss}} - k\sqrt{x_{\mathrm{ss}}} \\
\Leftrightarrow \sqrt{x_{\mathrm{ss}}} &= \frac{u_{\mathrm{ss}}}{k} \\
\Rightarrow x_{\mathrm{ss}} &= \frac{u_{\mathrm{ss}}^2}{k^2}
\end{aligned}
$$

   (b) Linearize the system for small deviations $(\delta x(t), \delta u(t)))$ from rest $(x_{\mathrm{ss}}, u_{\mathrm{ss}})$ to obtain an ODE of the following form:

$$\delta\dot{x}(t) = A\;\delta x(t) + B\;\delta u(t).$$

   The following applies to the linearization:

$$
\begin{aligned}
A = \frac{\delta f}{\delta x}(x_{\mathrm{ss}}, u_{\mathrm{ss}})\mathrm{und}B &= \frac{\delta f}{\delta u}(x_{\mathrm{ss}}, u_{\mathrm{ss}})\mathrm{mit}f(x,u) = u(t) - k\sqrt{x(t)} \\
A &= \frac{-k}{2\sqrt{x_{\mathrm{ss}}}} = \frac{-k^2}{2u_{\mathrm{ss}}}, B = 1 \\
\delta\dot{x}(t) &= \frac{-k^2}{2u_{\mathrm{ss}}}\delta x(t) + \delta u(t)
\end{aligned}
$$

   (c) Now assume $k = 0.60\frac{\sqrt{\mathrm{kg}}}{\mathrm{s}}$ and $u_{\mathrm{ss}} = 2.4\frac{\mathrm{kg}}{\mathrm{s}}$. Calculate x_{ss}, A, and B.

$$
\begin{aligned}
x_{\mathrm{ss}} &= \frac{u_{\mathrm{ss}}^2}{k^2} = \left(\frac{2.4\;\mathrm{kg/s}}{0.60\sqrt{\mathrm{kg}}/\mathrm{s}}\right)^2 = \left(4\sqrt{\mathrm{kg}}\right)^2 = 16\;\mathrm{kg} \\
A &= \frac{-k^2}{2u_{\mathrm{ss}}} = \frac{-\left(0.60\sqrt{\mathrm{kg}}/\mathrm{s}\right)^2}{2\cdot 2.4\;\mathrm{kg/s}} = \frac{-0.36}{4.8\;\mathrm{s}} = -0.075\frac{1}{\mathrm{s}} \\
B &= 1
\end{aligned}
$$

   (d) Now consider an extended sink that observes the water temperature $x_2(t)$ in addition to the water quantity $x_1(t)$. Initially, there is a quantity of water m_0 of temperature $T_0 = T_a$ in the basin. The inflowing water has the temperature T_h. Since the water in the basin also releases heat to the environment, heat losses occur with a heat loss rate of $k_2 \cdot C \cdot m(t) \cdot (x_2(t) - T_a)$, where k_2 is a constant with unit $1/s$, the constant C is the specific heat capacity of water with unit $J/(kg \cdot K)$, and T_a is the ambient temperature. This system is described by the ODE

$$\dot{x}(t) = \begin{bmatrix} -k_1\sqrt{x_1(t)} + u(t) \\ -k_2(x_2(t) - T_a) + \frac{T_{\mathrm{h}} - x_2(t)}{x_1(t)}u(t) \end{bmatrix}.$$

   Carry out steps a) to c) again under the assumption that $T_{\mathrm{h}} = 340\,\mathrm{K}, T_a = 300\,\mathrm{K}$, $k_1 = 0.60\frac{\sqrt{\mathrm{kg}}}{\mathrm{s}}, k_2 = 0.1\frac{1}{\mathrm{s}}$ and $u_{\mathrm{ss}} = 2.4\frac{\mathrm{kg}}{\mathrm{s}}$.
   In the state of equilibrium it holds that: $\dot{x}_{\mathrm{ss}} = f(x_{\mathrm{ss}}, u_{\mathrm{ss}}) = 0$

$$
\begin{aligned}
\begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} u_{\mathrm{ss}} - k_1\sqrt{x_{1\mathrm{ss}}} \\ -k_2(x_{2\mathrm{ss}} - T_0) + \frac{T_{\mathrm{h}} - x_{2\mathrm{ss}}}{x_{1\mathrm{ss}}}u_{\mathrm{ss}} \end{bmatrix} \\
\Leftrightarrow k_2 x_{2\mathrm{ss}} + \frac{u_{\mathrm{ss}}x_{2\mathrm{ss}}}{x_{1\mathrm{ss}}} &= k_2 T_0 + T_{\mathrm{h}}\frac{u_{\mathrm{ss}}}{x_{1\mathrm{ss}}} \\
\Rightarrow x_{2\mathrm{ss}} &= \frac{k_2 T_0 + T_{\mathrm{h}}\frac{u_{\mathrm{ss}}}{x_{1\mathrm{ss}}}}{k_2 + \frac{u_{\mathrm{ss}}}{x_{1\mathrm{ss}}}}
\end{aligned}
$$

The following applies to the linearization:

$$A = \frac{\delta f}{\delta x}(x_{\mathrm{ss}}, u_{\mathrm{ss}})\mathrm{und}B = \frac{\delta f}{\delta u}(x_{\mathrm{ss}}, u_{\mathrm{ss}})\mathrm{mit}f(x,u) = \begin{bmatrix} u(t) - k_1\sqrt{x_1(t)} \\ -k_2(x_2(t) - T_0) + \frac{T_{\mathrm{h}} - x_2(t)}{x_1(t)}u(t) \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{-k_1^2}{2u_{\mathrm{ss}}} & 0 \\ \frac{x_{2\mathrm{ss}} - T_{\mathrm{h}}}{x_{1\mathrm{ss}}^2}u_{\mathrm{ss}} & -k_2 - \frac{u_{\mathrm{ss}}}{x_{1\mathrm{ss}}} \end{bmatrix} B = \begin{bmatrix} 1 \\ \frac{T_{\mathrm{h}} - x_{2\mathrm{ss}}}{x_{1\mathrm{ss}}} \end{bmatrix}$$

With $T_{\mathrm{h}} = 340\,\mathrm{K}, T_0 = 300\,\mathrm{K}$ *and* $k_2 = 0.1\frac{1}{\mathrm{s}}$ *it follows:*

$$x_{2\mathrm{ss}} = \frac{0.1\frac{1}{\mathrm{s}} \cdot 300\,\mathrm{K} + 340\,\mathrm{K}\frac{2{,}4\frac{\mathrm{kg}}{\mathrm{s}}}{16\,\mathrm{kg}}}{0.1\frac{1}{\mathrm{s}} + \frac{2.4\frac{\mathrm{kg}}{\mathrm{s}}}{16\,\mathrm{kg}}} = 324\,\mathrm{K}$$

$$A = \begin{bmatrix} -0.075\frac{1}{\mathrm{s}} & 0 \\ -0.15\frac{\mathrm{K}}{\mathrm{kg\,s}} & -0.25\frac{1}{\mathrm{s}} \end{bmatrix} B = \begin{bmatrix} 1 \\ 1\frac{\mathrm{K}}{\mathrm{kg}} \end{bmatrix}$$